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Weighted quantitative isoperimetric inequalities in the Grushin space R^{h+1} with density $|x|^p$

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Abstract

In this paper, we prove weighted quantitative isoperimetric inequalities for the set $E_\alpha = \{(x,y) \in R^{h+1} : |y| < \int_{\arcsin |x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\,dt, |x| < 1\}$ in half-cylinders in the Grushin space R^{h+1} with density $|x|^p$, $p \geq 0$.

MSC: 52B60; 53C17

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1 Introduction

The study of isoperimetric problems in Carnot-Carathéodory spaces has been an active field over the past few decades. Pansu [1] first proved an isoperimetric inequality of the type $P_H(E) \geq C|E|^{\frac{3}{4}}$ $(C > 0)$ in the Heisenberg group $\mathbb{H}^1$ where $P_H(E)$ and $|E|$ denote Heisenberg perimeter and Lebesgue volume of E, respectively. In 1983 Pansu [2] conjectured that, up to a null set, a left translation and a dilation, the isoperimetric set in the Heisenberg group $\mathbb{H}^1$ is a bubble set as follows:

$$E_{\text{isop}} = \left\{ (z,t) \in \mathbb{H}^1 : |t| < \frac{1}{2}\left(\arccos |z| + |z|\sqrt{1 - |z|^2} \right), |z| < 1 \right\}. \tag{1}$$

The formula defining E_{isop} in (1) makes sense in $\mathbb{H}^n$ for $n \geq 2$ and Pansu's conjecture can be naturally extended to any dimension. Until today Pansu's conjecture has not completely been solved. It has been only supported by many partial results, where further hypotheses involving regularity or symmetry of the admissible sets are assumed; see [3–8]. For Carnot groups, one can only get an isoperimetric inequality [9] though we know the fact that isoperimetric sets exist [10].

Monti and Morbidelli [11] completely solved the isoperimetric problem in the Grushin plane R^2. Franceschi and Monti [12] studied isoperimetric problem for a class of x-spherical symmetry sets (here if $h = 1$, the assumption of x-spherical symmetry can be removed) in Grushin spaces R^{h+k}. In particular, they pointed out that when $k = 1$, up to a null set, a vertical translation and a dilation, the x-spherical symmetric isoperimetric set

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is

$$E_\alpha = \left\{ (x, y) \in R^{h+1} : |y| < \int_{\arcsin |x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\, dt, |x| < 1 \right\}. \tag{2}$$

In the case of $\alpha = 1$, the set E_α (2) is just the Pansu sphere in the Heisenberg group.

On the other hand, manifolds with density, a new category in geometry, have been widely studied. They arise naturally in geometry as quotients of Riemannian manifolds, in physics as spaces with different media, in probability as the famous Gauss space and in a number of other places as well (see [13, 14]). Morgan and Pratelli [15] studied the isoperimetric problems in Euclidean spaces R^n with density; see [16–20] and the references therein. The weighted Sobolev and Poincaré inequalities for Hörmander's vector fields were well studied in [21–23]. The weighted isoperimetric-type and Sobolev-type inequalities for hypersurfaces in the Carnot group with density were obtained in [24]. In [25] He and Zhao proved that the set E_α is also a weighted x-spherical symmetry isoperimetric set in the Grushin space R^{h+1} with density $|x|^p$, $p > -h + 1$.

Very recently, Franceschi *et al.* [26] obtained quantitative isoperimetric inequalities for the bubble set E_{isop} in half-cylinders in Heisenberg groups by the construction of sub-calibrations.

Motivated by the nice work mentioned above, in this paper we consider the quantitative isoperimetric inequalities for the set E_α in half-cylinders in the Grushin space R^{h+1} with density $|x|^p$, $p \geq 0$. These inequalities show that the weighted volume distance of a set F from the set E_α with the same weighted volume is controlled in terms of the difference of the weighted α-perimeter of F and the weighted α-perimeter of E_α. We get the following theorem.

Theorem 1.1 *Let F be any measurable set in the Grushin space R^{h+1} with density $e^\phi = |x|^p$, $p \geq 0$, where F satisfies $V_\phi(F) = V_\phi(E_\alpha)$. Let $C_\varepsilon = \{(x, y) \in R^{h+1} : |x| < 1, y > y_\varepsilon\}$ be half-cylinders, where $0 \leq \varepsilon < 1$ and $y_\varepsilon = f(1 - \varepsilon)$ with $f(r) = \int_{\arcsin r}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\, dt$.*

(i) *If $F \triangle E_\alpha \subset\subset C_0$, then we have*

$$P_{\alpha,\phi}(F) - P_{\alpha,\phi}(E_\alpha) \geq \frac{h+p}{120\omega_h^2} V_\phi(E_\alpha \triangle F)^3.$$

