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On the Laplacian spectral radii of Halin graphs

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Abstract

Let T be a tree with at least four vertices, none of which has degree 2, embedded in the plane. A Halin graph is a plane graph constructed by connecting the leaves of T into a cycle. Thus the cycle C forms the outer face of the Halin graph, with the tree inside it. Let G be a Halin graph with order n. Denote by $\mu(G)$ the Laplacian spectral radius of G. This paper determines all the Halin graphs with $\mu(G) \geq n - 4$. Moreover, we obtain the graphs with the first three largest Laplacian spectral radius among all the Halin graphs on n vertices.

MSC: 05C50; 05C12

Keywords: Halin graphs; Laplacian spectral radius
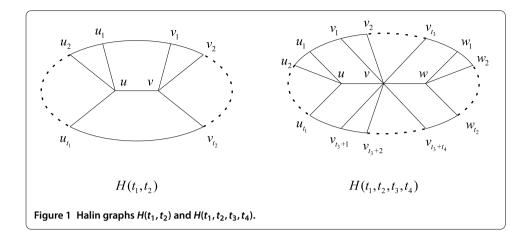
1 Introduction

In this paper, we consider simple and undirected connected graphs. Let $G = G(V, E)$ be a simple graph with n vertices and m edges. Let $N_G(v)$ be the set of vertices adjacent to v in G and $d(v) = |N_G(v)|$ be the degree of v. As usual, we denote by Δ and δ the maximum and minimum degree of G, respectively. Denote by $G[S]$ the induced subgraph of G. Let $G - v$ be the graph obtained from G by deleting the vertex $v \in V(G)$. Similarly, $G - e$ denote the graph obtained from G by deleting an edge $e \in G$. Let G_1 and G_2 be two vertex disjoint graphs. The graph $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The join of graphs G_1 and G_2 is the graph $G_1 \vee G_2$ obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2. As usual, we denote by P_n, C_n and K_n the path, cycle and complete graph on n vertices, respectively.

A *Halin graph* is a plane graph constructed as follows. Let T be a tree on at least four vertices. All vertices of T have degree 1 or at least 3. The vertices with degree 1 are called *leaves*. Let C be a cycle connecting the leaves of T in such a way that C forms the boundary of the unbounded face. We always say the tree T is the characteristic tree of G and the cycle C is the primary cycle. Moreover, the vertices of C are called *exterior* vertices and the other vertices are called *interior* vertices. The Halin graphs was introduced by Halin [1]. We call $K_1 \vee C_{n-1}$ the *wheel graph*, denoted by W_n. Clearly, W_n is the unique Halin graph with only one interior vertex. In particular, we use $H(t_1, t_2)$ and $H(t_1, t_2, t_3, t_4)$ to denote the Halin graphs with two interior vertices and three interior vertices, respectively (see Figure 1).

For a graph G, we assume $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of G and $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degree. Let $A(G)$ be the adjacency ma-

Figure 1 Halin graphs $H(t_1, t_2)$ and $H(t_1, t_2, t_3, t_4)$.

trix. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$. Obviously, $L(G)$ is a positive semi-definite symmetric matrix, and its eigenvalues are denoted by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$. Moreover, $\mu(G) = \mu_1(G)$ is called the *Laplacian spectral radius* of G. Let G^c be the complement graph of G. It is well known that

$$\mu_i(G^c) = n - \mu_{n-i}(G) \quad \text{for } i = 1, 2, \ldots, n-1.$$

Consequently, we obtain a trivial upper bound of the Laplacian spectral radius: $\mu(G) \leq n$. Let G be a Halin graph on n vertices, $\mu(G) \geq \Delta(G) + 1 \geq 4$, the equality holds if and only if $G \cong W_4$.

The Laplacian eigenvalues of G can be used in several physical and chemical problems. Many researchers pay attention to the Laplacian spectra of graphs (see [2–11]). Halin graph is very important in the mathematical literature. In this paper we study the Laplacian spectral radii of Halin graphs. The following are our main results.

Theorem 1.1 *Let G be a Halin graph on n vertices.*
  (i) $n \geq \mu(G) > n - 1$ *if and only if $G = W_n$.*
  (ii) $n - 1 \geq \mu(G) > n - 2$ *if and only if $G = H(n - 4, 2)$.*
  (iii) $n - 2 \geq \mu(G) > n - 3$ *if and only if $G \in \{H(n - 5, 3), H(2, 2, 1, 0)\}$.*
  (iv) $n - 3 \geq \mu(G) \geq n - 4$ *if and only if $G \in \{H(n - 6, 4), H(3, 2, 1, 1), H(n - 6, 2, 1, 0), H(2, 2, t_3, t_4)\}$ where $t_3 + t_4 \geq 2$.*
  (v) *If $G \notin \{W_n, H(n - 4, 2), H(n - 5, 3)\}$, $\mu(W_n) > \mu(H(n - 4, 2)) > \mu(H(n - 5, 3)) > \mu(G)$.*

