Onjai-uea and Phuengrattana *Journal of Inequalities and Applications* (2017) 2017:137
DOI 10.1186/s13660-017-1416-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On solving split mixed equilibrium problems and fixed point problems of hybrid-type multivalued mappings in Hilbert spaces

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Abstract
In this paper, we introduce and study iterative algorithms for solving split mixed equilibrium problems and fixed point problems of λ-hybrid multivalued mappings in real Hilbert spaces and prove that the proposed iterative algorithm converges weakly to a common solution of the considered problems. We also provide an example to illustrate the convergence behavior of the proposed iteration process.

MSC: 47N10; 47J20; 47J22; 46N10

Keywords: split mixed equilibrium problems; hybrid multivalued mappings; weak convergence; Hilbert spaces

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, $\varphi : C \to \mathbb{R}$ be a function, and $F : C \times C \to \mathbb{R}$ be a bifunction. The *mixed equilibrium problem* is to find $x \in C$ such that

$$F(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of mixed equilibrium problem is denoted by $MEP(F,\varphi)$. In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem, which is to find $x \in C$ such that $F(x,y) \geq 0, \forall y \in C$. The solution set of equilibrium problem is denoted by *EP(F)*.

The mixed equilibrium problem is very general in the sense that it includes, as special cases, optimization problems, variational inequality problems, minimization problems, fixed point problems, Nash equilibrium problems in noncooperative games, and others; see, *e.g.*, [1–4].

In 1994, Censor and Elfving [5] firstly introduced the following split feasibility problem in finite-dimensional Hilbert spaces: Let H_1, H_2 be two Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem is formulated as finding a point x^*

with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q.$$

The split feasibility problem can extensively be applied in fields such as intensity-modulated radiation therapy, signal processing and image reconstruction, then the split feasibility problem has received so much attention by so many scholars; see [6–9].

In 2013, Kazmi and Rizvi [10] introduced and studied the following split equilibrium problem: let $C \subseteq H_1$ and $Q \subseteq H_2$. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and let $A : H_1 \to H_2$ be a bounded linear operator. The *split equilibrium problem* is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C \text{ and such that } y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \forall y \in Q. \quad (1.2)$$

The solution set of the split equilibrium problem is denoted by

$$SEP(F_1, F_2) := \{x^* \in C : x^* \in EP(F_1) \text{ and } Ax^* \in EP(F_2)\}.$$

The authors gave an iterative algorithm to find the common element of sets of solution of the split equilibrium problem and hierarchical fixed point problem; for more details refer to [11, 12].

In 2016, Suantai *et al.* [13] proposed the iterative algorithm to solve the problems for finding a common elements the set of solution of the split equilibrium problem and the fixed point of a nonspreading multivalued mapping in Hilbert space, given sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in C \quad \text{arbitrarily}, \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)Su_n, \quad \forall n \in \mathbb{N}, \end{cases} \quad (1.3)$$

where $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0,\infty)$ and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of A, $C \subset H_1$, $Q \subset H_2$, $S : C \to K(C)$ is a $\frac{1}{2}$-nonspreading multivalued mapping, $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions. The authors showed that under certain conditions, the sequence $\{x_n\}$ converges weakly to an element of $F(S) \cap SEP(F_1, F_2)$.

Several iterative algorithms have been developed for solving split feasibility problems and related split equilibrium problems; see, *e.g.*, [14–16].

Motivated and inspired by the above results and related literature, we propose an iterative algorithm for finding a common element of the set of solutions of split mixed equilibrium problems and the set of fixed points of λ-hybrid multivalued mappings in real Hilbert spaces. Then we prove some weak convergence theorems which extend and improve the corresponding results of Kazmi and Rizvi [10] and Suantai *et al.* [13] and many others. We finally provide numerical examples for supporting our main result.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x \in H$ by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$, respectively. It is also well known [17] that a Hilbert space H satisfies *Opial's condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\longrightarrow\infty} \|x_n - x\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

The following two lemmas are useful for our main results.

