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On Hardy-type integral inequalities with the gamma function

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Abstract

By means of real analysis and weight functions, we obtain a few equivalent conditions of two kinds of Hardy-type integral inequalities with the non-homogeneous kernel and parameters. The constant factors related to the gamma function are proved to be the best possible. We also consider the operator expressions and some cases of homogeneous kernel.

MSC: 26D15

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1 Introduction

If $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(y)\,dy < \infty$, then we have the following Hilbert's integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy \right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. In 1925, by introducing one pair of conjugate exponents (p, q), Hardy [2] gave an extension of (1) as follows: For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^p(x)\,dx < \infty$ and $0 < \int_0^\infty g^q(y)\,dy < \infty$, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)\,dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)\,dy \right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1) and (2) are important in analysis and its applications (*cf.* [3, 4]).

In 1934, Hardy *et al.* gave an extension of (2) as follows: If $k_1(x,y)$ is a non-negative homogeneous function of degree -1, $k_p = \int_0^\infty k_1(u,1)u^{\frac{-1}{p}}\,du \in \mathbf{R}_+ = (0,\infty)$, then

$$\int_0^\infty \int_0^\infty k_1(x,y)f(x)g(y)\,dx\,dy < k_p \left(\int_0^\infty f^p(x)\,dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)\,dy \right)^{\frac{1}{q}}, \tag{3}$$

where the constant factor k_p is the best possible (*cf.* [3], Theorem 319). Additionally, a Hilbert-type integral inequality with the non-homogeneous kernel is proved as follows:

If $h(u) > 0$, $\phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}\,du \in \mathbf{R}_+$, then

$$\int_0^\infty \int_0^\infty h(xy)f(x)g(y)\,dx\,dy < \phi\left(\frac{1}{p}\right)\left(\int_0^\infty x^{p-2}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}, \tag{4}$$

where the constant factor $\phi(\frac{1}{p})$ is still the best possible (*cf.* [3], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang gave a best extension of (1) with the kernel $\frac{1}{(x+y)^\lambda}$ (*cf.* [5, 6]). In 2004, by introducing another pair conjugate exponents (r, s), Yang [7] gave an extension of (2) as follows: If $\lambda > 0$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $f(x)$, $g(y) \geq 0$, $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)\,dx < \infty$ and $0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)\,dy < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda}\,dx\,dy < \frac{\pi}{\lambda \sin(\pi/r)}\left(\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)\,dy\right)^{\frac{1}{q}}, \tag{5}$$

where the constant factor $\frac{\pi}{\lambda \sin(\pi/r)}$ is the best possible. For $\lambda = 0$, $r = q$, $s = p$, (5) reduces to (2); For $\lambda = 1$, $r = p$, $s = q$, (5) reduces to the dual form of (2) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty x^{p-2}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q-2}g^q(y)\,dy\right)^{\frac{1}{q}}. \tag{6}$$

For $p = q = 2$, both (2) and (7) reduce to (1).

In 2005, in [8] one also gave an extension of (2) and (5) with the kernel $\frac{1}{(x+y)^\lambda}$. Krnić *et al.* [9–14] provided some extensions and particular cases of (2), (3) and (4) with parameters.

In 2009, Yang gave an extension of (3) and (5) as follows (*cf.* [15, 16]): If $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R} = (-\infty, \infty)$, $k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y) \quad (u, x, y > 0),$$

and

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1)u^{\lambda_1-1}\,du \in \mathbf{R}_+,$$

then

$$\int_0^\infty \int_0^\infty k_\lambda(x, y)f(x)g(y)\,dx\,dy < k(\lambda_1)\left(\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\lambda_2)-1}g^q(y)\,dy\right)^{\frac{1}{q}}, \tag{7}$$

where the constant factor $k(\lambda_1)$ is the best possible. For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (7) reduces to (3). Additionally, an extension of (4) was given as follows:

$$\begin{aligned} &\int_0^\infty \int_0^\infty h(xy)f(x)g(y)\,dx\,dy \\ &\quad < \phi(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)\,dy\right)^{\frac{1}{q}}, \end{aligned} \tag{8}$$

where the constant factor $\phi(\sigma)$ is the best possible (*cf.* [17]). For $\sigma = \frac{1}{p}$, (8) reduces to (4).

Some equivalent inequalities of (7) and (8) are considered by [16]. In 2013, Yang [17] studied the equivalency between (7) and (8). In 2017, Hong [18] studied an equivalent condition between (7) with a few parameters.