(ii) *If $F \triangle E_\alpha \subset\subset C_\varepsilon$ with $0 < \varepsilon < 1$, then we have*

$$P_{\alpha,\phi}(F) - P_{\alpha,\phi}(E_\alpha) \geq \frac{\sqrt{\varepsilon}(h+p)}{8[(1-\varepsilon)^{\alpha+h} + (1-\varepsilon)^h \sqrt{\varepsilon}]\omega_h} V_\phi(E_\alpha \triangle F)^2.$$

Here $P_{\alpha,\phi}(E) = \sup\{\int_E \operatorname{div}_\alpha(|x|^p \varphi)\, dx\, dy : \varphi \in C_c^1(R^{h+1}; R^{h+1}), \max|\varphi| \leq 1\}$ and $V_\phi(E) = \int_E |x|^p\, dx\, dy$ are called the weighted α-perimeter and the weighted volume of E, respectively. Finally ω_h denotes the Euclidean volume of the unit ball.

When $p = 0$ in Theorem 1.1, we can obtain the quantitative isoperimetric inequalities for the set E_α in half-cylinders in Grushin spaces.

2 Preliminaries

The Grushin space $R^{h+1} = \{(x,y) : x \in R^h, y \in R\}$ is a Carnot-Carathéodory space with a system of vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, h \quad \text{and} \quad Y = |x|^\alpha \frac{\partial}{\partial y},$$

where $\alpha > 0$ is a given real number and $|x|$ is the standard Euclidean norm of x.

The α-perimeter of a measurable set $E \subset R^{h+1}$ in an open set $A \subset R^{h+1}$ is defined as

$$P_\alpha(E;A) = \sup\left\{ \int_E \operatorname{div}_\alpha \varphi \, dx \, dy : \varphi \in C_c^1(A; R^{h+1}), \|\varphi\|_\infty = \max_{(x,y)\in A} |\varphi(x,y)| \leq 1 \right\},$$

where the α-divergence of the vector field $\varphi : A \to R^{h+1}$ is given by

$$\operatorname{div}_\alpha \varphi = X_1\varphi_1 + \cdots + X_h\varphi_h + Y\varphi_{h+1}.$$

If $P_\alpha(E;A) < \infty$, by the Riesz representation theorem there exist a positive Radon measure μ_E on A and a μ_E-measurable function $\nu_E : A \to R^{h+1}$ such that $|\nu_E| = 1$ μ_E-a.e. on A and the generalized Gauss-Green formula

$$\int_E \operatorname{div}_\alpha \varphi \, dx \, dy = - \int_A \langle \varphi, \nu_E \rangle \, d\mu_E \tag{3}$$

holds for all $\varphi \in C_c^1(A; R^{h+1})$. Here and hereafter, $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product. The measure μ_E is called α-perimeter measure and the function ν_E is called measure theoretic inner unit α-normal of E.

Now we endow the Grushin space R^{h+1} with density e^ϕ and define the weighted α-perimeter of a measurable set $E \subset R^{h+1}$ in an open set $A \subset R^{h+1}$ as

$$P_{\alpha,\phi}(E;A) = \sup\left\{ \int_E (\operatorname{div}_{\alpha,\phi} \varphi)e^\phi \, dx \, dy : \varphi \in C_c^1(A; R^{h+1}), \right.$$

$$\left. \|\varphi\|_\infty = \max_{(x,y)\in A} |\varphi(x,y)| \leq 1 \right\}, \tag{4}$$

where $\operatorname{div}_{\alpha,\phi} \varphi = e^{-\phi} \operatorname{div}_\alpha(e^\phi \varphi)$ is called the weighted α-divergence of φ.

By the definition of $\operatorname{div}_{\alpha,\phi} \varphi$, (4) can also be rewritten as

$$P_{\alpha,\phi}(E;A) = \sup\left\{ \int_E \operatorname{div}_\alpha(e^\phi \varphi) \, dx \, dy : \varphi \in C_c^1(A; R^{h+1}), \right.$$

$$\left. \|\varphi\|_\infty = \max_{(x,y)\in A} |\varphi(x,y)| \leq 1 \right\}. \tag{5}$$

If $P_{\alpha,\phi}(E;A) < \infty$, then by (3) we have

$$\int_E \operatorname{div}_{\alpha,\phi} \varphi \, dV_\phi = - \int_A \langle \varphi, \nu_E \rangle \, d\mu_{E,\phi}, \tag{6}$$

where $dV_\phi = e^\phi \, dx \, dy$ is the weighted volume measure and $d\mu_{E,\phi} = e^\phi \, d\mu_E$ is called the weighted α-perimeter measure. For any open set $A \subset R^{h+1}$, we have $P_{\alpha,\phi}(E;A) = \mu_{E,\phi}(A)$. When $A = R^{h+1}$, let $P_{\alpha,\phi}(E) = P_{\alpha,\phi}(E; R^{h+1})$.