Lemma 2.1 *In a real Hilbert space H, the following inequalities hold*:

(1) $\|x-y\|^2 \leq \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \forall x,y \in H$;

(2) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \forall x,y \in H$;

(3) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0,1], \forall x,y \in H$;

(4) *If $\{x_n\}$ is a sequence in H which converges weakly to $z \in H$, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2, \quad \forall y \in H.$$

Lemma 2.2 ([18]) *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H. Let $u, v \in H$ be such that* $\lim_{n\to\infty} \|x_n - u\|$ *and* $\lim_{n\to\infty} \|x_n - v\|$ *exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

A single-valued mapping $T : C \longrightarrow H$ is called *δ-inverse strongly monotone* [19] if there exists a positive real number δ such that

$$\langle x-y, Tx-Ty\rangle \geq \delta\|Tx-Ty\|^2, \quad \forall x,y \in C.$$

For each $\gamma \in (0, 2\delta]$, we see that $I - \gamma T$ is a nonexpansive single-valued mapping, that is,

$$\big\|(I-\gamma T)x - (I-\gamma T)y\big\| \leq \|x-y\|, \quad \forall x,y \in C.$$

We denote by $CB(C)$ and $K(C)$ the collection of all nonempty closed bounded subsets and nonempty compact subsets of C, respectively. The *Hausdorff metric* $\mathcal{H}$ on $CB(C)$ is defined by

$$\mathcal{H}(A,B) := \max\Big\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(y,A)\Big\}, \quad \forall A,B \in CB(C),$$

where $\operatorname{dist}(x,B) = \inf\{d(x,y) : y \in B\}$ is the distance from a point x to a subset B. Let $S : C \to CB(C)$ be a multivalued mapping. An element $x \in C$ is called a *fixed point* of S if $x \in Sx$. The set of all fixed points of S is denoted by $F(S)$, that is, $F(S) = \{x \in C : x \in Sx\}$. Recall that a multivalued mapping $S : C \to CB(C)$ is called

(i) *nonexpansive* if

$$\mathcal{H}(Sx,Sy) \leq \|x-y\|, \quad x,y \in C;$$

(ii) *quasi-nonexpansive* if $F(S) \neq \emptyset$ and

$$\mathcal{H}(Sx, Sp) \leq \|x - p\|, \quad \forall x \in C, \forall p \in F(S);$$

(iii) *nonspreading* [13] if

$$2\mathcal{H}(Sx, Sy)^2 \leq \operatorname{dist}(y, Sx)^2 + \operatorname{dist}(x, Sy)^2, \quad \forall x, y \in C;$$

(iv) λ-*hybrid* [20] if there exists $\lambda \in \mathbb{R}$ such that

$$(1 + \lambda)\mathcal{H}(Sx, Sp)^2 \leq (1 - \lambda)\|x - y\|^2 + \lambda \operatorname{dist}(y, Sx)^2 + \lambda \operatorname{dist}(x, Sy)^2, \quad \forall x, y \in C.$$

We note that 0-hybrid is nonexpansive, 1-hybrid is nonspreading, and if S is λ-hybrid with $F(S) \neq \emptyset$, then S is quasi-nonexpansive. It is well known [20] that if S is λ-hybrid, then $F(S)$ is closed. In addition, if S satisfies the condition: $Sp = \{p\}$ for all $p \in F(S)$, then $F(S)$ is also convex.

The following result is a demiclosedness principle for λ-hybrid multivalued mapping in a real Hilbert space.

Lemma 2.3 ([20]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to K(C)$ be a λ-hybrid multivalued mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $y_n \in Sx_n$ with $x_n - y_n \to 0$, then $x \in Sx$.*

For solving the mixed equilibrium problem, we assume that the bifunction $F_1 : C \times C \to \mathbb{R}$ satisfies the following assumption:

Assumption 2.4 Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ be the bifunction, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous satisfies the following conditions:

(A1) $F_1(x, x) = 0$ for all $x \in C$;
(A2) F_1 is monotone, *i.e.*, $F_1(x, y) + F_1(y, x) \leq 0, \forall x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F_1(tz + (1 - t)x, y) \leq F_1(x, y)$;
(A4) for each $x \in C$, $y \mapsto F_1(x, y)$ is convex and lower semicontinuous;
(B1) for each $x \in H_1$ and fixed $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,

$$F_1(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

Lemma 2.5 ([21]) *Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfies Assumption* 2.4 *and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap \operatorname{dom} \varphi \neq \emptyset$. For $r > 0$ and $x \in H_1$. Define a mapping $T_r^{F_1} : H_1 \to C$ as follows*:

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\},$$

for all $x \in H_1$. Assume that either (B1) *or* (B2) *holds. Then the following conclusions hold*:

(1) *for each $x \in H_1$, $T_r^{F_1} \neq \emptyset$*;

(2) *$T_r^{F_1}$ is single-valued*;

(3) *$T_r^{F_1}$ is firmly nonexpansive, i.e., for any $x, y \in H_1$,*

$$\|T_r^{F_1}x - T_r^{F_1}y\|^2 \leq \langle T_r^{F_1}x - T_r^{F_1}y, x - y\rangle;$$

