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Some new results on convex sequences

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Abstract
In the present paper, we obtained a main theorem related to factored infinite series. Some new results are also deduced.

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1 Introduction

Let $\sum a_n$ be a given infinite series with (s_n) as the sequence of partial sums. In [1], Borwein introduced the (C,α,β) methods in the following form: Let $\alpha+\beta\neq -1,-2,\ldots$. Then the (C,α,β) mean is defined by

$$u_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}s_v, \tag{1}$$

where

$$A_n^{\alpha+\beta}=O\bigl(n^{\alpha+\beta}\bigr),\qquad A_0^{\alpha+\beta}=1\quad\text{and}\quad A_{-n}^{\alpha+\beta}=0\quad\text{for } n>0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C,\alpha,\beta,\sigma;\delta|_k$, $k\geq 1$, $\delta\geq 0$, $\alpha+\beta>-1$, and $\sigma\in R$, if (see [2])

$$\sum_{n=1}^{\infty}n^{\sigma(\delta k+k-1)}\frac{|t_n^{\alpha,\beta}|^k}{n^k}<\infty, \tag{3}$$

where $t_n^{\alpha,\beta}$ is the (C,α,β) transform of the sequence (na_n). It should be noted that, for $\beta=0$, the $|C,\alpha,\beta,\sigma;\delta|_k$ summability method reduces to the $|C,\alpha,\sigma;\delta|_k$ summability method (see [3]). Let us consider the sequence $(\theta_n^{\alpha,\beta})$ which is defined by (see [4])

$$\theta_n^{\alpha,\beta}=\begin{cases}|t_n^{\alpha,\beta}|, & \alpha=1,\beta>-1,\\ \max_{1\leq v\leq n}|t_v^{\alpha,\beta}|, & 0<\alpha<1,\beta>-1.\end{cases} \tag{4}$$

2 The main result

Here, we shall prove the following theorem.

Theorem *If* (λ_n) *is a convex sequence (see* [5]) *such that the series* $\sum \frac{\lambda_n}{n}$ *is convergent and let* $(\theta_n^{\alpha,\beta})$ *be a sequence defined as in* (4). *If the condition*

$$\sum_{n=1}^{m} n^{\sigma(\delta k+k-1)} \frac{(\theta_n^{\alpha,\beta})^k}{n^{k-1}} = O(m) \quad \text{as } m \to \infty \tag{5}$$

holds, then the series $\sum a_n \lambda_n$ *is summable* $|C,\alpha,\beta,\sigma;\delta|_k$, $k \geq 1$, $0 \leq \delta < \alpha \leq 1$, $\sigma \in R$, *and* $(\alpha+\beta+1)k - \sigma(\delta k+k-1) > 1$.

One should note that, if we set $\sigma = 1$, *then we obtain a well-known result of Bor (see* [6]).

We will use the following lemmas for the proof of the theorem given above.

Lemma 1 ([4]) *If* $0 < \alpha \leq 1$, $\beta > -1$, *and* $1 \leq \nu \leq n$, *then*

$$\left|\sum_{p=0}^{\nu} A_{n-p}^{\alpha-1} A_p^{\beta} a_p\right| \leq \max_{1 \leq m \leq \nu} \left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p\right|. \tag{6}$$

Lemma 2 ([7]) *If* (λ_n) *is a convex sequence such that the series* $\sum \frac{\lambda_n}{n}$ *is convergent, then* $n\Delta\lambda_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} (n+1)\Delta^2 \lambda_n$ *is convergent.*

3 Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} A_\nu^{\beta} \nu a_\nu \lambda_\nu.$$

First applying Abel's transformation and then using Lemma 1, we have

$$\begin{aligned}
T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}} \sum_{\nu=1}^{n-1} \Delta\lambda_\nu \sum_{p=1}^{\nu} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} A_\nu^{\beta} \nu a_\nu, \\
\left|T_n^{\alpha,\beta}\right| &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{\nu=1}^{n-1} |\Delta\lambda_\nu| \left|\sum_{p=1}^{\nu} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p\right| + \frac{|\lambda_n|}{A_n^{\alpha+\beta}} \left|\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} A_\nu^{\beta} \nu a_\nu\right| \\
&\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{\nu=1}^{n-1} A_\nu^{\alpha} A_\nu^{\beta} \theta_\nu^{\alpha,\beta} |\Delta\lambda_\nu| + |\lambda_n| \theta_n^{\alpha,\beta} \\
&= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}.
\end{aligned}$$