Let Σ be a hypersurface in the Grushin space R^{h+1} with density e^{ϕ}. Σ can be locally given by the zero set of a function $u \in C^1$ such that $|\nabla_{\alpha} u| \neq 0$ on Σ, where $\nabla_{\alpha} u = (X_1 u, \ldots, X_h u, Yu)$ is called the α-gradient of u. For a set $E = \{(x,y) \in R^{h+1} : u(x,y) > 0\}$, the inner unit α-normal in equation (6) is given on $\Sigma = \partial E$ by the vector

$$\nu_E = \frac{\nabla_{\alpha} u(x,y)}{|\nabla_{\alpha} u(x,y)|}.$$

Then we define the weighted α-mean curvature of Σ as

$$H_{\Sigma,\phi} = -\frac{1}{h} \operatorname{div}_{\alpha,\phi} \nu_E = -\frac{1}{h}\left(\operatorname{div}_{\alpha} \nu_E + \langle \nu_E, \nabla_{\alpha}\phi\rangle\right). \tag{7}$$

Remark 2.1 Noticing that the α-mean curvature of Σ is defined by $H_{\Sigma} = -\frac{1}{h}\operatorname{div}_{\alpha}\nu_E$, then from (7) we have

$$H_{\Sigma,\phi} = H_{\Sigma} - \frac{1}{h}\langle \nu_E, \nabla_{\alpha}\phi\rangle.$$

To prove Theorem 1.1, we need the following lemma.

Lemma 2.1 *Let the Grushin space R^{h+1} be endowed with density $e^{\phi} = |x|^p$. For any $0 \leq \varepsilon < 1$, let $C_{\varepsilon} = \{(x,y) \in R^{h+1} : |x| < 1, y > y_{\varepsilon}\}$ be half-cylinders, where $y_{\varepsilon} = f(1 - \varepsilon)$ with $f(r) = \int_{\arcsin r}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\,dt$. There exists a continuous function $u : C_{\varepsilon} \to R$ with level sets $\Sigma_s = \{(x,y) \in C_{\varepsilon} : u(x,y) = s\}, s \in R$, such that*
  (i) *$u \in C^1(C_{\varepsilon} \cap E_{\alpha}) \cap C^1(C_{\varepsilon} \setminus E_{\alpha})$ and $\frac{\nabla_{\alpha} u(x,y)}{|\nabla_{\alpha} u(x,y)|}$ is continuously defined on $C_{\varepsilon} \setminus \{x = 0\}$;*
  (ii) *$\bigcup_{s>1} \Sigma_s = C_{\varepsilon} \cap E_{\alpha}$ and $\bigcup_{s\leq 1} \Sigma_s = C_{\varepsilon} \setminus E_{\alpha}$;*
  (iii) *Σ_s is a hypersurface of class C^2 with constant weighted α-mean curvature, that is,*

$$H_{\Sigma_s,\phi} = \frac{1}{s}\left(1 + \frac{p}{h}\right) \quad \textit{for } s > 1$$

  and

$$H_{\Sigma_s,\phi} = 1 + \frac{p}{h} \quad \textit{for } s \leq 1;$$

  (iv) *for any point $(x, f(|x|) - y) \in \Sigma_s$ with $s > 1$, we have*

$$1 - \frac{h}{h+p}H_{\Sigma_s,\phi}\big(x, f(|x|) - y\big) \geq \frac{1}{5}y^2 \quad \textit{when } \varepsilon = 0 \tag{8}$$

  and

$$1 - \frac{h}{h+p}H_{\Sigma_s,\phi}\big(x, f(|x|) - y\big) \geq \frac{\sqrt{\varepsilon}}{(1-\varepsilon)^{\alpha} + \sqrt{\varepsilon}}y \quad \textit{when } 0 < \varepsilon < 1. \tag{9}$$

Proof The profile function of the set E_{α} is the function $f : [0,1] \to R$,

$$f(r) = \int_{\arcsin r}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\,dt. \tag{10}$$

Its first and second derivatives are

$$f'(r) = -\frac{r^{\alpha+1}}{\sqrt{1-r^2}}, \qquad f''(r) = \frac{r^\alpha [\alpha r^2 - (\alpha+1)]}{(1-r^2)^{\frac{3}{2}}}. \tag{11}$$