(4) *$F(T_r^{F_1}) = MEP(F_1, \varphi)$*;

(5) *$MEP(F_1, \varphi)$ is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.4 *and $\phi : Q \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function such that $Q \cap \operatorname{dom}\phi \neq \emptyset$, where Q is a nonempty closed and convex subset of a Hilbert space H_2. For each $s > 0$ and $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows*:

$$T_s^{F_2}(v) = \left\{w \in Q : F_2(w,d) + \phi(d) - \phi(w) + \frac{1}{r}\langle d - w, w - v\rangle \geq 0, \forall d \in Q\right\}.$$

Then we have the following:

(6) *for each $v \in H_2$, $T_s^{F_2} \neq \emptyset$*;

(7) *$T_s^{F_2}$ is single-valued*;

(8) *$T_s^{F_2}$ is firmly nonexpansive*;

(9) *$F(T_s^{F_2}) = MEP(F_2, \phi)$*;

(10) *$MEP(F_2, \phi)$ is closed and convex.*

3 Main results

In this section, we prove the weak convergence theorems for finding a common element of the set of solutions of split mixed equilibrium problems and the set of fixed points of λ-hybrid multivalued mappings in real Hilbert spaces and give a numerical example to support our main result.

We introduce the definition of split mixed equilibrium problems in real Hilbert spaces as follows.

Definition 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions, let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ and $\phi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions such that $C \cap \operatorname{dom}\varphi \neq \emptyset$ and $Q \cap \operatorname{dom}\phi \neq \emptyset$, and let $A : H_1 \to H_2$ be a bounded linear operator. The *split mixed equilibrium problem* is to find $x^* \in C$ such that

$$F_1(x^*, x) + \varphi(x) - \varphi(x^*) \geq 0, \quad \forall x \in C, \tag{3.1}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad F_2(y^*, y) + \phi(y) - \phi(y^*) \geq 0, \quad \forall y \in Q. \tag{3.2}$$

The solution set of the split mixed equilibrium problem (3.1) and (3.2) is denoted by

$$SMEP(F_1, \varphi, F_2, \phi) := \{x^* \in C : x^* \in MEP(F_1, \varphi) \text{ and } Ax^* \in MEP(F_2, \phi)\}.$$

We now get our main result.

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator and $S : C \to K(C)$ a λ-hybrid multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption* 2.4, *let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ and $\phi : Q \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex functions such that $C \cap \operatorname{dom}\varphi \neq \emptyset$ and $Q \cap \operatorname{dom}\phi \neq \emptyset$, respectively, and F_2 is upper semicontinuous in the first argument. Assume that $\Theta = F(S) \cap SMEP(F_1, \varphi, F_2, \phi) \neq \emptyset$ and $Sp = \{p\}$ for all $p \in F(S)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)w_n, \quad w_n \in Su_n, \\ x_{n+1} = \beta_n w_n + (1 - \beta_n)z_n, \quad z_n \in Sy_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.3}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$, and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(C1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C2) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(C3) $0 < \liminf_{n\to\infty} r_n$.

Then the sequence $\{x_n\}$ generated by (3.3) *converges weakly to $p \in \Theta$.*

Proof First, we show that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping. Since $T_{r_n}^{F_2}$ is firmly nonexpansive and $I - T_{r_n}^{F_2}$ is 1-inverse strongly monotone, we see that

$$\begin{aligned} \left\| A^*\left(I - T_{r_n}^{F_2}\right)Ax - A^*\left(I - T_{r_n}^{F_2}\right)Ay \right\|^2 &= \left\langle A^*\left(I - T_{r_n}^{F_2}\right)(Ax - Ay), A^*\left(I - T_{r_n}^{F_2}\right)(Ax - Ay) \right\rangle \\ &= \left\langle \left(I - T_{r_n}^{F_2}\right)(Ax - Ay), AA^*\left(I - T_{r_n}^{F_2}\right)(Ax - Ay) \right\rangle \\ &\leq L\left\langle \left(I - T_{r_n}^{F_2}\right)(Ax - Ay), \left(I - T_{r_n}^{F_2}\right)(Ax - Ay) \right\rangle \\ &= L\left\| \left(I - T_{r_n}^{F_2}\right)(Ax - Ay) \right\|^2 \\ &\leq L\left\langle Ax - Ay, \left(I - T_{r_n}^{F_2}\right)(Ax - Ay) \right\rangle \\ &= L\left\langle x - y, A^*\left(I - T_{r_n}^{F_2}\right)Ax - A^*\left(I - T_{r_n}^{F_2}\right)Ay \right\rangle \end{aligned}$$

for all $x, y \in H_1$. This implies that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping. Since $\gamma \in (0, \frac{1}{L})$, it follows that $I - \gamma A^*(I - T_{r_n}^{F_2})A$ is a nonexpansive mapping.

