Kim et al. *Journal of Inequalities and Applications* (2017) 2017:84
DOI 10.1186/s13660-017-1359-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fourier series of sums of products of ordered Bell and poly-Bernoulli functions

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Abstract
In this paper, we study three types of sums of products of ordered Bell and poly-Bernoulli functions and derive their Fourier series expansion. In addition, we express those functions in terms of Bernoulli functions.

MSC: 11B68; 11B83; 42A16

Keywords: Fourier series; ordered Bell function; poly-Bernoulli function

1 Introduction

The *ordered Bell polynomials* $b_m(x)$ are defined by the generating function

$$\frac{1}{2-e^t}e^{xt} = \sum_{m=0}^{\infty} b_m(x)\frac{t^m}{m!}. \tag{1}$$

Thus they form an Appell sequence. For $x = 0$, $b_m = b_m(0)$, $(m \geq 0)$ are called *ordered Bell numbers* which have been studied in various counting problems in number theory and enumerative combinatorics (see [1, 4, 5, 16, 17, 19]). The ordered Bell numbers are all positive integers, as we can see, for example, from

$$b_m = \sum_{n=0}^{m} n!S_2(m,n) = \sum_{n=0}^{\infty} \frac{n^m}{2^{n+1}} \quad (m \geq 0).$$

The first few ordered Bell polynomials are as follows:

$$\begin{aligned} &b_0(x) = 1, \qquad b_1(x) = x+1, \qquad b_2(x) = x^2+2x+3, \\ &b_3(x) = x^3+3x^2+9x+13, \qquad b_4(x) = x^4+4x^3+18x^2+52x+75, \\ &b_5(x) = x^5+5x^4+30x^3+130x^2+375x+541. \end{aligned}$$

From (1), we can derive

$$\begin{aligned} &\frac{d}{dx}b_m(x) = mb_{m-1}(x) \quad (m \geq 1), \\ &-b_m(x+1)+2b_m(x) = x^m \quad (m \geq 0). \end{aligned}$$

From these, in turn, we have

$$
\begin{aligned}
&-b_m(1) + 2b_m = \delta_{m,0} \quad (m \geq 0), \\
&\int_0^1 b_m(x)\,dx = \frac{1}{m+1}\bigl(b_{m+1}(1) - m_{m+1}\bigr) \\
&\qquad = \frac{1}{m+1} b_{m+1}.
\end{aligned}
$$

For any integer r, the *poly-Bernoulli polynomials of index* r $\mathbb{B}_m^{(r)}(x)$ are given by the generating function (see [2, 3, 7–10, 12, 13, 20])

$$
\frac{Li_r(1-e^{-t})}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} \mathbb{B}_m^{(r)}(x) \frac{t^m}{m!},
$$

where $Li_r(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^r}$ is the polylogarithmic function for $r \geq 1$ and a rational function for $r \leq 0$.

We note here that

$$
\frac{d}{dx}\bigl(Li_{r+1}(x)\bigr) = \frac{1}{x} Li_r(x).
$$

Also, we need the following for later use.

$$
\begin{aligned}
&\frac{d}{dx}\mathbb{B}_m^{(r)}(x) = m\mathbb{B}_{m-1}^{(r)}(x) \quad (m \geq 1), \\
&\mathbb{B}_m^{(1)}(x) = B_m(x), \qquad \mathbb{B}_0^{(r)}(x) = 1, \qquad \mathbb{B}_m^{(0)}(x) = x^m, \\
&\mathbb{B}_m^{(0)} = \delta_{m,0}, \qquad \mathbb{B}_m^{(r+1)}(1) - \mathbb{B}_m^{(r+1)}(0) = \mathbb{B}_{m-1}^{(r)} \quad (m \geq 1), \\
&\int_0^1 \mathbb{B}_m^{(r)}(x)\,dx = \frac{1}{m+1}\bigl(\mathbb{B}_{m+1}^{(r)}(1) - \mathbb{B}_{m+1}^{(r)}(0)\bigr) \\
&\qquad = \frac{1}{m+1}\mathbb{B}_m^{(r-1)}.
\end{aligned}
$$

Here the *Bernoulli polynomials* $B_m(x)$ are given by the generating function

$$
\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.
$$

For any real number x, we let

$$
\langle x \rangle = x - \lfloor x \rfloor \in [0,1)
$$

denote the fractional part of x.