We define the function $g : [0,1] \to R$,

$$g(r) = (\alpha+1)f(r) - rf'(r)$$
$$= (\alpha+1)\int_{\arcsin r}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\,dt + \frac{r^{\alpha+2}}{\sqrt{1-r^2}}. \tag{12}$$

Its derivative is

$$g'(r) = \frac{r^{\alpha+1}}{(1-r^2)^{\frac{3}{2}}} > 0. \tag{13}$$

Now we construct a foliation of C_ε. In $C_\varepsilon \setminus E_\alpha$, the leaves Σ_s of the foliation are vertical translations of the top part of the boundary ∂E_α. In $C_\varepsilon \cap E_\alpha$, the leaves Σ_s are constructed as follows: the surface ∂E_α is dilated by a factor larger than 1 where dilation is defined by $(x,y) \to (\lambda x, \lambda^{\alpha+1} y)$ $(\forall \lambda > 0)$, and then it is translated downwards in such a way that the surface $\{y = y_\varepsilon = f(1-\varepsilon)\}$ is also the leaf at last.

We construct a function u on the set $C_\varepsilon \setminus E_\alpha$ as

$$u(x,y) = f(|x|) - y + 1, \quad (x,y) \in C_\varepsilon \setminus E_\alpha. \tag{14}$$

Let $\Sigma_s = \{(x,y) \in C_\varepsilon \setminus E_\alpha : u(x,y) = s\}$. Then we have $s \le 1$ and $\Sigma_1 = \partial E_\alpha$. From (14), we know $u \in C^1(C_\varepsilon \setminus E_\alpha)$ and $\bigcup_{s \le 1} \Sigma_s = C_\varepsilon \setminus E_\alpha$.

In the following we will define the function u on the set $D_\varepsilon = C_\varepsilon \cap E_\alpha$. Setting $r = |x|$ and $r_\varepsilon = 1 - \varepsilon$, we let $F_\varepsilon : D_\varepsilon \times (1,\infty) \to R$ be a function

$$F_\varepsilon(x,y,s) = s^{\alpha+1}\left[f\left(\frac{r}{s}\right) - f\left(\frac{r_\varepsilon}{s}\right) \right] + y_\varepsilon - y. \tag{15}$$

For any $(x,y) \in D_\varepsilon$ we have

$$\lim_{s \to 1^+} F_\varepsilon(x,y,s) = f(r) - y > 0, \qquad \lim_{s \to \infty} F_\varepsilon(x,y,s) = y_\varepsilon - y < 0.$$

On the other hand, using (12) and (13) we have

$$\partial_s F_\varepsilon = s^\alpha \left[g\left(\frac{r}{s}\right) - g\left(\frac{r_\varepsilon}{s}\right) \right] < 0. \tag{16}$$

So there exists a unique $s > 1$ such that $F_\varepsilon(x,y,s) = 0$ for any $(x,y) \in D_\varepsilon$. Furthermore we can define a function $u : D_\varepsilon \to R, s = u(x,y)$ determined by the equation $F_\varepsilon(x,y,s) = 0$. Obviously we have $u \in C^1(C_\varepsilon \cap E_\alpha)$ and $C_\varepsilon \cap E_\alpha = \bigcup_{s>1} \Sigma_s$, where $\Sigma_s = \{(x,y) \in C_\varepsilon \cap E_\alpha : s = u(x,y)$ is determined by $F_\varepsilon(x,y,s) = 0\}$.

By (15), we find

$$\partial_{x_i} F_\varepsilon(x,y,s) = \frac{s^\alpha x_i}{r} f'\left(\frac{r}{s}\right), \quad i = 1,\dots,h; \qquad \partial_y F_\varepsilon(x,y,s) = -1. \tag{17}$$

Using (11), (16) and (17), we obtain

$$\partial_{x_i} u(x,y) = \frac{x_i r^\alpha}{s^\alpha \sqrt{s^2 - r^2}[g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})]}, \qquad \partial_y u(x,y) = \frac{1}{s^\alpha [g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})]}. \tag{18}$$

Then we have

$$X_i u = \partial_{x_i} u = \frac{x_i r^\alpha}{s^\alpha \sqrt{s^2 - r^2}[g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})]}, \qquad Y u = |x|^\alpha \partial_y u = \frac{r^\alpha}{s^\alpha [g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})]},$$

and the square length of the α-gradient of u on D_ε is

$$|\nabla_\alpha u|^2 = \sum_{i=1}^{h} (X_i u)^2 + (Y u)^2 = \frac{r^{2\alpha}}{s^{2\alpha-2}(s^2 - r^2)[g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})]^2}.$$