2 Preliminaries
In order to prove the theorem, we present some lemmas which will be used frequently in the proof.

Lemma 2.1 ([7]) *Let G be a connected graph on n vertices with at least one edge. Then $\mu(G) \geq \Delta(G) + 1$ with equality holding if and only if $\Delta(G) = n - 1$.*

Lemma 2.2 ([12]) *Let G be a graph and $q(G)$ be the signless Laplacian spectral radius. Then $\mu(G) \leq q(G)$. Moreover, if G is connected, then the equality holds if and only if G is a bipartite graph.*

Lemma 2.3 ([13]) *Let G be a simple connected graph with n vertices and degree sequence* $d_1 \geq d_2 \geq \cdots \geq d_n$. *Then*

$$q(G) \leq \min_{1 \leq i \leq n} \left\{ \frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_1 + 1)^2 + 8(i-1)(d_1 - d_i)}}{2} \right\}.$$

Lemma 2.4 ([2]) *Let G be a connected graph. Then*

$$\mu(G) \leq \max\{d(u) + d(v) \mid uv \in E(G)\}.$$

Moreover, the equality holds if and only if G is a regular bipartite graph or a semiregular bipartite graph.

For a graph G, we denote by $m(v)$ the average of degrees of the vertices adjacent to v, that is,

$$m(v) = \frac{\sum_{u \in N(v)} d(u)}{d(v)}.$$

As usual, $d(v)m(v)$ is called the 2-*degree* of vertex v.

Lemma 2.5 ([6, 8]) *Let G be a simple graph. Then*

$$\mu(G) \leq \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} \,\middle|\, uv \in E(G) \right\}.$$

If G is connected, then equality holds if and only if G is a regular bipartite graph or a semiregular bipartite graph.

Lemma 2.6 ([14]) *Let G be a Halin graph with k interior vertices. Then* $|E(G)| = 2n - k - 1$ *and* $n \geq 2k + 2$.

First, we discuss the Halin graphs with at least four interior vertices.

Lemma 2.7 *Let G be a Halin graph with k interior vertices. If* $k \geq 4$, *then* $\mu(G) < n - 4$.

Proof Let G be a Halin graph with the primary cycle C. It follows from Lemma 2.6 that $n \geq 2k + 2 \geq 10$. Consider any edge $uv \in E(G)$. If $u, v \in V(C)$, then $d(u) + d(v) = 6 \leq n - 4$. If $u \in V(C)$ and $v \notin V(C)$. Suppose that $d(v) = t + 1$. Note that $t + 1 + 3(k-1) - 2(k-1) = t + k \geq t + 4$, there are at least $t + 4$ vertices in C. Then $n - k = |V(C)| \geq t + 4$, and thus $d(u) + d(v) = t + 1 + 3 \leq n - k \leq n - 4$. If $u, v \notin V(C)$, and let $d(u) = t_1 + 1$ and $d(v) = t_2 + 1$. Similarly, $t_1 + 1 + t_2 + 1 + 3(k-2) - 2(k-1) = t_1 + t_2 + k - 2 \geq t_1 + t_2 + 2$, so there are at least $t_1 + t_2 + 2$ vertices in C. Then $n - k = |V(C)| \geq t_1 + t_2 + 2$, and therefore $d(u) + d(v) = t_1 + 1 + t_2 + 1 \leq n - k \leq n - 4$. In each case, we always have $d(u) + d(v) \leq n - 4$. It follows from Lemma 2.4 that $\mu(G) < n - 4$. □

Next, we consider the Halin graphs with three interior vertices. Let $G = H(t_1, t_2, t_3, t_4)$ and $t_1 \geq t_2 \geq 2$. Let u, v and w be three interior vertices. For simplicity, we may take $t = t_3 + t_4 \geq 1$. It is clear that $d(u) = t_1 + 1$, $d(v) = t + 2$, $d(w) = t_2 + 1$ and $n = t + t_1 + t_2 + 3$.

Lemma 2.8 *Let* $G = H(t_1, t_2, t_3, t_4)$ *be a Halin graph with* $t_1 \geq t_2 \geq 3$. *Then* $\mu(G) < n - 4$.

Proof For $t_2 \geq 4$, it can easily be seen that $n \geq 12$ and

$$
\begin{cases}
t = n - (t_1 + t_2 + 3) \leq n - 11, \\
t_1 = n - (t + t_2 + 3) \leq n - 8, \\
t_2 = n - (t + t_1 + 3) \leq n - 8.
\end{cases}
$$

Consider all types of edges in G. Let $u' \in N(u) \cap V(C)$, $v' \in N(v) \cap V(C)$ and $w' \in N(w) \cap V(C)$. It is obvious that $d(u') = d(v') = d(w') = 3$. Then it follows that

$$d(u) + d(v) = t_1 + t + 3 = n - t_2 \leq n - 4;$$

$$d(v) + d(w) = t_2 + t + 3 = n - t_1 \leq n - 4;$$

$$d(u) + d(u') = t_1 + 1 + 3 = t_1 + 4 \leq n - 8 + 4 = n - 4;$$

$$d(w) + d(w') = t_2 + 1 + 3 = t_2 + 4 \leq n - 8 + 4 = n - 4;$$

$$d(v) + d(v') = t + 2 + 3 \leq n - 11 + 5 = n - 6.$$

If xy is an edge in C, then $d(x) + d(y) = 6 \leq n - 4$. Consequently, we have $d(x) + d(y) \leq n - 4$ for each edge $xy \in E(G)$. Then it follows from Lemma 2.4 that $\mu(G) < n - 4$ in this case.