Remark 1 (*cf.* [17]) If $h(xy) = 0$, for $xy > 1$, $\phi(\sigma) = \int_0^1 h(u)u^{\sigma-1}\,du = \phi_1(\sigma) \in \mathbf{R}_+$, then (8) reduces to the following Hardy-type integral inequality with the non-homogeneous kernel:

$$\begin{aligned} &\int_0^\infty g(y)\left(\int_0^{\frac{1}{y}} h(xy)f(x)\,dx\right)dy \\ &\quad < \phi_1(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)\,dy\right)^{\frac{1}{q}}; \end{aligned} \tag{9}$$

if $h(xy) = 0$, for $xy < 1$, $\phi(\sigma) = \int_1^\infty h(u)u^{\sigma-1}\,du = \phi_2(\sigma) \in \mathbf{R}_+$, then (8) reduces to the following another kind of Hardy-type integral inequality with the non-homogeneous kernel:

$$\begin{aligned} &\int_0^\infty g(y)\left(\int_{\frac{1}{y}}^\infty h(xy)f(x)\,dx\right)dy \\ &\quad < \phi_2(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)\,dy\right)^{\frac{1}{q}}. \end{aligned} \tag{10}$$

In this paper, by real analysis and the weight functions, we obtain a few equivalent conditions of two kinds of Hardy-type integral inequalities with the non-homogeneous kernel and parameters as $\frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}$. The constant factors related to the gamma function are proved to be the best possible. We also consider the operator expressions and some cases of homogeneous kernel.

2 Two lemmas

For $\beta > -1$, $\sigma + \mu = \lambda \in \mathbf{R}$, we set

$$h(u) := \frac{(\min\{u,1\})^\alpha|\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} \quad (u > 0).$$

Then, for $\sigma > -\alpha$, setting $v = -(\sigma+\alpha)\ln u$, we find

$$\begin{aligned} k_1(\sigma) &:= \int_0^1 \frac{(\min\{u,1\})^\alpha|\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}}u^{\sigma-1}\,du \\ &= \int_0^1 u^{\sigma+\alpha-1}(-\ln u)^\beta\,du = \frac{1}{(\sigma+\alpha)^{\beta+1}}\int_0^\infty v^\beta e^{-v}\,dv = \frac{\Gamma(\beta+1)}{(\sigma+\alpha)^{\beta+1}} \in \mathbf{R}_+. \end{aligned} \tag{11}$$

For $\mu > -\alpha$, we find

$$k_2(\sigma) := \int_1^\infty \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-1}\,du$$
$$= \int_0^1 v^{\mu+\alpha-1}(-\ln v)^\beta\,dv = \frac{\Gamma(\beta+1)}{(\mu+\alpha)^{\beta+1}} = k_1(\mu) \in \mathbf{R}_+, \quad (12)$$

where $\Gamma(\eta) := \int_0^\infty v^{\eta-1}e^{-v}\,dv$ ($\eta > 0$) is the gamma function (*cf.* [19]).

Lemma 1 *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma_1 \in R$, $\beta > -1$, $\sigma > -\alpha$, there exists a constant M_1, such that, for any non-negative measurable functions $f(x)$ and $g(y)$ in $(0,\infty)$, the following inequality*:

$$\int_0^\infty g(y)\left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx\right] dy$$
$$\leq M_1\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)\,dy\right]^{\frac{1}{q}} \quad (13)$$

holds true, then we have $\sigma_1 = \sigma$, and then $M_1 \geq k_1(\sigma)$.

Proof If $\sigma_1 > \sigma$, then, for $n \geq \frac{1}{\sigma_1-\sigma}$ ($n \in \mathbf{N}$), we set the following functions:

$$f_n(x) := \begin{cases} x^{\sigma+\frac{1}{pn}-1}, & 0 < x \leq 1, \\ 0, & x > 1, \end{cases} \qquad g_n(y) := \begin{cases} 0, & 0 < y < 1, \\ y^{\sigma_1-\frac{1}{qn}-1}, & y \geq 1, \end{cases}$$

and find

$$J_1 := \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y)\,dy\right]^{\frac{1}{q}}$$
$$= \left(\int_0^1 x^{\frac{1}{n}-1}\,dx\right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\frac{1}{n}-1}\,dy\right)^{\frac{1}{q}} = n.$$

Setting $u = xy$, we obtain

$$I_1 := \int_0^\infty g_n(y)\left(\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f_n(x)\,dx\right) dy$$
$$= \int_1^\infty \left(\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha (-\ln xy)^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} x^{\sigma+\frac{1}{pn}-1}\,dx\right) y^{\sigma_1-\frac{1}{qn}-1}\,dy$$
$$= \int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\,dy \int_0^1 \frac{(\min\{u,1\})^\alpha (-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma+\frac{1}{pn}-1}\,du.$$

Then by (13), we have

$$\int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\,dy \int_0^1 \frac{(\min\{u,1\})^\alpha (-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma+\frac{1}{pn}-1}\,du$$
$$= I_1 \leq M_1 J_1 = M_1 n < \infty. \quad (14)$$

Since $(\sigma_1 - \sigma) - \frac{1}{n} \geq 0$, it follows that $\int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\,dy = \infty$. By (14), in view of $\int_0^{\frac{1}{y}} \frac{(\min\{u,1\})^\alpha(-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma+\frac{1}{pn}-1}\,du > 0$, we find $\infty < \infty$, which is a contradiction.