Finally, we recall the following facts about Bernoulli functions $B_m(\langle x \rangle)$:

(a) for $m \geq 2$,

$$
B_m\bigl(\langle x \rangle\bigr) = -m! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m};
$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Here we will study three types of sums of products of ordered Bell and poly-Bernoulli functions and derive their Fourier series expansion. In addition, we will express those functions in terms of Bernoulli functions.

(1) $\alpha_m(\langle x\rangle) = \sum_{k=0}^{m} b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle), (m \geq 1)$;

(2) $\beta_m(\langle x\rangle) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle), (m \geq 1)$;

(3) $\gamma_m(\langle x\rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle), (m \geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [18, 21]).

As to $\gamma_m(\langle x\rangle)$, we note that the next polynomial identity follows immediately from Theorems 4.1 and 4.2, which is in turn derived from the Fourier series expansion of $\gamma_m(\langle x\rangle)$:

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(x)\mathbb{B}_{m-k}^{(r+1)}(x) \\ &\quad = \frac{1}{m}\sum_{s=0}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}\left(\mathbb{B}_{m-s}^{(r)} + b_{m-s+1}\right)\right)B_s(x), \end{aligned}$$

where $H_l = \sum_{j=1}^{l} \frac{1}{j}$ are the harmonic numbers and

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)} b_k\mathbb{B}_{l-k}^{(r+1)} + 2\sum_{k=1}^{l-1} \frac{1}{k(l-k)} b_k\mathbb{B}_{l-k-1}^{(r)},$$

with $\Lambda_1 = 0$.

The polynomial identities can be derived also for the functions $\alpha_m(\langle x\rangle)$ and $\beta_m(\langle x\rangle)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. We refer the reader to [6, 11, 14, 15] for the recent papers on related works.

2 Fourier series of functions of the first type

Let

$$\alpha_m(x) = \sum_{k=0}^{m} b_k(x)\mathbb{B}_{m-k}^{(r+1)}(x),$$

where r, m are integers with $m \geq 1$. Then we will study the function

$$\alpha_m\big(\langle x\rangle\big) = \sum_{k=0}^{m} b_k\big(\langle x\rangle\big)\mathbb{B}_{m-k}^{(r+1)}\big(\langle x\rangle\big) \quad (m \geq 1),$$

defined on $\mathbb{R}$ which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi inx},$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m\bigl(\langle x\rangle\bigr)e^{-2\pi inx}\,dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi inx}\,dx. \end{aligned}$$

Before proceeding further, we observe the following

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=0}^{m}\bigl\{kb_{k-1}(x)\mathbb{B}_{m-k}^{(r+1)}(x) + (m-k)b_k\mathbb{B}_{m-k-1}^{(r+1)}(x)\bigr\} \\ &= \sum_{k=1}^{m} kb_{k-1}(x)\mathbb{B}_{m-k}^{(r+1)}(x) + \sum_{k=0}^{m-1}(m-k)b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x) \\ &= \sum_{k=0}^{m-1}(k+1)b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x) + \sum_{k=0}^{m-1}(m-k)b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x) \\ &= (m+1)\alpha_{m-1}(x). \end{aligned}$$

Thus $(\frac{\alpha_{m+1}(x)}{m+2})' = \alpha_m(x)$, and so $\int_0^1 \alpha_m(x)\,dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0))$. For $m \geq 1$, we put

$$\begin{aligned} \Delta_m &= \alpha_m(1) - \alpha_m(0) \\ &= \sum_{k=0}^{m} b_k(1)\mathbb{B}_{m-k}^{(r+1)}(1) - \sum_{k=0}^{m} b_k\mathbb{B}_{m-k}^{(r+1)} \\ &= \sum_{k=0}^{m-1}(2b_k - \delta_{k,0})\bigl(\mathbb{B}_{m-k}^{(r+1)} + \mathbb{B}_{m-k-1}^{(r)}\bigr) + 2b_m - \delta_{m,0} - \sum_{k=0}^{m} b_k\mathbb{B}_{m-k}^{(r+1)} \\ &= 2\sum_{k=0}^{m-1} b_k\mathbb{B}_{m-k}^{(r+1)} + 2\sum_{k=0}^{m-1} b_k\mathbb{B}_{m-k-1}^{(r)} - \mathbb{B}_m^{(r+1)} - \mathbb{B}_{m-1}^{(r)} + b_m - \sum_{k=0}^{m-1} b_k\mathbb{B}_{m-k}^{(r+1)} \\ &= \sum_{k=1}^{m} b_k\mathbb{B}_{m-k}^{(r+1)} + 2\sum_{k=0}^{m-1} b_k\mathbb{B}_{m-k-1}^{(r)} - \mathbb{B}_{m-1}^{(r)}. \end{aligned}$$

Thus, $\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0$ and $\int_0^1 \alpha_m(x)\,dx = \frac{1}{m+2}\Delta_{m+1}$.