Note that $|\nabla_\alpha u| = 0$ if and only if $x = 0$. So for any $(x,y) \in D_\varepsilon$ with $x \neq 0$, we have

$$\frac{X_i u}{|\nabla_\alpha u|} = -\frac{x_i}{s}, \quad i = 1, \ldots, h; \qquad \frac{Y u}{|\nabla_\alpha u|} = -\frac{\sqrt{s^2 - r^2}}{s}. \tag{19}$$

If $(x,y) \in D_\varepsilon$ tends to $(\overline{x}, \overline{y}) \in \partial E_\alpha$ with $\overline{x} \neq 0$ and $\overline{y} > 0$, then $s = u(x,y)$ converges to 1. From (19), we have

$$\lim_{(x,y) \to (\overline{x}, \overline{y})} \frac{\nabla_\alpha u(x,y)}{|\nabla_\alpha u(x,y)|} = \left(-\overline{x}_1, \ldots, -\overline{x}_h, \sqrt{1 - |\overline{x}|^2}\right) = \frac{\nabla_\alpha u(\overline{x}, \overline{y})}{|\nabla_\alpha u(\overline{x}, \overline{y})|},$$

where the right hand side is computed by the definition (14) of u. The above equality shows that $\frac{\nabla_\alpha u}{|\nabla_\alpha u|}$ is continuous on $C_\varepsilon \setminus \{x = 0\}$.

In the case of $e^\phi = |x|^p$, we get $\phi = p \ln|x|$ and $\nabla_\alpha \phi = (\frac{p}{|x|^2} x_1, \ldots, \frac{p}{|x|^2} x_h, 0)$ for $x \neq 0$. From (14), we know that the inner unit α-normal of Σ_s with $s \leq 1$ is

$$\nu_{\Sigma_s} = \left(-x_1, \ldots, -x_h, -\sqrt{1 - |x|^2}\right).$$

So the weighted α-mean curvature $H_{\Sigma_s, \phi}$ of Σ_s with $s \leq 1$ is given by

$$H_{\Sigma_s, \phi} = \frac{1}{h}\left(-\operatorname{div}_\alpha \nu_{E_\alpha} - \langle \nu_{E_\alpha}, \nabla_\alpha \phi \rangle\right) = 1 + \frac{p}{h}.$$

From (19) we know that the inner unit α-normal of Σ_s with $s > 1$ is

$$\nu_{\Sigma_s} = \left(-\frac{x_1}{s}, \ldots, -\frac{x_h}{s}, -\frac{\sqrt{s - |x|^2}}{s}\right).$$

So the weighted α-mean curvature $H_{\Sigma_s, \phi}$ of Σ_s with $s > 1$ is given by

$$H_{\Sigma_s, \phi} = \frac{1}{s}\left(1 + \frac{p}{h}\right).$$

Fixing a point x with $|x| < 1 - \varepsilon$ and for $0 \leq y < f(|x|) - y_\varepsilon$, we define the function

$$h_x(y) = u\left(x, f(|x|) - y\right) = s = \left(1 + \frac{p}{h}\right)\frac{1}{H_{\Sigma_s, \phi}}, \tag{20}$$

where $s \geq 1$ is uniquely determined by $(x, f(|x|) - y) \in \Sigma_s$. Then the function $y \rightarrow h_x(y)$ is increasing and $h_x(0) = 1$.

From (18) and (20), we know

$$h_x'(y) = -\partial_y u(x, f(|x|) - y) = \frac{1}{(h_x(y))^\alpha [g(\frac{r_\varepsilon}{h_x(y)}) - g(\frac{r}{h_x(y)})]},$$

for all $0 \leq y < f(|x|) - y_\varepsilon$.

By (13), g is strictly increasing. So $h_x(y)$ satisfies

$$h_x'(y) \geq \frac{1}{h_x^\alpha(y)[g(\frac{r_\varepsilon}{h_x(y)}) - g(0)]}. \tag{21}$$

On the other hand, for any $s > 1$ we have

$$
\begin{aligned}
s^\alpha \left[g\left(\frac{r_\varepsilon}{s}\right) - g(0) \right] &= s^\alpha \int_0^{\frac{r_\varepsilon}{s}} g'(r)\, dr \\
&= s^\alpha \int_0^{\frac{r_\varepsilon}{s}} \frac{r^{\alpha+1}}{(1 - r^2)^{\frac{2}{3}}}\, dr \\
&\leq r_\varepsilon^\alpha \int_0^{\frac{r_\varepsilon}{s}} \frac{r}{(1 - r^2)^{\frac{2}{3}}}\, dr \\
&= r_\varepsilon^\alpha \left[\left(1 - \left(\frac{r_\varepsilon}{s}\right)^2\right)^{-\frac{1}{2}} - 1 \right] \\
&\leq r_\varepsilon^\alpha \frac{s}{\sqrt{s^2 - r_\varepsilon^2}} \\
&\leq r_\varepsilon^\alpha \frac{1}{\sqrt{s - r_\varepsilon}}. \tag{22}
\end{aligned}
$$