If $\sigma_1 < \sigma$, then, for $n \geq \frac{1}{\sigma-\sigma_1}$ ($n \in \mathbf{N}$), we set the following functions:

$$\tilde{f}_n(x) := \begin{cases} 0, & 0 < x < 1, \\ x^{\sigma-\frac{1}{pn}-1}, & x \geq 1, \end{cases} \qquad \tilde{g}_n(y) := \begin{cases} y^{\sigma_1+\frac{1}{qn}-1}, & 0 < y \leq 1, \\ 0, & y > 1, \end{cases}$$

and find

$$\begin{aligned} \tilde{J}_1 &:= \left[\int_0^\infty x^{p(1-\sigma)-1}\tilde{f}_n^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1}\tilde{g}_n^q(y)\,dy\right]^{\frac{1}{q}} \\ &= \left(\int_1^\infty x^{-\frac{1}{n}-1}\,dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1}\,dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

Setting $u = xy$, we obtain

$$\begin{aligned} \tilde{I}_1 &:= \int_0^\infty \tilde{f}_n(x)\left(\int_0^{\frac{1}{x}} \frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}\tilde{g}_n(y)\,dy\right)dx \\ &= \int_1^\infty \left(\int_0^{\frac{1}{x}} \frac{(\min\{xy,1\})^\alpha(-\ln xy)^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} y^{\sigma_1+\frac{1}{qn}-1}\,dy\right) x^{\sigma-\frac{1}{pn}-1}\,dx \\ &= \int_1^\infty x^{(\sigma-\sigma_1)-\frac{1}{n}-1}\,dx \int_0^1 \frac{(\min\{u,1\})^\alpha(-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma_1+\frac{1}{qn}-1}\,du. \end{aligned}$$

Then by the Fubini theorem (*cf.* [20]) and (13), we have

$$\begin{aligned} &\int_1^\infty x^{(\sigma-\sigma_1)-\frac{1}{n}-1}\,dx \int_0^1 \frac{(\min\{u,1\})^\alpha(-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma_1+\frac{1}{qn}-1}\,du \\ &\quad = \tilde{I}_1 = \int_0^\infty \tilde{g}_n(y)\left(\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}\tilde{f}_n(x)\,dx\right)dy \\ &\quad \leq M_1\tilde{J}_1 = M_1 n < \infty. \end{aligned} \tag{15}$$

Since $(\sigma_1 - \sigma) - \frac{1}{n} \geq 0$, it follows that $\int_1^\infty x^{(\sigma-\sigma_1)-\frac{1}{n}-1}\,dx = \infty$. By (15), in view of

$$\int_0^1 \frac{(\min\{u,1\})^\alpha(-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma_1+\frac{1}{qn}-1}\,du > 0$$

we find $\infty < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we reduce (15) as follows:

$$M_1 \geq \int_0^1 \frac{(\min\{u,1\})^\alpha(-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma+\frac{1}{qn}-1}\,du. \tag{16}$$

Since

$$\left\{\frac{(\min\{u,1\})^\alpha(-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma+\frac{1}{qn}-1}\right\}_{n=1}^\infty$$

is non-negative and increasing in (0, 1], by Levi theorem (*cf.* [20]), we find

$$M_1 \geq \lim_{n\to\infty}\int_0^1 \frac{(\min\{u,1\})^\alpha(-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}}u^{\sigma+\frac{1}{qn}-1}\,du$$
$$= \int_0^1 \lim_{n\to\infty}\frac{(\min\{u,1\})^\alpha(-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}}u^{\sigma+\frac{1}{qn}-1}\,du = k_1(\sigma).$$

The lemma is proved. □

Lemma 2 *If* $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $\sigma_1 \in R$, $\beta > -1$, $\mu = \lambda - \sigma > -\alpha$, *there exists a constant* M_2, *such that, for any non-negative measurable functions* $f(x)$ *and* $g(y)$ *in* $(0,\infty)$, *the following inequality*:

$$\int_0^\infty g(y)\left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}f(x)\,dx\right]dy$$
$$\leq M_2\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{17}$$

holds true, then we have $\sigma_1 = \sigma$, *and then* $M_2 \geq k_1(\mu)$.

Proof If $\sigma_1 < \sigma$, then, for $n \geq \frac{1}{\sigma-\sigma_1}$ ($n \in \mathbf{N}$), we set two functions $\tilde{f}_n(x)$ and $\tilde{g}_n(y)$ as in Lemma 1, and find

$$\tilde{J}_1 = \left[\int_0^\infty x^{p(1-\sigma)-1}\tilde{f}_n^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1}\tilde{g}_n^q(y)\,dy\right]^{\frac{1}{q}} = n.$$

Setting $u = xy$, we obtain

$$\tilde{I}_2 := \int_0^\infty \tilde{g}_n(y)\left(\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}\tilde{f}_n(x)\,dx\right)dy$$
$$= \int_0^1\left(\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha(\ln xy)^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}x^{\sigma-\frac{1}{pn}-1}\,dx\right)y^{\sigma_1+\frac{1}{qn}-1}\,dy$$
$$= \int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\,dy\int_1^\infty \frac{(\min\{u,1\})^\alpha(\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}}u^{\sigma-\frac{1}{pn}-1}\,du,$$

and then by (17), we obtain

$$\int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\,dy\int_1^\infty \frac{(\min\{u,1\})^\alpha(\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}}u^{\sigma-\frac{1}{pn}-1}\,du$$
$$= \tilde{I}_2 \leq M_2\tilde{J}_1 = M_2 n < \infty. \tag{18}$$