Now, we want to determine the Fourier coefficients $A_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\bigl[\alpha_m(x)e^{-2\pi inx}\bigr]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}\,dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}\,dx - \frac{1}{2\pi in}\bigl(\alpha_m(1)-\alpha_m(0)\bigr)\\
&= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m\\
&= \frac{m+1}{2\pi in}\left(\frac{m}{2\pi in}A_n^{(m-2)} - \frac{1}{2\pi in}\Delta_{m-1}\right) - \frac{1}{2\pi in}\Delta_m\\
&= \frac{(m+1)_2}{(2\pi in)^2}A_n^{(m-2)} - \sum_{j=1}^{2}\frac{(m+1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1}\\
&= \cdots\\
&= \frac{(m+1)_m}{(2\pi in)^m}A_n^{(0)} - \sum_{j=1}^{m}\frac{(m+1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1}\\
&= -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1},
\end{aligned}
$$

where we note that $A_n^{(0)} = \int_0^1 e^{-2\pi inx}\,dx = 0$.

Case 2: $n = 0$.

$$
A_0^{(m)} = \int_0^1 \alpha_m(x)\,dx = \frac{1}{m+2}\Delta_{m+1}.
$$

$\alpha_m(\langle x\rangle)$, $(m \geq 1)$ is piecewise C^∞. Moreover, $\alpha_m(\langle x\rangle)$ is continuous for those positive integers m with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Delta_m \neq 0$.

Assume first that m is a positive integer with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise C^∞ and continuous. Thus the Fourier series of $\alpha_m(\langle x\rangle)$ converges uniformly to $\alpha_m(\langle x\rangle)$, and

$$
\begin{aligned}
\alpha_m\bigl(\langle x\rangle\bigr) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr)\\
&\quad + \Delta_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}
$$

Now, we can state our first theorem.

Theorem 2.1 *For each positive integer l, we let*

$$
\Delta_l = \sum_{k=1}^{l} b_k\mathbb{B}_{l-k}^{(r+1)} + 2\sum_{k=0}^{l-1} b_k\mathbb{B}_{l-k-1}^{(r)} - \mathbb{B}_{l-1}^{(r)}.
$$

Assume that $\Delta_m = 0$ for a positive integer m. Then we have the following

(a) $\sum_{k=0}^{m} b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle)$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m} b_k\bigl(\langle x\rangle\bigr)\mathbb{B}_{m-k}^{(r+1)}\bigl(\langle x\rangle\bigr)\\&\quad=\frac{1}{m+2}\Delta_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},\end{aligned}$$

for all $x\in\mathbb{R}$, *where the convergence is uniform.*

(b)

$$\sum_{k=0}^{m} b_k\bigl(\langle x\rangle\bigr)\mathbb{B}_{m-k}^{(r+1)}\bigl(\langle x\rangle\bigr)=\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr),$$

for all $x\in\mathbb{R}$, *where* $B_j(\langle x\rangle)$ *is the Bernoulli function.*

Assume next that $\Delta_m\neq 0$ for a positive integer m. Then $\alpha_m(0)\neq\alpha_m(1)$. So $\alpha_m(\langle x\rangle)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(\langle x\rangle)$ converges pointwise to $\alpha_m(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\bigl(\alpha_m(0)+\alpha_m(1)\bigr)=\alpha_m(0)+\frac{1}{2}\Delta_m,$$

for $x\in\mathbb{Z}$.

Now, we can state our second theorem.

Theorem 2.2 *For each positive integer* l, *we let*

$$\Delta_l=\sum_{k=1}^{l} b_k\mathbb{B}_{l-k}^{(r+1)}+2\sum_{k=0}^{l-1} b_k\mathbb{B}_{l-k-1}^{(r)}-\mathbb{B}_{l-1}^{(r)}.$$

Assume that $\Delta_m\neq 0$ *for a positive integer* m. *Then we have the following.*

(a)

$$\begin{aligned}&\frac{1}{m+2}\Delta_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\&\quad=\begin{cases}\sum_{k=0}^{m} b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m} b_k\mathbb{B}_{m-k}^{(r+1)}+\frac{1}{2}\Delta_m, & \text{for } x\in\mathbb{Z};\end{cases}\end{aligned}$$