In order to complete the proof of the theorem by using Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\sigma(\delta k+k-1)} \frac{|T_{n,r}^{\alpha,\beta}|^k}{n^k} < \infty, \quad \text{for } r = 1,2.$$

For $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, and we obtain

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\sigma(\delta k+k-1)} \frac{|T_{n,1}^{\alpha,\beta}|^k}{n^k} &\leq \sum_{n=2}^{m+1} n^{\sigma(\delta k+k-1)-k} \left| \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \Delta\lambda_v \right|^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{(\alpha+\beta+1)k-\sigma(\delta k+k-1)}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} \Delta\lambda_v \left(\theta_v^{\alpha,\beta}\right)^k \right\} \\
&\quad \times \left\{ \sum_{v=1}^{n-1} \Delta\lambda_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta\lambda_v \left(\theta_v^{\alpha,\beta}\right)^k \sum_{n=v+1}^{m+1} \frac{1}{n^{(\alpha+\beta+1)k-\sigma(\delta k+k-1)}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta\lambda_v \left(\theta_v^{\alpha,\beta}\right)^k \int_v^{\infty} \frac{dx}{x^{(\alpha+\beta+1)k-\sigma(\delta k+k-1)}} \\
&= O(1) \sum_{v=1}^{m} \Delta\lambda_v v^{\sigma(\delta k+k-1)} \frac{(\theta_v^{\alpha,\beta})^k}{v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(\Delta\lambda_v) \sum_{p=1}^{v} p^{\sigma(\delta k+k-1)} \frac{(\theta_p^{\alpha,\beta})^k}{p^{k-1}} \\
&\quad + O(1) \Delta\lambda_m \sum_{v=1}^{m} v^{\sigma(\delta k+k-1)} \frac{(\theta_v^{\alpha,\beta})^k}{v^{k-1}} \\
&= O(1) \sum_{v=1}^{m} v \Delta^2 \lambda_v + O(1) m \Delta\lambda_m = O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

by virtue of hypotheses of the theorem and Lemma 2. Similarly, we have

$$\begin{aligned}
\sum_{n=1}^{m} n^{\sigma(\delta k+k-1)} \frac{|T_{n,2}^{\alpha,\beta}|^k}{n^k} &= O(1) \sum_{n=1}^{m} \frac{\lambda_n}{n} n^{\sigma(\delta k+k-1)} \frac{(\theta_n^{\alpha,\beta})^k}{n^{k-1}} \\
&= O(1) \sum_{n=1}^{m-1} \Delta\left(\frac{\lambda_n}{n}\right) \sum_{v=1}^{n} v^{\sigma(\delta k+k-1)} \frac{(\theta_v^{\alpha,\beta})^k}{v^{k-1}} \\
&\quad + O(1) \frac{\lambda_m}{m} \sum_{n=1}^{m} n^{\sigma(\delta k+k-1)} \frac{(\theta_n^{\alpha,\beta})^k}{n^{k-1}} \\
&= O(1) \sum_{n=1}^{m-1} \Delta\lambda_n + O(1) \sum_{n=1}^{m-1} \frac{\lambda_{n+1}}{n+1} + O(1)\lambda_m \\
&= O(1) \sum_{n=1}^{m-1} \Delta\lambda_n + O(1) \sum_{n=2}^{m-1} \frac{\lambda_n}{n} + O(1)\lambda_m \\
&= O(1)(\lambda_1 - \lambda_m) + O(1) \sum_{n=1}^{m-1} \frac{\lambda_n}{n} + O(1)\lambda_m \\
&= O(1) \quad \text{as } m \to \infty
\end{aligned}$$

in view of hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

4 Conclusions

By selecting proper values for α, β, δ, and σ, we have some new results concerning the $|C,1|_k$, $|C,\alpha|_k$, and $|C,\alpha;\delta|_k$ summability methods.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author carried out all work of this article and the main theorem. The author read and approved the final manuscript.

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