Now, we divide the proof into five steps as follows:

Step 1. Show that $\{x_n\}$ is bounded.

Let $q \in \Theta$. Then we have $q = T_{r_n}^{F_1}q$ and $q = (I - \gamma A^*(I - T_{r_n}^{F_2})A)q$. By nonexpansiveness of $I - \gamma A^*(I - T_{r_n}^{F_2})A$, it implies that

$$\begin{aligned} \|u_n - q\| &= \left\| T_{r_n}^{F_1}\left(I - \gamma A^*\left(I - T_{r_n}^{F_2}\right)A\right)x_n - T_{r_n}^{F_1}\left(I - \gamma A^*\left(I - T_{r_n}^{F_2}\right)A\right)q \right\| \\ &\leq \left\| \left(I - \gamma A^*\left(I - T_{r_n}^{F_2}\right)A\right)x_n - \left(I - \gamma A^*\left(I - T_{r_n}^{F_2}\right)A\right)q \right\| \\ &\leq \|x_n - q\|. \end{aligned} \tag{3.4}$$

This implies that

$$\|w_n - q\| = \operatorname{dist}(w_n, Sq) \leq H(Su_n, Sq) \leq \|u_n - q\| \leq \|x_n - q\|, \tag{3.5}$$

and so

$$\begin{aligned} \|y_n - q\| &= \big\|\alpha_n x_n + (1-\alpha_n)w_n - q\big\| \\ &\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|w_n - q\| \\ &= \|x_n - q\|. \end{aligned} \tag{3.6}$$

It follows that

$$\|z_n - q\| = \operatorname{dist}(z_n, Sq) \leq H(Sy_n, Sq) \leq \|y_n - q\| \leq \|x_n - q\|. \tag{3.7}$$

By (3.5) and (3.7), we have

$$\begin{aligned} \|x_{n+1} - q\| &= \big\|\beta_n w_n + (1-\beta_n)z_n - q\big\| \\ &\leq \beta_n\|w_n - q\| + (1-\beta_n)\|z_n - q\| \\ &= \|x_n - q\|. \end{aligned} \tag{3.8}$$

This implies that $\{\|x_n - q\|\}$ is decreasing and bounded below, thus $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists for all $q \in \Theta$.

Step 2. Show that $\lim_{n\to\infty}\|w_n - z_n\| = 0$.

From Lemma 2.1(3), (3.5), (3.7), and $Sq = \{q\}$, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \big\|\beta_n w_n + (1-\beta_n)z_n - q\big\|^2 \\ &\leq \beta_n\|w_n - q\|^2 + (1-\beta_n)\|z_n - q\|^2 - \beta_n(1-\beta_n)\|w_n - z_n\|^2 \\ &\leq \|x_n - q\|^2 - \beta_n(1-\beta_n)\|w_n - z_n\|^2. \end{aligned} \tag{3.9}$$

This implies that

$$\beta_n(1-\beta_n)\|w_n - z_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

From Condition (C1) and $\lim_{n\to\infty}\|x_n - q\|$ exists, we have

$$\lim_{n\to\infty}\|w_n - z_n\| = 0. \tag{3.10}$$

Step 3. Show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$ and $\lim_{n\to\infty}\|w_n - u_n\| = 0$.

For $q \in \Theta$, we see that

$$\begin{aligned} \|u_n - q\|^2 &= \big\|T_{r_n}^{F_1}\big(I - \gamma A^*\big(I - T_{r_n}^{F_2}\big)A\big)x_n - T_{r_n}^{F_1}q\big\|^2 \\ &\leq \big\|\big(I - \gamma A^*\big(I - T_{r_n}^{F_2}\big)A\big)x_n - q\big\|^2 \\ &\leq \|x_n - q\|^2 + \gamma^2\big\|A^*\big(I - T_{r_n}^{F_2}\big)Ax_n\big\|^2 + 2\gamma\big\langle q - x_n, A^*\big(I - T_{r_n}^{F_2}\big)Ax_n\big\rangle \end{aligned}$$