Since $(\sigma_1-\sigma)+\frac{1}{n} \leq 0$, it follows that $\int_1^\infty y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\,dy = \infty$. By (18), in view of

$$\int_1^\infty \frac{(\min\{u,1\})^\alpha(\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}}u^{\sigma-\frac{1}{pn}-1}\,du > 0,$$

we have $\infty < \infty$, which is a contradiction.

If $\sigma_1 > \sigma$, then, for $n \geq \frac{1}{\sigma_1 - \sigma}$ ($n \in \mathbf{N}$), we set two functions $f_n(x)$ and $g_n(y)$ as in Lemma 1, and we find

$$J_1 = \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y)\,dy\right]^{\frac{1}{q}} = n.$$

Setting $u = xy$, we obtain

$$\begin{aligned} I_2 &:= \int_0^\infty f_n(x)\left(\int_{\frac{1}{x}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} g_n(y)\,dy\right)dx \\ &= \int_0^1 \left[\int_{\frac{1}{x}}^\infty \frac{(\min\{xy,1\})^\alpha (\ln xy)^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} y^{\sigma_1 - \frac{1}{qn} - 1}\,dy\right] x^{\sigma + \frac{1}{pn} - 1}\,dx \\ &= \int_0^1 x^{(\sigma-\sigma_1)+\frac{1}{n}-1}\,dx \int_1^\infty \frac{(\min\{u,1\})^\alpha (\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma_1 - \frac{1}{qn} - 1}\,du, \end{aligned}$$

and then by Fubini theorem (*cf.* [20]) and (17), we have

$$\begin{aligned} &\int_0^1 x^{(\sigma-\sigma_1)+\frac{1}{n}-1}\,dx \int_1^\infty \frac{(\min\{u,1\})^\alpha (\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma_1 - \frac{1}{qn} - 1}\,du \\ &\quad = I_2 = \int_0^\infty g_n(y)\left(\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha (\ln xy)^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f_n(x)\,dx\right)dy \\ &\quad \leq M_2 J_1 = M_2 n. \end{aligned} \tag{19}$$

Since $(\sigma - \sigma_1) + \frac{1}{n} \leq 0$, it follows that $\int_0^1 x^{(\sigma-\sigma_1)+\frac{1}{n}-1}\,dx = \infty$. By (19), in view of

$$\int_1^\infty \frac{(\min\{u,1\})^\alpha (\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma_1 - \frac{1}{qn} - 1}\,du > 0,$$

we have $\infty < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we reduce (19) as follows:

$$M_2 \geq \int_1^\infty \frac{(\min\{u,1\})^\alpha (\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma - \frac{1}{qn} - 1}\,du. \tag{20}$$

Since

$$\left\{\frac{(\min\{u,1\})^\alpha (\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma - \frac{1}{qn} - 1}\right\}_{n=1}^\infty$$

is non-negative and increasing in $[1,\infty)$, still by the Levi theorem (*cf.* [20]), we have

$$\begin{aligned} M_2 &\geq \lim_{n\to\infty} \int_1^\infty \frac{(\min\{u,1\})^\alpha (\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma - \frac{1}{qn} - 1}\,du \\ &= \int_1^\infty \lim_{n\to\infty} \frac{(\min\{u,1\})^\alpha (\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma - \frac{1}{qn} - 1}\,du = k_1(\mu). \end{aligned}$$

The lemma is proved. □

3 Main results and corollaries

Theorem 1 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma_1 \in R$, $\beta > -1$, $\sigma > -\alpha$, *then the following conditions are equivalent*:

(i) *There exists a constant* M_1, *such that, for any* $f(x) \geq 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty,$$

we have the following Hardy-type integral inequality of the first kind with the non-homogeneous kernel:

$$\begin{aligned} J &:= \left\{ \int_0^\infty y^{p\sigma_1 - 1} \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx \right]^p dy \right\}^{\frac{1}{p}} \\ &< M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx \right]^{\frac{1}{p}}. \end{aligned} \tag{21}$$

(ii) *There exists a constant* M_1, *such that, for any* $f(x), g(y) \geq 0$, *satisfying* $0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty$, *and* $0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)\,dy < \infty$, *we have the following inequality*:

$$\begin{aligned} I &:= \int_0^\infty g(y) \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx \right] dy \\ &< M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)\,dy \right]^{\frac{1}{q}}. \end{aligned} \tag{22}$$

(iii) $\sigma_1 = \sigma$.

If condition (iii) *holds true, then* $M_1 \geq k_1(\sigma)$ *and the constant factor*

$$M_1 = k_1(\sigma) = \frac{\Gamma(\beta+1)}{(\sigma+\alpha)^{\beta+1}}$$

in (21) *and* (22) *is the best possible*.