(b)

$$\begin{aligned}&\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr)\\&\quad=\sum_{k=0}^{m} b_k\bigl(\langle x\rangle\bigr)\mathbb{B}_{m-k}^{(r+1)}\bigl(\langle x\rangle\bigr),\quad \text{for } x\notin\mathbb{Z};\end{aligned}$$

$$\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle)$$
$$=\sum_{k=0}^{m}b_k\mathbb{B}_{m-k}^{(r+1)}+\frac{1}{2}\Delta_m,\quad \textit{for } x\in\mathbb{Z}.$$

3 Fourier series of functions of the second type

Let $\beta_m(x)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(x)\mathbb{B}_{m-k}^{(r+1)}(x)$, where r,m are integers with $m\geq 1$. Then we will investigate the function

$$\beta_m(\langle x\rangle)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},$$

where

$$B_n^{(m)}=\int_0^1\beta_m(\langle x\rangle)e^{-2\pi inx}\,dx$$
$$=\int_0^1\beta_m(x)e^{-2\pi inx}\,dx.$$

Before proceeding further, we note the following.

$$\beta_m'(x)=\sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}b_{k-1}(x)\mathbb{B}_{m-k}^{(r+1)}(x)+\frac{m-k}{k!(m-k)!}b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x)\right\}$$
$$=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}b_{k-1}(x)\mathbb{B}_{m-k}^{(r+1)}(x)+\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x)$$
$$=\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x)+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x)$$
$$=2\beta_{m-1}(x).$$

Thus

$$\left(\frac{\beta_{m+1}(x)}{2}\right)'=\beta_m(x),\quad\text{and}\quad\int_0^1\beta_m(x)\,dx=\frac{1}{2}\big(\beta_{m+1}(1)-\beta_{m+1}(0)\big).$$

For $m\geq 1$, we put

$$\Omega_m=\beta_m(1)-\beta_m(0)$$
$$=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(1)\mathbb{B}_{m-k}^{(r+1)}(1)-\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k\mathbb{B}_{m-k}^{(r+1)}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!}(2b_k - \delta_{k,0})\big(\mathbb{B}_{m-k}^{(r+1)} + \mathbb{B}_{m-k-1}^{(r)}\big) \\
&\quad + \frac{1}{m!}(2b_m - \delta_{m,0}) - \sum_{k=0}^{m} \frac{1}{k!(m-k)!} b_k \mathbb{B}_{m-k}^{(r+1)} \\
&= 2\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k \mathbb{B}_{m-k}^{(r+1)} + 2\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k \mathbb{B}_{m-k-1}^{(r)} \\
&\quad - \frac{1}{m!}\mathbb{B}_{m}^{(r+1)} - \frac{1}{m!}\mathbb{B}_{m-1}^{(r)} + \frac{1}{m!} b_m - \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k \mathbb{B}_{m-k}^{(r+1)} \\
&= \sum_{k=1}^{m} \frac{1}{k!(m-k)!} b_k \mathbb{B}_{m-k}^{(r+1)} + 2\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} b_k \mathbb{B}_{m-k-1}^{(r)} - \frac{1}{m!}\mathbb{B}_{m-1}^{(r)}.
\end{aligned}
$$

Hence

$$
\beta_m(0) = \beta_m(1) \quad \Longleftrightarrow \quad \Omega_m = 0, \quad \text{and} \quad \int_0^1 \beta_m(x)\,dx = \frac{1}{2}\Omega_{m+1}.
$$

We now would like to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$
\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x}\,dx \\
&= -\frac{1}{2\pi i n}\big[\beta_m(x) e^{-2\pi i n x}\big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x) e^{-2\pi i n x}\,dx \\
&= -\frac{1}{2\pi i n}\big(\beta_m(1) - \beta_m(0)\big) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x}\,dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m \\
&= \frac{2}{2\pi i n}\left(\frac{2}{2\pi i n} B_n^{(m-2)} - \frac{1}{2\pi i n}\Omega_{m-1}\right) - \frac{1}{2\pi i n}\Omega_m \\
&= \left(\frac{2}{2\pi i n}\right)^2 B_n^{(m-2)} - \sum_{j=1}^{2} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \\
&= \cdots \\
&= \left(\frac{2}{2\pi i n}\right)^m B_n^{(0)} - \sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \\
&= -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}.
\end{aligned}
$$

Case 2: $n = 0$.