$$
\begin{aligned}
&\leq \|x_n - q\|^2 + \gamma^2 \langle Ax_n - T_{r_n}^{F_2} Ax_n, AA^*(I - T_{r_n}^{F_2}) Ax_n \rangle \\
&\quad + 2\gamma \langle A(q - x_n), Ax_n - T_{r_n}^{F_2} Ax_n \rangle \\
&\leq \|x_n - q\|^2 + L\gamma^2 \langle Ax_n - T_{r_n}^{F_2} Ax_n, Ax_n - T_{r_n}^{F_2} Ax_n \rangle \\
&\quad + 2\gamma \langle A(q - x_n) + (Ax_n - T_{r_n}^{F_2} Ax_n) \\
&\quad - (Ax_n - T_{r_n}^{F_2} Ax_n), Ax_n - T_{r_n}^{F_2} Ax_n \rangle \\
&\leq \|x_n - q\|^2 + L\gamma^2 \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \\
&\quad + 2\gamma \left(\langle Ap - T_{r_n}^{F_2} Ax_n, Ax_n - T_{r_n}^{F_2} Ax_n \rangle - \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \right) \\
&\leq \|x_n - q\|^2 + L\gamma^2 \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \\
&\quad + 2\gamma \left(\frac{1}{2} \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 - \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \right) \\
&= \|x_n - q\|^2 + \gamma(L\gamma - 1) \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2.
\end{aligned}
$$

Thus, by (3.5) and (3.7), we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \beta_n \|w_n - q\|^2 + (1 - \beta_n) \|z_n - q\|^2 \\
&\leq \beta_n \|u_n - q\|^2 + (1 - \beta_n) \|x_n - q\|^2 \\
&\leq \beta_n \left(\|x_n - q\|^2 + \gamma(L\gamma - 1) \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \right) + (1 - \beta_n) \|x_n - q\|^2 \\
&\leq \|x_n - q\|^2 + \gamma(L\gamma - 1)\beta_n \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2. \qquad (3.11)
\end{aligned}
$$

Therefore, we have

$$
-\gamma(L\gamma - 1)\beta_n \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.
$$

Since $\gamma(L\gamma - 1) < 0$, it follows by Condition (C1) and the existence of $\lim_{n\to\infty} \|x_n - q\|$ that

$$
\lim_{n\to\infty} \|Ax_n - T_{r_n}^{F_2} Ax_n\| = 0. \qquad (3.12)
$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive and $I - \gamma A^*(I - T_{r_n}^{F_2})A$ is nonexpansive, we have

$$
\begin{aligned}
\|u_n - q\|^2 &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1} q\|^2 \\
&\leq \langle T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1} q, (I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - q \rangle \\
&= \langle u_n - q, (I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - q \rangle \\
&= \frac{1}{2} \left(\|u_n - q\|^2 + \|(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - q\|^2 \right. \\
&\quad \left. - \|u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n\|^2 \right) \\
&\leq \frac{1}{2} \left(\|u_n - q\|^2 + \|x_n - q\|^2 - \left(\|u_n - x_n\|^2 + \gamma^2 \|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 \right. \right. \\
&\quad \left. \left. - 2\gamma \langle u_n - x_n, A^*(I - T_{r_n}^{F_2})Ax_n \rangle \right) \right),
\end{aligned}
$$

which implies that

$$\begin{aligned}\|u_n - q\|^2 &\leq \|x_n - q\|^2 - \|u_n - x_n\|^2 + 2\gamma\langle u_n - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle \\ &\leq \|x_n - q\|^2 - \|u_n - x_n\|^2 + 2\gamma\|u_n - x_n\|\|A^*(I - T_{r_n}^{F_2})Ax_n\|. \end{aligned} \tag{3.13}$$

This implies by (3.5) and (3.7) that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \beta_n\|w_n - q\|^2 + (1-\beta_n)\|z_n - q\|^2 \\ &\leq \beta_n\|u_n - q\|^2 + (1-\beta_n)\|x_n - q\|^2 \\ &\leq \beta_n\big(\|x_n - q\|^2 - \|u_n - x_n\|^2 + 2\gamma\|u_n - x_n\|\|A^*(I - T_{r_n}^{F_2})Ax_n\|\big) \\ &\quad + (1-\beta_n)\|x_n - q\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\beta_n\|u_n - x_n\|^2 &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2\gamma\beta_n\|u_n - x_n\|\|A^*(I - T_{r_n}^{F_2})Ax_n\| \\ &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2\gamma\beta_n M\|A^*(I - T_{r_n}^{F_2})Ax_n\|,\end{aligned}$$

where $M = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. This implies by Condition (C1), (3.12), and the existence of $\lim_{n\to\infty}\|x_n - q\|$ that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.14}$$