Proof (i)⇒(ii). By Hölder's inequality (*cf.* [21]), we have

$$\begin{aligned} I &= \int_0^\infty \left[y^{\sigma_1 - \frac{1}{p}} \int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx \right] \left(y^{\frac{1}{p}-\sigma_1} g(y) \right) dy \\ &\leq J \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)\,dy \right]^{\frac{1}{q}}. \end{aligned} \tag{23}$$

Then by (21), we have (22).

(ii)⇒(iii). By Lemma 1, we have $\sigma_1 = \sigma$.

(iii)⇒(i). Setting $u = xy$, we obtain the following weight function:

$$\begin{aligned} \omega_1(\sigma, y) &:= y^\sigma \int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} x^{\sigma-1}\,dx \\ &= \int_0^1 \frac{(\min\{u,1\})^\alpha (-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-1}\,du = k_1(\sigma) \quad (y > 0). \end{aligned} \tag{24}$$

By Hölder's inequality with weight and (24), for $y \in (0,\infty)$, we have

$$\begin{aligned}
&\left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx\right]^p \\
&= \left\{\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \left[\frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x)\right]\left[\frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}}\right] dx\right\}^p \\
&\leq \int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1} f^p(x)}{x^{(\sigma-1)p/q}}\,dx \\
&\quad \times \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}}\,dx\right]^{p-1} \\
&= \left[\frac{\omega_1(\sigma,y)}{y^{q(\sigma-1)+1}}\right]^{p-1} \int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)\,dx \\
&= \bigl(k_1(\sigma)\bigr)^{p-1} y^{-p\sigma+1} \int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)\,dx. \qquad (25)
\end{aligned}$$

If (25) obtains the form of equality for a $y \in (0,\infty)$, then (*cf.* [21]), there exist constants A and B, such that they are not all zero, and

$$A\frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B\frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \quad \text{a.e. in } \mathbf{R}_+.$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). It follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(\sigma-1)} \frac{B}{Ax} \quad \text{a.e. in } \mathbf{R}_+,$$

which contradicts the fact that $0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty$. Hence, (25) assumes the form of strict inequality. Hence, for $\sigma_1 = \sigma$, by (25) and by the Fubini theorem (*cf.* [20]), we obtain

$$\begin{aligned}
J &< \bigl(k_1(\sigma)\bigr)^{\frac{1}{q}} \left\{\int_0^\infty \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1} f^p(x)}{x^{(\sigma-1)p/q}}\,dx\right] dy\right\}^{\frac{1}{p}} \\
&= \bigl(k_1(\sigma)\bigr)^{\frac{1}{q}} \left\{\int_0^\infty \left[\int_0^{\frac{1}{x}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1}\,dy}{x^{(\sigma-1)(p-1)}}\right] f^p(x)\,dx\right\}^{\frac{1}{p}} \\
&= \bigl(k_1(\sigma)\bigr)^{\frac{1}{q}} \left[\int_0^\infty \omega_1(\sigma,x) x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \\
&= k_1(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}.
\end{aligned}$$

Setting $M_1 \geq k_1(\sigma)$, (21) follows.

Therefore, conditions (i), (ii) and (iii) are equivalent.

When condition (iii) follows, if there exists a constant factor $M_1 \geq k_1(\sigma)$, such that (22) is valid, then by Lemma 1, we have $M_1 \geq k_1(\sigma)$. Hence, the constant factor $M_1 = k_1(\sigma)$ in (22) is the best possible. The constant factor $M_1 = k_1(\sigma)$ in (21) is still the best possible. Otherwise, by (23) (for $\sigma_1 = \sigma$), we can conclude that the constant factor $M_1 = k_1(\sigma)$ in (22) is not the best possible. □

Setting $y = \frac{1}{Y}$, $G(Y) = Y^{\lambda-2}g(\frac{1}{Y})$, $\mu_1 = \lambda - \sigma_1$ in Theorem 1, then replacing Y by y and $G(Y)$ by $g(y)$, we have

Corollary 1 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu_1 \in R$, $\beta > -1$, $\sigma = \lambda - \mu > -\alpha$, *then the following conditions are equivalent*:

(i) *There exists a constant* M_1, *such that*, *for any* $f(x) \geq 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty,$$

we have the following Hardy-type inequality of the first kind with the homogeneous kernel:

$$\left\{\int_0^\infty y^{p\mu_1-1}\left[\int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} < M_1\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}. \tag{26}$$

(ii) *There exists a constant* M_1, *such that*, *for any* $f(x), g(y) \geq 0$, *satisfying* $0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty$, *and* $0 < \int_0^\infty y^{q(1-\mu_1)-1} g^q(y)\,dy < \infty$, *we have the following inequality*:

$$\int_0^\infty g(y)\left[\int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)\,dx\right] dy < M_1\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\mu_1)-1} g^q(y)\,dy\right]^{\frac{1}{q}}. \tag{27}$$

(iii) $\mu_1 = \mu$.