$$
B_0^{(m)} = \int_0^1 \beta_m(x)\,dx = \frac{1}{2}\Omega_{m+1}.
$$

$\beta_m(\langle x\rangle)$, $(m \geq 1)$ is piecewise C^∞. Moreover, $\beta_m(\langle x\rangle)$ is continuous for those positive integers m with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

Assume first that m is a positive integer with $\Omega_m = 0$. Then $\beta_m(0) = \beta_m(1)$. Thus $\beta_m(\langle x\rangle)$ is piecewise C^∞ and continuous. Hence the Fourier series of $\beta_m(\langle x\rangle)$ converges uniformly to $\beta_m(\langle x\rangle)$, and

$$\begin{aligned}
\beta_m(\langle x\rangle) &= \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx} \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle) \\
&\quad + \Omega_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now, we can state our first result.

Theorem 3.1 *For each positive integer l, we let*

$$\Omega_l = \sum_{k=1}^{l}\frac{1}{k!(l-k)!}b_k\mathbb{B}_{l-k}^{(r+1)} + 2\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}b_k\mathbb{B}_{l-k-1}^{(r)} - \frac{1}{l!}\mathbb{B}_{l-1}^{(r)}.$$

Assume that $\Omega_m = 0$ for a positive integer m. Then we have the following.

(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle) \\
&\quad = \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx},
\end{aligned}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle) = \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle),$$

for all $x \in \mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.

Assume next that $\Omega_m \neq 0$ for a positive integer m. Then $\beta_m(0) \neq \beta_m(1)$. So $\beta_m(\langle x\rangle)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\big(\beta_m(0)+\beta_m(1)\big)=\beta_m(0)+\frac{1}{2}\Omega_m,$$

for $x \in \mathbb{Z}$.

Now, we can state our second theorem.

Theorem 3.2 *For each positive integer l, we let*

$$\Omega_l=\sum_{k=1}^{l}\frac{1}{k!(l-k)!}b_k\mathbb{B}_{l-k}^{(r+1)}+2\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}b_k\mathbb{B}_{l-k-1}^{(r)}-\frac{1}{l!}\mathbb{B}_{l-1}^{(r)}.$$

Assume that $\Omega_m \neq 0$ for a positive integer m. Then we have the following.

(a)

$$\frac{1}{2}\Omega_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle), & \textit{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k\mathbb{B}_{m-k}^{(r+1)}+\frac{1}{2}\Omega_m, & \textit{for } x\in\mathbb{Z}.\end{cases}$$

(b)

$$\frac{1}{2}\Omega_{m+1}+\sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j\big(\langle x\rangle\big)$$
$$=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k\big(\langle x\rangle\big)\mathbb{B}_{m-k}^{(r+1)}\big(\langle x\rangle\big),\quad \textit{for } x\notin\mathbb{Z};$$
$$\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j\big(\langle x\rangle\big)$$
$$=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}b_k\mathbb{B}_{m-k}^{(r+1)}+\frac{1}{2}\Omega_m,\quad \textit{for } x\in\mathbb{Z}.$$

4 Fourier series of functions of the third type

Let

$$\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(x)\mathbb{B}_{m-k}^{(r+1)}(x),$$

where r, m are integers with $m \geq 2$. Then we will consider the function

$$\gamma_m\big(\langle x\rangle\big)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k\big(\langle x\rangle\big)\mathbb{B}_{m-k}^{(r+1)}\big(\langle x\rangle\big),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi inx},$$

where

$$\begin{aligned} C_n^{(m)} &= \int_0^1 \gamma_m\bigl(\langle x\rangle\bigr)e^{-2\pi inx}\,dx \\ &= \int_0^1 \gamma_m(x)e^{-2\pi inx}\,dx. \end{aligned}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned} \gamma_m'(x) &= \sum_{k=1}^{m-1}\frac{1}{m-k}b_{k-1}(x)\mathbb{B}_{m-k}^{(r+1)}(x) + \sum_{k=1}^{m-1}\frac{1}{k}b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x) \\ &= \sum_{k=0}^{m-2}\frac{1}{m-k-1}b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x) + \sum_{k=1}^{m-1}\frac{1}{k}b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x) \\ &= \sum_{k=1}^{m-2}\left(\frac{1}{m-k-1}+\frac{1}{k}\right)b_k(x)\mathbb{B}_{m-k-1}^{(r+1)}(x) + \frac{1}{m-1}\mathbb{B}_{m-1}^{(r+1)}(x) + \frac{1}{m-1}b_{m-1}(x) \\ &= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1}\mathbb{B}_{m-1}^{(r+1)}(x) + \frac{1}{m-1}b_{m-1}(x). \end{aligned}$$