When $\varepsilon = 0$, we have $r_\varepsilon = 1$. So (22) turns into

$$s^\alpha \left[g\left(\frac{r_\varepsilon}{s}\right) - g(0) \right] \leq \frac{1}{\sqrt{s-1}}.$$

By (21), we get

$$h_x'(y) \geq \sqrt{h_x(y) - 1}, \quad y \geq 0. \tag{23}$$

Integrating (23) with $h_x(0) = 1$, we get

$$h_x(y) \geq 1 + \frac{1}{4}y^2.$$

Thus we obtain

$$1 - \frac{h}{h+p} H_{\Sigma_s, \phi}(x, f(|x|) - y) = 1 - \frac{1}{h_x(y)} \geq \frac{y^2}{4 + y^2}.$$

Noticing

$$y < f(|x|) \le f(0) = \int_0^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\,dt \le \int_0^{\frac{\pi}{2}} \sin(t)\,dt = 1,$$

we have

$$1 - \frac{h}{h+p} H_{\Sigma_s,\phi}\big(x, f(|x|) - y\big) \ge \frac{1}{5}y^2. \tag{24}$$

When $0 < \varepsilon < 1$, (22) turns into

$$s^\alpha \left[g\left(\frac{r_\varepsilon}{s}\right) - g(0) \right] \le \frac{(1-\varepsilon)^\alpha}{\sqrt{s-1+\varepsilon}}.$$

So by (21), we have

$$h_x'(y) \ge \frac{\sqrt{\varepsilon}}{(1-\varepsilon)^\alpha}, \quad y \ge 0. \tag{25}$$

Integrating (25) with $h_x(0) = 1$, we have

$$h_x(y) \ge 1 + \frac{\sqrt{\varepsilon}}{(1-\varepsilon)^\alpha} y.$$

Noticing $y < f(|x|) - y_\varepsilon \le 1$, so we have

$$
\begin{aligned}
1 - \frac{h}{h+p} H_{\Sigma_s,\phi}\big(x, f(|x|) - y\big) &= 1 - \frac{1}{h_x(y)} \\
&\ge 1 - \frac{1}{1 + \frac{\sqrt{\varepsilon}}{(1-\varepsilon)^\alpha} y} \\
&\ge \frac{\sqrt{\varepsilon}}{(1-\varepsilon)^\alpha + \sqrt{\varepsilon}} y.
\end{aligned}
\tag{26}
$$

$\square$

3 Proof of Theorem 1.1

Let $u : C_\varepsilon \to R$ be the function given by Lemma 2.1 and let $\Sigma_s = \{(x,y) \in C_\varepsilon : u(x,y) = s\}$ be leaves of the foliation, $s \in R$. We define the vector field $X : C_\varepsilon \setminus \{x = 0\} \to R^{h+1}$ by

$$X = -\frac{\nabla_\alpha u}{|\nabla_\alpha u|}.$$

Then X satisfies the following properties:

(i) $|X| = 1$;

(ii) for $(x,y) \in \partial E_\alpha \cap C_\varepsilon$, we have $X(x,y) = -\nu_{E_\alpha}(x,y)$ where $\nu_{E_\alpha}(x,y)$ is the unit inner α-normal to ∂E_α;

(iii) for any point $(x,y) \in \Sigma_s$ with $s \le 1$, we have

$$\mathrm{div}_{\alpha,\phi} X(x,y) = h + p. \tag{27}$$

For any point $(x, y) \in \Sigma_s$ with $s > 1$, we have

$$\operatorname{div}_{\alpha,\phi} X(x, y) = \frac{1}{s}(h + p) < h + p. \tag{28}$$

Let $F \subset R^{h+1}$ be a set with finite weighted α-perimeter such that $V_\phi(F) = V_\phi(E_\alpha)$ and $F \triangle E_\alpha \subset\subset C_\varepsilon$. By Theorem 2.2.2 in [27], without loss of generality we can assume that the boundary ∂F of F is C^∞.