If condition (iii) *holds true*, *then we have* $M_1 \geq k_1(\sigma)$, *and the constant* $M_1 = k_1(\sigma)$ *in* (26) *and* (27) *is the best possible*.

Remark 2 On the other hand, setting $y = \frac{1}{Y}$, $G(Y) = Y^{\lambda-2}g(\frac{1}{Y})$, $\sigma_1 = \lambda - \mu_1$ in Corollary 1, then replacing Y by y and $G(Y)$ by $g(y)$, we have Theorem 1. Hence, Theorem 1 and Corollary 1 are equivalent.

Similarly, we obtain the following weight function:

$$\omega_2(\sigma, y) := y^\sigma \int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} x^{\sigma-1}\,dx = \int_1^\infty \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-1}\,du = k_1(\mu) \quad (y > 0),$$

and then in view of Lemma 2 and in the same way, we have

Theorem 2 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma_1 \in R$, $\beta > -1$, $\mu = \lambda - \sigma > -\alpha$, *then the following conditions are equivalent*:

(i) *There exists a constant* M_2, *such that*, *for any* $f(x) \geq 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty,$$

we have the following Hardy-type inequality of the second kind with the non-homogeneous kernel:

$$\left\{\int_0^\infty y^{p\sigma_1-1}\left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} < M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}. \tag{28}$$

(ii) *There exists a constant* M_2, *such that*, *for any* $f(x), g(y) \geq 0$, *satisfying* $0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty$, *and* $0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)\,dy < \infty$, *we have the following inequality*:

$$\int_0^\infty g(y)\left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx\right] dy < M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)\,dy\right]^{\frac{1}{q}}. \tag{29}$$

(iii) $\sigma_1 = \sigma$.

If condition (iii) *holds true*, *then we have* $M_2 \geq k_1(\mu)$ *and the constant factor*

$$M_2 = k_1(\mu) = \frac{\Gamma(\beta+1)}{(\mu+\alpha)^{\beta+1}}$$

in (28) *and* (29) *is the best possible*.

Setting $y = \frac{1}{Y}$, $G(Y) = Y^{\lambda-2} g(\frac{1}{Y})$, $\mu_1 = \lambda - \sigma_1$ in Theorem 2, then replacing Y by y and $G(Y)$ by $g(y)$, we have

Corollary 2 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu_1 \in R$, $\beta > -1$, $\mu = \lambda - \sigma > -\alpha$, *then the following conditions are equivalent*:

(i) *There exists a constant* M_2, *such that*, *for any* $f(x) \geq 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty,$$

we have the following Hardy-type inequality of the second kind with the homogeneous kernel:

$$\left\{\int_0^\infty y^{p\mu_1-1}\left[\int_y^\infty \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} < M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}. \tag{30}$$

(ii) *There exists a constant* M_2, *such that, for any* $f(x), g(y) \geq 0$, *satisfying* $0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty$, *and* $0 < \int_0^\infty y^{q(1-\mu_1)-1} g^q(y)\,dy < \infty$, *we have the following inequality*:

$$\int_0^\infty g(y)\left[\int_y^\infty \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)\,dx\right] dy < M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\mu_1)-1} g^q(y)\,dy\right]^{\frac{1}{q}}. \quad (31)$$

(iii) $\mu_1 = \mu$.

If condition (iii) *holds true, then we have* $M_2 \geq k_1(\mu)$, *and the constant* $M_2 = k_1(\mu)$ *in* (30) *and* (31) *is the best possible*.

Remark 3 Theorem 2 and Corollary 2 are still equivalent.

4 Operator expressions

For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma, \lambda > 0$, $\mu = \lambda - \sigma$, we set the following functions: $\varphi(x) := x^{p(1-\sigma)-1}$, $\psi(y) := y^{q(1-\sigma)-1}$, $\phi(y) := y^{q(1-\mu)-1}$, wherefrom, $\psi^{1-p}(y) = y^{p\sigma-1}$, $\phi^{1-p}(y) = y^{p\mu-1}$ $(x, y \in \mathbf{R}_+)$. Define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}_+) := \left\{f : \|f\|_{p,\varphi} := \left(\int_0^\infty \varphi(x)\left|f(x)\right|^p dx\right)^{\frac{1}{p}} < \infty\right\},$$

wherefrom

$$L_{q,\psi}(\mathbf{R}_+) := \left\{g : \|g\|_{q,\psi} := \left(\int_0^\infty \psi(y)\left|g(y)\right|^q dy\right)^{\frac{1}{q}} < \infty\right\},$$

$$L_{q,\phi}(\mathbf{R}_+) := \left\{g : \|g\|_{q,\phi} := \left(\int_0^\infty \phi(y)\left|g(y)\right|^q dy\right)^{\frac{1}{q}} < \infty\right\},$$

$$L_{p,\psi^{1-p}}(\mathbf{R}_+) = \left\{h : \|h\|_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y)\left|h(y)\right|^p dy\right)^{\frac{1}{p}} < \infty\right\},$$

$$L_{q,\phi^{1-p}}(\mathbf{R}_+) = \left\{h : \|h\|_{p,\phi^{1-p}} = \left(\int_0^\infty \phi^{1-p}(y)\left|h(y)\right|^p dy\right)^{\frac{1}{p}} < \infty\right\}.$$