From this, we see that

$$\left(\frac{1}{m}\left(\gamma_{m+1}(x) - \frac{1}{m(m+1)}\mathbb{B}_{m+1}^{(r+1)}(x) - \frac{1}{m(m+1)}b_{m+1}(x)\right)\right)' = \gamma_m(x),$$

and

$$\begin{aligned} &\int_0^1 \gamma_m(x)\,dx \\ &\quad= \frac{1}{m}\left[\gamma_{m+1}(x) - \frac{1}{m(m+1)}\mathbb{B}_{m+1}^{(r+1)}(x) - \frac{1}{m(m+1)}b_{m+1}(x)\right]_0^1 \\ &\quad= \frac{1}{m}\biggl(\gamma_{m+1}(1) - \gamma_m(0) - \frac{1}{m(m+1)}\bigl(\mathbb{B}_{m+1}^{(r+1)}(1) - \mathbb{B}_{m+1}^{(r+1)}(0)\bigr) \\ &\qquad - \frac{1}{m(m+1)}\bigl(b_{m+1}(1) - b_{m+1}(0)\bigr)\biggr) \\ &\quad= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}b_{m+1}\right). \end{aligned}$$

For $m \geq 2$, we let

$$\begin{aligned} \Lambda_m &= \gamma_m(1) - \gamma_m(0) \\ &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\bigl(b_k(1)\mathbb{B}_{m-k}^{(r+1)}(1) - b_k\mathbb{B}_{m-k}^{(r+1)}\bigr) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \big((2b_k - \delta_{k,0}) \big(\mathbb{B}_{m-k}^{(r+1)} + \mathbb{B}_{m-k-1}^{(r)} \big) - b_k \mathbb{B}_{m-k}^{(r+1)} \big) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k \mathbb{B}_{m-k}^{(r+1)} + 2 \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k \mathbb{B}_{m-k-1}^{(r)}.
\end{aligned}
$$

Then

$$
\gamma_m(0) = \gamma_m(1) \quad \Longleftrightarrow \quad \Lambda_m = 0,
$$

and

$$
\int_0^1 \gamma_m(x)\,dx = \frac{1}{m} \left(\Lambda_{m+1} - \frac{1}{m(m+1)} \mathbb{B}_m^{(r)} - \frac{1}{m(m+1)} b_{m+1} \right).
$$

Now, we would like to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$. For this computation, we need to know the following:

$$
\int_0^1 \mathbb{B}_l^{(r+1)}(x) e^{-2\pi i n x}\,dx = \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi i n)^k} \mathbb{B}_{l-k}^{(r)}, & \text{for } n \neq 0, \\ \frac{1}{l+1} \mathbb{B}_l^{(r)}, & \text{for } n = 0, \end{cases}
$$

$$
\int_0^1 b_l(x) e^{-2\pi i n x}\,dx = \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi i n)^k} b_{l-k+1}, & \text{for } n \neq 0, \\ \frac{1}{l+1} b_{l+1}, & \text{for } n = 0, \end{cases}
$$

$$
\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x}\,dx \\
&= -\frac{1}{2\pi i n} \big[\gamma_m(x) e^{-2\pi i n x} \big]_0^1 + \frac{1}{2\pi i n} \int_0^1 \gamma_m'(x) e^{-2\pi i n x}\,dx \\
&= -\frac{1}{2\pi i n} \big(\gamma_m(1) - \gamma_m(0) \big) \\
&\quad + \frac{1}{2\pi i n} \int_0^1 \left((m-1)\gamma_{m-1}(x) + \frac{1}{m-1} \mathbb{B}_{m-1}^{(r+1)}(x) + \frac{1}{m-1} b_{m-1}(x) \right) e^{-2\pi i n x}\,dx \\
&= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n} \Lambda_m + \frac{1}{2\pi i n(m-1)} \int_0^1 \mathbb{B}_{m-1}^{(r+1)}(x) e^{-2\pi i n x}\,dx \\
&\quad + \frac{1}{2\pi i n(m-1)} \int_0^1 b_{m-1}(x) e^{-2\pi i n x}\,dx \\
&= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n} \Lambda_m - \frac{1}{2\pi i n(m-1)} \Theta_m - \frac{1}{2\pi i n(m-1)} \Phi_m,
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k \mathbb{B}_{m-k}^{(r+1)} + 2 \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k \mathbb{B}_{m-k-1}^{(r)}, \\
\Theta_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \mathbb{B}_{m-k-1}^{(r)}, \qquad \Phi_m = \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} b_{m-k}.
\end{aligned}
$$