For $\delta > 0$, let $E_\alpha^\delta = \{(x, y) \in E_\alpha : |x| > \delta\}$. By (28) and (6), we have

$$
\begin{aligned}
V_\phi\left(E_\alpha^\delta \setminus F\right) &= \int_{E_\alpha^\delta \setminus F} |x|^p \, dx \, dy \\
&\geq \int_{E_\alpha^\delta \setminus F} \frac{\operatorname{div}_{\alpha,\phi} X}{h + p} |x|^p \, dx \, dy \\
&= \frac{1}{h + p} \left\{ \int_{\partial F \cap E_\alpha^\delta} \langle X, \nu_F \rangle \, d\mu_{F,\phi} - \int_{\partial E_\alpha^\delta \cap F} \langle X, \nu_{E_\alpha^\delta} \rangle \, d\mu_{E_\alpha^\delta,\phi} \right\}.
\end{aligned}
$$

Letting $\delta \to 0^+$ and using the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
V_\phi(E_\alpha \setminus F) &= \int_{E_\alpha \setminus F} |x|^p \, dx \, dy \\
&\geq \int_{E_\alpha \setminus F} \frac{\operatorname{div}_{\alpha,\phi} X}{h + p} |x|^p \, dx \, dy \\
&= \frac{1}{h + p} \left\{ \int_{\partial F \cap E_\alpha} \langle X, \nu_F \rangle \, d\mu_{F,\phi} - \int_{\partial E_\alpha \setminus F} \langle X, \nu_{E_\alpha} \rangle \, d\mu_{E_\alpha,\phi} \right\} \\
&\geq \frac{1}{h + p} \left\{ \int_{\partial E_\alpha \setminus F} d\mu_{E_\alpha,\phi} - \int_{\partial F \cap E_\alpha} d\mu_{F,\phi} \right\} \\
&= \frac{1}{h + p} \left\{ P_{\alpha,\phi}(E_\alpha; C_\varepsilon \setminus F) - P_{\alpha,\phi}(F; E_\alpha) \right\}. \tag{29}
\end{aligned}
$$

By a similar computation, we also have

$$
\begin{aligned}
V_\phi(F \setminus E_\alpha) \\
&= \int_{F \setminus E_\alpha} |x|^p \, dx \, dy \\
&= \int_{F \setminus E_\alpha} \frac{\operatorname{div}_{\alpha,\phi} X}{h + p} |x|^p \, dx \, dy \\
&= \frac{1}{h + p} \left\{ -\int_{\partial F \setminus E_\alpha} \langle X, \nu_F \rangle \, d\mu_{F,\phi} + \int_{\partial E_\alpha \cap F} \langle X, \nu_{E_\alpha} \rangle \, d\mu_{E_\alpha,\phi} \right\} \\
&\leq \frac{1}{h + p} \left\{ \int_{\partial F \setminus E_\alpha} d\mu_{F,\phi} - \int_{\partial E_\alpha \cap F} d\mu_{E_\alpha,\phi} \right\} \\
&= \frac{1}{h + p} \left\{ P_{\alpha,\phi}(F; C_\varepsilon \setminus E_\alpha) - P_{\alpha,\phi}(E_\alpha; F) \right\}. \tag{30}
\end{aligned}
$$

On the other hand, we have

$$\int_{E_\alpha \setminus F} \frac{\text{div}_{\alpha,\phi} X}{h+p} |x|^p \, dx \, dy = \int_{E_\alpha \setminus F} \left[1 + \left(\frac{\text{div}_{\alpha,\phi} X}{h+p} - 1 \right) \right] |x|^p \, dx \, dy$$

$$= V_\phi(E_\alpha \setminus F) - \int_{E_\alpha \setminus F} \left(1 - \frac{\text{div}_{\alpha,\phi} X}{h+p} \right) |x|^p \, dx \, dy. \tag{31}$$

From (29), (30) and (31), we obtain

$$\frac{1}{h+p} \left\{ P_{\alpha,\phi}(E_\alpha; C_\varepsilon \setminus F) - P_{\alpha,\phi}(F; E_\alpha) \right\}$$

$$\leq \int_{E_\alpha \setminus F} \frac{\text{div}_{\alpha,\phi} X}{h+p} |x|^p \, dx \, dy$$

$$= V_\phi(E_\alpha \setminus F) - \int_{E_\alpha \setminus F} \left(1 - \frac{\text{div}_{\alpha,\phi} X}{h+p} \right) |x|^p \, dx \, dy$$

$$= V_\phi(F \setminus E_\alpha) - \int_{E_\alpha \setminus F} \left(1 - \frac{\text{div}_{\alpha,\phi} X}{h+p} \right) |x|^p \, dx \, dy$$

$$\leq \frac{1}{h+p} \left\{ P_{\alpha,\phi}(F; C_\varepsilon \setminus E_\alpha) - P_{\alpha,\phi}(E_\alpha; F) \right\} - \int_{E_\alpha \setminus F} \left(1 - \frac{\text{div}_{\alpha,\phi} X}{h+p} \right) |x|^p \, dx \, dy.$$