(a) In view of Theorem 1 ($\sigma_1 = \sigma$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_1(y) := \int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx \quad (y \in \mathbf{R}_+),$$

by (21), we have

$$\|h_1\|_{p,\psi^{1-p}} = \left[\int_0^\infty \psi^{1-p}(y) h_1^p(y)\,dy\right]^{\frac{1}{p}} < M_1\|f\|_{p,\varphi} < \infty. \quad (32)$$

Definition 1 Define a Hardy-type integral operator of the first kind with the non-homogeneous kernel $T_1^{(1)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\psi^{1-p}}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_1^{(1)}f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying for any $y \in \mathbf{R}_+$, $T_1^{(1)}f(y) = h_1(y)$.

In view of (32), it follows that $\|T_1^{(1)}f\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq M_1\|f\|_{p,\varphi}$, and then the operator $T_1^{(1)}$ is bounded satisfying

$$\|T_1^{(1)}\| = \sup_{f(\neq\theta)\in L_{p,\varphi}(R_+)} \frac{\|T_1^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_1.$$

If we define the formal inner product of $T_1^{(1)}f$ and g as follows:

$$\left(T_1^{(1)}f, g\right) := \int_0^\infty \left(\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx\right) g(y)\,dy,$$

then we can rewrite Theorem 1 (for $\sigma_1 = \sigma$) as follows.

Theorem 3 *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > -1$, $\sigma > -\alpha$, then the following conditions are equivalent*:

(i) *There exists a constant M_1, such that, for any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$, $\|f\|_{p,\varphi} > 0$, we have the following inequality*:

$$\left\|T_1^{(1)}f\right\|_{p,\psi^{1-p}} < M_1\|f\|_{p,\varphi}. \tag{33}$$

(ii) *There exists a constant M_1, such that, for any $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, we have the following inequality*:

$$\left(T_1^{(1)}f, g\right) < M_1\|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{34}$$

We still have $\|T_1^{(1)}\| = k_1(\sigma) \leq M_1$.

(b) In view of Corollary 1 ($\mu_1 = \mu$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_2(y) := \int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)\,dx \quad (y \in \mathbf{R}_+),$$

by (26), we have

$$\|h_2\|_{p,\phi^{1-p}} = \left[\int_0^\infty \phi^{1-p}(y) h_2^p(y)\,dy\right]^{\frac{1}{p}} < M_1\|f\|_{p,\varphi} < \infty. \tag{35}$$

Definition 2 Define a Hardy-type integral operator of the first kind with the homogeneous kernel $T_1^{(2)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\phi^{1-p}}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_1^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R}_+)$, satisfying for any $y \in \mathbf{R}_+$, $T_1^{(2)}f(y) = h_2(y)$.

In view of (35), it follows that $\|T_1^{(2)}f\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq M_1\|f\|_{p,\varphi}$, and then the operator $T_1^{(2)}$ is bounded satisfying

$$\|T_1^{(2)}\| = \sup_{f(\neq\theta)\in L_{p,\varphi}(R_+)} \frac{\|T_1^{(2)}f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_1.$$

If we define the formal inner product of $T_1^{(2)}f$ and g as follows:

$$\left(T_1^{(2)}f, g\right) := \int_0^\infty \left(\int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)\,dx\right) g(y)\,dy,$$

then we can rewrite Corollary 1 (for $\mu_1 = \mu$) as follows.

Corollary 3 *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \beta > -1, \sigma = \lambda - \mu > -\alpha$, *then the following conditions are equivalent*:

(i) *There exists a constant* M_1, *such that, for any* $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$, $\|f\|_{p,\varphi} > 0$, *we have the following inequality*:

$$\left\| T_1^{(2)}f \right\|_{p,\phi^{1-p}} < M_1 \|f\|_{p,\varphi}. \tag{36}$$

(ii) *There exists a constant* M_1, *such that, for any* $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+), g \in L_{q,\phi}(\mathbf{R}_+)$, $\|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0$, *we have the following inequality*:

$$\left(T_1^{(2)}f, g\right) < M_1 \|f\|_{p,\varphi} \|g\|_{q,\phi}. \tag{37}$$

We still have $\|T_1^{(2)}\| = k_1(\sigma) \leq M_1$.

Remark 4 Theorem 3 and Corollary 3 are equivalent.

(c) In view of Theorem 2 ($\sigma_1 = \sigma$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$H_1(y) := \int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx \quad (y \in \mathbf{R}_+),$$

by (28), we have

$$\|H_1\|_{p,\psi^{1-p}} = \left[\int_0^\infty \psi^{1-p}(y) H_1^p(y)\,dy\right]^{\frac{1}{p}} < M_2 \|f\|_{p,\varphi} < \infty. \tag{38}$$

Definition 3 Define a Hardy-type integral operator of the second kind with the non-homogeneous kernel $T_2^{(1)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\psi^{1-p}}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_2^{(1)}f = H_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying for any $y \in \mathbf{R}_+$, $T_2^{(1)}f(y) = H_1(y)$.