$$\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m - \frac{1}{2\pi in(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} - \frac{1}{2\pi in(m-2)}\Theta_{m-1} - \frac{1}{2\pi in(m-2)}\Phi_{m-1}\right) \\
&\quad - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m - \frac{1}{2\pi in(m-1)}\Phi_m \\
&= \frac{(m-1)_2}{(2\pi in)^2}C_n^{(m-2)} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} \\
&\quad - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= \cdots \\
&= -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1}.
\end{aligned}$$

We note here that

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&\quad= \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j}\frac{(m-j)_{k-1}}{(2\pi in)^k}b_{m-j-k+1} \\
&\quad= \sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}(m-j)}b_{m-j-k+1} \\
&\quad= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{k=1}^{m-j}\frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}}b_{m-j-k+1} \\
&\quad= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{s=j+1}^{m}\frac{(m-1)_{s-2}}{(2\pi in)^s}b_{m-s+1} \\
&\quad= \sum_{s=2}^{m}\frac{(m-1)_{s-2}}{(2\pi in)^s}b_{m-s+1}\sum_{j=1}^{s-1}\frac{1}{m-j} \\
&\quad= \sum_{s=1}^{m}\frac{(m-1)_{s-2}}{(2\pi in)^s}b_{m-s+1}(H_{m-1}-H_{m-s}) \\
&\quad= \frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}b_{m-s+1}.
\end{aligned}$$

Putting everything together, we get

$$C_n^{(m)} = -\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left\{\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\left(\mathbb{B}_{m-s}^{(r)} + b_{m-s+1}\right)\right\}.$$

Case 2: $n = 0$.

$$\begin{aligned} C_0^{(m)} &= \int_0^1 \gamma_m(x)\,dx \\ &= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}b_{m+1}\right). \end{aligned}$$

$\gamma_m(\langle x\rangle)$, $m \geq 2$ is piecewise C^∞. Moreover, $\gamma_m(\langle x\rangle)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. Thus $\gamma_m(\langle x\rangle)$ is piecewise C^∞ and continuous. Hence the Fourier series of $\gamma_m(\langle x\rangle)$ converges uniformly to $\gamma_m(\langle x\rangle)$, and

$$\begin{aligned} \gamma_m\bigl(\langle x\rangle\bigr) &= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}b_{m+1}\right) \\ &\quad + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\bigl(\mathbb{B}_{m-s}^{(r)} + b_{m-s+1}\bigr)\right)\right\}e^{2\pi inx} \\ &= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}b_{m+1}\right) \\ &\quad + \frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\bigl(\mathbb{B}_{m-s}^{(r)} + b_{m-s+1}\bigr)\right)\left(-s!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right) \\ &= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbb{B}_m^{(r)} - \frac{1}{m(m+1)}b_{m+1}\right) \\ &\quad + \frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\bigl(\mathbb{B}_{m-s}^{(r)} + b_{m-s+1}\bigr)\right)B_s\bigl(\langle x\rangle\bigr) \\ &\quad + \Lambda_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases} \\ &= \frac{1}{m}\sum_{\substack{s=0 \\ s\neq 1}}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\bigl(\mathbb{B}_{m-s}^{(r)} + b_{m-s+1}\bigr)\right)B_s\bigl(\langle x\rangle\bigr) \\ &\quad + \Lambda_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned}$$

Now, we can state our first result.

Theorem 4.1 *For each integer $l \geq 2$, we let*

$$\Lambda_l = \sum_{k=1}^{l-1}\frac{1}{k(l-k)}b_k\mathbb{B}_{l-k}^{(r+1)} + 2\sum_{k=1}^{l-1}\frac{1}{k(l-k)}b_k\mathbb{B}_{l-k-1}^{(r)},$$

with $\Lambda_1 = 0$. Assume that $\Lambda_m = 0$ for an integer $m \geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k(\langle x\rangle)\mathbb{B}_{m-k}^{(r+1)}(\langle x\rangle)$ *has Fourier series expansion*