From (3.5), (3.7), and the definition of $\{y_n\}$, we obtain

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \beta_n\|w_n - q\|^2 + (1-\beta_n)\|z_n - q\|^2 \\ &\leq \beta_n\|x_n - q\|^2 + (1-\beta_n)\|y_n - q\|^2 \\ &= \beta_n\|x_n - q\|^2 + (1-\beta_n)\big(\alpha_n\|x_n - q\|^2 + (1-\alpha_n)\|w_n - q\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - w_n\|^2\big) \\ &\leq \beta_n\|x_n - q\|^2 + (1-\beta_n)\big(\|x_n - q\|^2 - \alpha_n(1-\alpha_n)\|x_n - w_n\|^2\big) \\ &= \|x_n - q\|^2 - \alpha_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2.\end{aligned}$$

This implies that

$$\alpha_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

From Conditions (C1), (C2), and the existence of $\lim_{n\to\infty}\|x_n - q\|$, we have

$$\lim_{n\to\infty}\|w_n - x_n\| = 0. \tag{3.15}$$

By (3.14) and (3.15), we get

$$\|w_n - u_n\| \leq \|w_n - x_n\| + \|x_n - u_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Step 4. Show that $\omega_w(x_n) \subset \Theta$, where $\omega_w(x_n) = \{x \in H_1 : x_{n_i} \rightharpoonup x, \{x_{n_i}\} \subset \{x_n\}\}$. Since $\{x_n\}$ is bounded and H_1 is reflexive, $\omega_w(x_n)$ is nonempty. Let $p \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ converging weakly to p. From (3.14), it implies that $u_{n_i} \rightharpoonup p$ as $i \to \infty$. By (3.16) and Lemma 2.3, we have $p \in F(S)$.

Next, we show that $p \in MEP(F_1, \varphi)$. Since $u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n$, we have

$$F_1(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\bigl\langle y - u_n, u_n - x_n - \gamma A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle \geq 0, \quad \forall y \in C,$$

which implies that

$$F_1(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\bigl\langle y - u_n, \gamma A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle \geq 0, \quad \forall y \in C.$$

From Assumption 2.4(A2), we have

$$\begin{aligned}
&\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\bigl\langle y - u_n, \gamma A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle \\
&\quad \geq -F_1(u_n, y) \geq F_1(y, u_n), \quad \forall y \in C,
\end{aligned}$$

and hence

$$\begin{aligned}
&\varphi(y) - \varphi(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle - \frac{1}{r_{n_i}}\bigl\langle y - u_{n_i}, \gamma A^*\bigl(I - T_{r_{n_i}}^{F_2}\bigr)Ax_{n_i}\bigr\rangle \geq F_1(y, u_{n_i}), \\
&\quad \forall y \in C.
\end{aligned}$$

This implies by $u_{n_i} \rightharpoonup p$, Condition (C3), (3.12), (3.14), Assumption 2.4(A2), and the proper lower semicontinuity of φ that

$$F_1(y, p) + \varphi(p) - \varphi(y) \leq 0, \quad \forall y \in C.$$

Put $y_t = ty + (1 - t)p$ for all $t \in (0, 1]$ and $y \in C$. Consequently, we get $y_t \in C$ and hence $F_1(y_t, p) + \varphi(p) - \varphi(y_t) \leq 0$. So, by Assumption 2.4(A1)-(A4), we have

$$\begin{aligned}
0 &= F_1(y_t, y_t) - \varphi(y_t) + \varphi(y_t) \\
&\leq tF_1(y_t, y) + (1 - t)F_1(y_t, p) + t\varphi(y) + (1 - t)\varphi(p) - \varphi(y_t) \\
&\leq t\bigl(F_1(y_t, y) + \varphi(y) - \varphi(y_t)\bigr).
\end{aligned}$$

Hence, we have

$$F_1(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0, \quad \forall y \in C.$$

Letting $t \to 0$, by Assumption 2.4(A3) and the proper lower semicontinuity of φ, we have

$$F_1(p, y) + \varphi(y) - \varphi(p) \geq 0, \quad \forall y \in C.$$

This implies that $p \in MEP(F_1, \varphi)$.

Since A is a bounded linear operator, we have $Ax_{n_i} \rightharpoonup Ap$. Then it follows from (3.12) that

$$T_{r_{n_i}}^{F_2} Ax_{n_i} \rightharpoonup Ap \quad \text{as } i \to \infty. \tag{3.17}$$

By the definition of $T_{r_{n_i}}^{F_2} Ax_{n_i}$, we have

$$F_2\bigl(T_{r_{n_i}}^{F_2} Ax_{n_i}, y\bigr) + \phi(y) - \phi\bigl(T_{r_{n_i}}^{F_2} Ax_{n_i}\bigr) + \frac{1}{r_{n_i}} \bigl\langle y - T_{r_{n_i}}^{F_2} Ax_{n_i}, T_{r_{n_i}}^{F_2} Ax_{n_i} - Ax_{n_i} \bigr\rangle \geq 0, \quad \forall y \in Q.$$

Since F_2 is upper semicontinuous in the first argument, it implies by (3.17) that

$$F_2(Ap, y) + \phi(y) - \phi(Ap) \geq 0, \quad \forall y \in Q.$$

This shows that $Ap \in MEP(F_2, \phi)$. Therefore, $p \in SMEP(F_1, \varphi, F_2, \phi)$ and hence $p \in \Theta$.