In view of (38), it follows that $\|T_2^{(1)}f\|_{p,\psi^{1-p}} = \|H_1\|_{p,\psi^{1-p}} \leq M_2 \|f\|_{p,\varphi}$, and then the operator $T_2^{(1)}$ is bounded satisfying

$$\left\| T_2^{(1)} \right\| = \sup_{f(\neq\theta)\in L_{p,\varphi}(R_+)} \frac{\|T_2^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_2.$$

If we define the formal inner product of $T_2^{(1)}f$ and g as follows:

$$\left(T_2^{(1)}f, g\right) := \int_0^\infty \left(\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)\,dx\right) g(y)\,dy,$$

then we can rewrite Theorem 2 (for $\sigma_1 = \sigma$) as follows.

Theorem 4 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > -1$, $\mu = \lambda - \sigma > -\alpha$, *then the following conditions are equivalent*:

(i) *There exists a constant* M_2, *such that, for any* $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$, $\|f\|_{p,\varphi} > 0$, *we have the following inequality*:

$$\left\| T_2^{(1)} f \right\|_{p,\psi^{1-p}} < M_2 \|f\|_{p,\varphi}. \tag{39}$$

(ii) *There exists a constant* M_2, *such that, for any* $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, *we have the following inequality*:

$$\left(T_2^{(1)} f, g \right) < M_2 \|f\|_{p,\varphi} \|g\|_{q,\psi}. \tag{40}$$

We still have $\| T_2^{(1)} \| = k_1(\mu) \leq M_2$.

(d) In view of Corollary 2 ($\mu_1 = \mu$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$H_2(y) := \int_y^\infty \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)\, dx \quad (y \in \mathbf{R}_+),$$

by (30), we have

$$\|H_2\|_{p,\phi^{1-p}} = \left[\int_0^\infty \phi^{1-p}(y) H_2^p(y)\, dy \right]^{\frac{1}{p}} < M_2 \|f\|_{p,\varphi} < \infty. \tag{41}$$

Definition 4 Define a Hardy-type integral operator of the second kind with the homogeneous kernel $T_2^{(2)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\phi^{1-p}}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_2^{(2)} f = H_2 \in L_{p,\phi^{1-p}}(\mathbf{R}_+)$, satisfying for any $y \in \mathbf{R}_+$, $T_2^{(2)} f(y) = H_2(y)$.

In view of (41), it follows that $\| T_2^{(2)} f \|_{p,\phi^{1-p}} = \|H_2\|_{p,\phi^{1-p}} \leq M_2 \|f\|_{p,\varphi}$, and then the operator $T_2^{(2)}$ is bounded satisfying

$$\left\| T_2^{(2)} \right\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(R_+)} \frac{\| T_2^{(2)} f \|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_2.$$

If we define the formal inner product of $T_1^{(2)} f$ and g as follows:

$$\left(T_2^{(2)} f, g \right) := \int_0^\infty \left(\int_y^\infty \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)\, dx \right) g(y)\, dy,$$

then we can rewrite Corollary 2 (for $\mu_1 = \mu$) as follows.

Corollary 4 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > -1$, $\mu = \lambda - \sigma > -\alpha$, *then the following conditions are equivalent*:

(i) *There exists a constant* M_2, *such that, for any* $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$, $\|f\|_{p,\varphi} > 0$, *we have the following inequality*:

$$\left\| T_2^{(2)} f \right\|_{p,\phi^{1-p}} < M_2 \|f\|_{p,\varphi}. \tag{42}$$

(ii) *There exists a constant M_2, such that, for any $f(x), g(y) \geq 0$, $f \in L_{p,\varphi}(\mathbf{R}_+)$, $g \in L_{q,\phi}(\mathbf{R}_+)$, $\|f\|_{p,\varphi}$, $\|g\|_{q,\phi} > 0$, we have the following inequality*:

$$\left(T_2^{(2)}f, g\right) < M_2\|f\|_{p,\varphi}\|g\|_{q,\phi}. \tag{43}$$

We still have $\|T_2^{(2)}\| = k_1(\mu) \leq M_2$.

Remark 5 Theorem 4 and Corollary 4 are equivalent.

5 Conclusions

In this paper, by means of real analysis and weight functions a few equivalent conditions of two kinds of Hardy-type integral inequalities with the non-homogeneous kernel and parameters are obtained by Theorem 1, 2. The constant factors related to the gamma function are proved to be the best possible. We also consider the operator expressions in Theorem 3, 4. The dependent cases of homogeneous kernel are assumed by Corollary 1-4. The method of weight functions is very important, it is the key to help us proving the main inequalities with the best possible constant factor. The lemmas provide an extensive account of this type of inequalities.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. JL participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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