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k\bigl(\langle x\rangle\bigr)\mathbb{B}_{m-k}^{(r+1)}\bigl(\langle x\rangle\bigr)\\&\quad=\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\mathbb{B}_m^{(r)}-\frac{1}{m(m+1)}b_{m+1}\right)\\&\qquad+\sum_{\substack{n=-\infty\\n\neq0}}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}\bigl(\mathbb{B}_{m-s}^{(r)}+b_{m-s+1}\bigr)\right)\right\}e^{2\pi inx},\end{aligned}$$

for all $x\in\mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k\bigl(\langle x\rangle\bigr)\mathbb{B}_{m-k}^{(r+1)}\bigl(\langle x\rangle\bigr)\\&\quad=\frac{1}{m}\sum_{\substack{s=0\\s\neq1}}^{m}\binom{m}{s}\left(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}\bigl(\mathbb{B}_{m-s}^{(r)}+b_{m-s+1}\bigr)\right)B_s\bigl(\langle x\rangle\bigr),\end{aligned}$$

for all $x\in\mathbb{R}$, where $B_s(\langle x\rangle)$ is the Bernoulli function.

Assume next that m is an integer ≥ 2 with $\Lambda_m\neq 0$. Then $\gamma_m(0)\neq\gamma_m(1)$. Hence $\gamma_m(\langle x\rangle)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Then the Fourier series of $\gamma_m(\langle x\rangle)$ converges pointwise to $\gamma_m(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\begin{aligned}\frac{1}{2}\bigl(\gamma_m(0)+\gamma_m(1)\bigr)&=\gamma_m(0)+\frac{1}{2}\Lambda_m\\&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}b_k\mathbb{B}_{m-k}^{(r+1)}+\frac{1}{2}\Lambda_m,\end{aligned}$$

for $x\in\mathbb{Z}$.

Now, we can state our second result.

Theorem 4.2 *For each integer $l\geq 2$, let*

$$\Lambda_l=\sum_{k=1}^{l-1}\frac{1}{k(l-k)}b_k\mathbb{B}_{l-k}^{(r+1)}+2\sum_{k=1}^{l-1}\frac{1}{k(l-k)}b_k\mathbb{B}_{l-k-1}^{(r)},$$

with $\Lambda_1=0$. Assume that $\Lambda_m\neq 0$ for an integer $m\geq 2$. Then we have the following.

(a)

$$\begin{aligned}&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\mathbb{B}_m^{(r)}-\frac{1}{m(m+1)}b_{m+1}\right)\\&\quad+\sum_{\substack{n=-\infty\\n\neq0}}^{\infty}\left\{-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}\bigl(\mathbb{B}_{m-s}^{(r)}+b_{m-s+1}\bigr)\right)\right\}e^{2\pi inx}\end{aligned}$$

$$= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(\langle x \rangle) \mathbb{B}_{m-k}^{(r+1)}(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k \mathbb{B}_{m-k}^{(r+1)} + \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m}\sum_{s=0}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\left(\mathbb{B}_{m-s}^{(r)} + b_{m-s+1}\right)\right)B_s(\langle x \rangle)$$

$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k(\langle x \rangle) \mathbb{B}_{m-k}^{(r+1)}(\langle x \rangle), \quad \text{for } x \notin \mathbb{Z};$$

$$\frac{1}{m}\sum_{\substack{s=0 \\ s\neq 1}}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}\left(\mathbb{B}_{m-s}^{(r)} + b_{m-s+1}\right)\right)B_s(\langle x \rangle)$$

$$= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} b_k \mathbb{B}_{m-k}^{(r+1)} + \frac{1}{2}\Lambda_m, \quad \text{for } x \in \mathbb{Z}.$$

5 Results and discussion

In this paper, we study three types of sums of products of ordered Bell and poly-Bernoulli functions and derive their Fourier series expansion. In addition, we express those functions in terms of Bernoulli functions. The Fourier series expansion of the ordered Bell and poly-Bernoulli functions are useful in computing the special values of the poly-zeta and multiple zeta function. For details, one is referred to [3, 7–18]. It is expected that the Fourier series of the ordered Bell functions will find some applications in connection with a certain generalization of the Euler zeta function and the higher-order generalized Frobenius-Euler numbers and polynomials.

6 Conclusion

In this paper, we considered the Fourier series expansion of the ordered Bell and poly-Bernoulli functions which are obtained by extending by periodicity of period 1 the ordered Bell and poly-Bernoulli polynomials on $[0,1)$. The Fourier series are explicitly determined.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Acknowledgements

The first author has been appointed a chair professor at Tianjin Polytechnic University by Tianjin City in China from August 2015 to August 2019.

Publisher's Note

Received: 26 February 2017 Accepted: 7 April 2017 Published online: 28 April 2017

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