Step 5. Show that $\{x_n\}$ converges weakly to an element of Θ. It is sufficient to show that $\omega_w(x_n)$ is a singleton set. Let $p, q \in \omega_w(x_n)$ and $\{x_{n_k}\}$, $\{x_{n_m}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ and $x_{n_m} \rightharpoonup q$. From (3.14), we also have $u_{n_k} \rightharpoonup p$ and $u_{n_m} \rightharpoonup q$. By (3.16) and Lemma 2.3, we see that $p, q \in F(S)$. Applying Lemma 2.2, we obtain $p = q$. This completes the proof. □

If $\varphi = \phi = 0$ in (3.1) and (3.2), then the split mixed equilibrium problem reduces to split equilibrium problem. So, the following result can be obtained from Theorem 3.2 immediately.

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator and $S : C \to K(C)$ a λ-hybrid multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption* 2.4, *and F_2 is upper semicontinuous in the first argument. Assume that $\Theta = F(S) \cap SEP(F_1, F_2) \neq \emptyset$ and $Sp = \{p\}$ for all $p \in F(S)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n) w_n, \quad w_n \in S u_n, \\ x_{n+1} = \beta_n w_n + (1 - \beta_n) z_n, \quad z_n \in S y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.18}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0, \infty)$, and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(C1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C2) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(C3) $0 < \liminf_{n\to\infty} r_n$.

Then the sequence $\{x_n\}$ generated by (3.18) *converges weakly to $p \in \Theta$.*

Remark 3.4

(i) Theorems 3.2 and 3.3 extend the corresponding one of Suantai *et al.* [13] and Kazmi and Rizvi [10] to λ-hybrid multivalued mapping and to a split mixed equilibrium problem. In fact, we present a new iterative algorithm for finding a common element of the set of solutions of split mixed equilibrium problems and the set of fixed points of λ-hybrid multivalued mappings in a real Hilbert space.

(ii) It is well known that the class of λ-hybrid multivalued mappings contains the classes of nonexpansive multivalued mappings, nonspreading multivalued mappings. Thus, Theorems 3.2 and 3.3 can be applied to these classes of mappings.

We give an example to illustrate Theorem 3.2 as follows.

Example 3.5 Let $H_1 = \mathbb{R}$, $H_2 = \mathbb{R}$, $C = [-3,0]$, and $Q = (-\infty,0]$. Let $A : H_1 \longrightarrow H_2$ defined by $Ax = \frac{x}{2}$ for each $x \in H_1$. Then $A^*y = \frac{y}{2}$ for each $y \in H_2$. So, $L = \frac{1}{2}$ is the spectral radius of A^*A. Define a multivalued mapping $S : C \longrightarrow K(C)$ by

$$Sx = \begin{cases} [-\frac{|x|}{|x|+1}, 0], & x \in [-3,-2); \\ \{0\}, & x \in [-2,0]. \end{cases}$$

It easy to see that S is 1-hybrid multivalued mapping with $F(S) = \{0\}$ and $S(0) = \{0\}$. For each $x, y \in C$, define the bifunction $F_1 : C \times C \longrightarrow \mathbb{R}$ by $F_1(x,y) = xy + y - x - x^2$ and define $\varphi(x) = 0$ for each $x \in C$. For each $u, v \in Q$, define the bifunction $F_2 : Q \times Q \longrightarrow \mathbb{R}$ by $F_2(u,v) = uv + 10v - 10u - u^2$ and define $\phi(u) = 0$ for each $u \in Q$.

Choose $\alpha_n = \frac{n}{5n+1}$, $\beta_n = \frac{n}{9n+1}$, $r_n = \frac{n}{n+1}$, and $\gamma = \frac{1}{15}$. It is easy to check that F_1, F_2, $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ satisfy all conditions in Theorem 3.2.