It is equivalent to

$$P_{\alpha,\phi}(F) - P_{\alpha,\phi}(E_\alpha) \geq (h+p) \int_{E_\alpha \setminus F} \left(1 - \frac{\text{div}_{\alpha,\phi} X}{h+p} \right) |x|^p \, dx \, dy. \tag{32}$$

For any x with $|x| < 1 - \varepsilon$, we define the vertical sections $E_\alpha^x = \{y : (x,y) \in E_\alpha\}$ and $F^x = \{y : (x,y) \in F\}$. By the Fubini theorem, we have

$$\int_{E_\alpha \setminus F} \left(1 - \frac{\text{div}_{\alpha,\phi} X}{h+p} \right) |x|^p \, dx \, dy$$

$$= \int_{\{|x|<1-\varepsilon\}} \int_{E_\alpha^x \setminus F^x} \left(1 - \frac{\text{div}_{\alpha,\phi} X}{h+p} \right) |x|^p \, dy \, dx.$$

Letting $m(x) = \mathcal{L}^1(E_\alpha^x \setminus F^x)$, where $\mathcal{L}^1$ denotes 1-dimensional Lebesgue measure, then we obtain

$$\int_{E_\alpha \setminus F} \left(1 - \frac{\text{div}_{\alpha,\phi} X}{h+p} \right) |x|^p \, dx \, dy = \int_{\{|x|<1-\varepsilon\}} \int_{f(|x|)-m(x)}^{f(|x|)} \left(1 - \frac{\text{div}_{\alpha,\phi} X}{h+p} \right) |x|^p \, dy \, dx$$

$$= \int_{\{|x|<1-\varepsilon\}} \int_0^{m(x)} \left(1 - \frac{1}{h_x(y)} \right) |x|^p \, dy \, dx, \tag{33}$$

where $h_x(y) = u(x, f(|x|) - y)$ is the function introduced in (20).

So from (32) and (33) we have

$$P_{\alpha,\phi}(F) - P_{\alpha,\phi}(E_\alpha) \geq (h+p) \int_{\{|x|<1-\varepsilon\}} \int_0^{m(x)} \left(1 - \frac{1}{h_x(y)} \right) |x|^p \, dy \, dx. \tag{34}$$

When $\varepsilon = 0$, by (8) in Lemma 2.1 and the Hölder inequality, (34) turns into

$$
\begin{aligned}
P_{\alpha,\phi}(F) - P_{\alpha,\phi}(E_\alpha) &\geq (h+p) \int_{\{|x|<1\}} \int_0^{m(x)} \frac{1}{5} y^2 \, dy |x|^p \, dx \\
&\geq \frac{h+p}{15} \int_{\{|x|<1\}} \big(m(x)\big)^3 |x|^{3p} \, dx \\
&\geq \frac{h}{15\omega_h^2} \left(\int_{\{|x|<1\}} m(x) |x|^p \, dx \right)^3 \\
&= \frac{h+p}{120\omega_h^2} V_\phi(E_\alpha \triangle F)^3.
\end{aligned}
\tag{35}
$$

When $0 < \varepsilon < 1$, by (9) in Lemma 2.1, and the Hölder inequality, (34) turns into

$$
\begin{aligned}
P_{\alpha,\phi}(F) - P_{\alpha,\phi}(E_\alpha) &\geq (h+p) \int_{\{|x|<1-\varepsilon\}} \int_0^{m(x)} \frac{\sqrt{\varepsilon}}{(1-\varepsilon)^\alpha + \sqrt{\varepsilon}} y \, dy |x|^p \, dx \\
&\geq \frac{\sqrt{\varepsilon}(h+p)}{2[(1-\varepsilon)^\alpha + \sqrt{\varepsilon}]} \int_{\{|x|<1-\varepsilon\}} \big(m(x)\big)^2 |x|^{2p} \, dx \\
&\geq \frac{\sqrt{\varepsilon}(h+p)}{2[(1-\varepsilon)^{\alpha+h} + (1-\varepsilon)^h \sqrt{\varepsilon}]\omega_h} \left(\int_{\{|x|<1-\varepsilon\}} m(x) |x|^p \, dx \right)^2 \\
&= \frac{\sqrt{\varepsilon}(h+p)}{8[(1-\varepsilon)^{\alpha+h} + (1-\varepsilon)^h \sqrt{\varepsilon}]\omega_h} V_\phi(E_\alpha \triangle F)^2.
\end{aligned}
\tag{36}
$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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