For each $x \in C$, we compute $T_r^{F_2}Ax$. Find z such that

$$\begin{aligned} 0 &\leq F_2(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - Ax\rangle \\ &= zy + 10y - 10z - z^2 + \frac{1}{r}\left\langle y - z, z - \frac{x}{2}\right\rangle \\ &= (z+10)(y-z) + \frac{1}{r}(y-z)\left(z - \frac{x}{2}\right) \\ &= (y-z)\left((z+10) + \frac{1}{r}\left(z - \frac{x}{2}\right)\right) \end{aligned}$$

for all $y \in Q$. Thus, by Lemma 2.5(2), it follows that $z = \frac{x-20r}{2(1+r)}$. That is, $T_r^{F_2}Ax = \frac{x-20r}{2(1+r)}$ for each $x \in C$. Furthermore, we get

$$\begin{aligned} \left(I - \gamma A^*\left(I - T_r^{F_2}\right)A\right)x &= x - \frac{1}{15}A^*\left(Ax - T_r^{F_2}Ax\right) \\ &= x - \frac{1}{15}A^*\left(\frac{x}{2} - \frac{x-20r}{2(1+r)}\right) \\ &= x - \frac{1}{15}\left(\frac{x}{4} - \frac{x-20r}{4(1+r)}\right) \\ &= x\left(1 - \frac{\gamma}{60}\right) - \frac{\gamma(x-20r)}{60(1+r)}. \end{aligned}$$

Table 1 Numerical results of Example 3.5 for the algorithm (3.19)

n	x_n	$\|x_n - x_{n-1}\|$
1	−3.0000000e+00	-
2	−6.8786127e–02	2.9312139e+00
3	0.0000000e+00	6.8786127e–02
4	0.0000000e+00	0.0000000e+00

Next, we find $u \in C$ such that $F_1(u,v) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle v-u, u-s\rangle \geq 0$ for all $v \in C$, where $s = (I - \gamma A^*(I - T_r^{F_2})A)x$. Note that

$$\begin{aligned} 0 \leq F_1(u,v) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle v-u, u-s\rangle &= uv + v - u - u^2 + \frac{1}{r}\langle v-u, u-s\rangle \\ &= (u+1)(v-u) + \frac{1}{r}(v-u)(u-s) \\ &= (v-u)\biggl((u+1) + \frac{1}{r}(u-s)\biggr). \end{aligned}$$

Thus, by Lemma 2.5(2), it follows that

$$u = \frac{s-r}{1+r} = \frac{59x - 60r}{60(1+r)} - \frac{x - 20r}{60(1+r)^2}.$$

Then the algorithm (3.3) becomes

$$\begin{cases} u_n = \frac{59x_n - 60r_n}{60(1+r_n)} - \frac{x_n - 20r_n}{60(1+r_n)^2}, \quad r_n = \frac{n}{n+1}, \\ y_n = \frac{n}{5n+1}x_n + (1 - \frac{n}{5n+1})w_n, \\ x_{n+1} = \frac{n}{9n+1}w_n + (1 - \frac{n}{9n+1})z_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.19}$$

where

$$w_n = \begin{cases} [-\frac{|u_n|}{|u_n|+1}, 0], & u_n \in [-3,-2); \\ \{0\}, & u_n \in [-2,0], \end{cases} \qquad z_n = \begin{cases} [-\frac{|y_n|}{|y_n|+1}, 0], & y_n \in [-3,-2); \\ \{0\}, & y_n \in [-2,0]. \end{cases}$$

We choose $w_n = -\frac{|u_n|}{|u_n|+1}$ if $u_n \in [-3,-2)$ and $z_n = -\frac{|y_n|}{|y_n|+1}$ if $y_n \in [-3,-2)$. By using SciLab, we compute the iterates of (3.19) for the initial point $x_1 = -3$. The numerical experiment's results of our iteration for approximating the point 0 are given in Table 1.

4 Conclusions

The results presented in this paper extend and generalize the work of Suantai *et al.* [13] and Kazmi and Rizvi [10]. The main aim of this paper is to propose an iterative algorithm to find an element for solving a class of split mixed equilibrium problems and fixed point problems for λ-hybrid multivalued mappings under weaker conditions. Some sufficient conditions for the weak convergence of such proposed algorithm are given. Also, in order to show the significance of the considered problem, some important applications are discussed.

Acknowledgements
The authors are thankful to the referees for careful reading and the useful comments and suggestions.

Funding
This work was supported by the Research Center for Pure and Applied Mathematics, Research and Development Institute, Nakhon Pathom Rajabhat University, Nakhon Pathom, Thailand. The second author was also supported by the Thailand Research Fund under the project RTA5780007.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 25 March 2017 Accepted: 5 June 2017 Published online: 15 June 2017

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