If $t_2 = 3$, then $t_1 \geq t_2 = 3$ and $n = t + t_1 + t_2 + 3 \geq t_1 + 7$. In this case, we use the bound in Lemma 2.5 to prove the result. Let $u' \in N(u) \cap V(C)$, $v' \in N(v) \cap V(C)$ and $w' \in N(w) \cap V(C)$. Note that

$$
\begin{cases}
d(u) = t_1 + 1, \\
d(v) = t + 2 = n - t_1 - 4, \\
d(w) = 4, \\
d(u') = d(v') = d(w') = 3.
\end{cases}
$$

The 2-degree of each vertex is as follows:

$$
\begin{cases}
d(u)m(u) = n + 2t_1 - 4, \\
d(v)m(v) = 3n - 2t_1 - 13, \\
d(w)m(w) = n - t_1 + 5, \\
d(u')m(u') = t_1 + 7, \\
d(v')m(v') = n - t_1 + 2, \\
d(w')m(w') = 10.
\end{cases}
$$

For all types of edges in G, consider the index in Lemma 2.5. Let $e = xy$ be an edge of G. Put

$$f(e) = f(xy) = \frac{d(x)(d(x) + m(x)) + d(y)(d(y) + m(y))}{d(x) + d(y)}.$$

For simplicity, we use type $u'u'$ to denote the edges $u_i u_j \in E(G)$ where $u_i, u_j \in N(u) \cap V(C)$. Similarly, we define the symbol $v'v'$ and $w'w'$. Then we will prove that the inequality $f(e) \leq n - 4$ holds. Note that each edge of G belongs to the one below (the types $u'w'$ and $v'v'$ may not exist).

- uv:

$$f(uv) = \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)}$$

$$= \frac{(t_1 + 1)^2 + n + 2t_1 - 4 + (n - t_1 - 4)^2 + 3n - 2t_1 - 13}{n - 3}.$$

Then $f(uv) \leq n - 4$ if and only if $(2t_1 - 3)n \geq 2t_1^2 + 10t_1 - 12$. Since $n \geq t_1 + 7$, it is easy to verify that $(2t_1 - 3)n \geq (2t_1 - 3)(t_1 + 7) \geq 2t_1^2 + 10t_1 - 12$ when $t_1 \geq 9$. So we have $f(uv) \leq n - 4$ when $t_1 \geq 9$.

If $t_1 = 8$ and $n \geq 16$, then $(2t_1 - 3)n \geq 208 > 196 = 2t_1^2 + 10t_1 - 12$. Hence $f(uv) \leq n - 4$.

If $t_1 = 7$ and $n \geq 15$, then $(2t_1 - 3)n \geq 2t_1^2 + 10t_1 - 12$. Hence $f(uv) \leq n - 4$.

An argument similar to the above shows that $f(uv) \leq n - 4$ when $\left\{ \begin{smallmatrix} t_1 = 6, \\ n \geq 14, \end{smallmatrix} \right.$ $\left\{ \begin{smallmatrix} t_1 = 5, \\ n \geq 13, \end{smallmatrix} \right.$ $\left\{ \begin{smallmatrix} t_1 = 4, \\ n \geq 12, \end{smallmatrix} \right.$ and $\left\{ \begin{smallmatrix} t_1 = 3, \\ n \geq 12. \end{smallmatrix} \right.$

Thus we conclude that inequality $f(uv) \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 9, \\ t_1 = 8 \quad \text{and} \quad n \geq 16, \\ t_1 = 7 \quad \text{and} \quad n \geq 15, \\ t_1 = 6 \quad \text{and} \quad n \geq 14, \\ t_1 = 5 \quad \text{and} \quad n \geq 13, \\ t_1 = 4 \quad \text{and} \quad n \geq 12, \\ t_1 = 3 \quad \text{and} \quad n \geq 12. \end{cases}$$

- vw:

$$f(vw) = \frac{(n - t_1 - 4)^2 + 3n - 2t_1 - 13 + 16 + n - t_1 + 5}{n - t_1}.$$

Then $f(vw) \leq n - 4$ if and only if $t_1 n \geq t_1^2 + t_1 + 24$. The inequality $f(vw) \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 12. \end{cases}$$

- uu':

$$f(uu') = \frac{(t_1 + 1)^2 + n + 2t_1 - 4 + 9 + t_1 + 7}{t_1 + 4}.$$

Then $f(uu') \leq n - 4$ if and only if $(t_1 + 3)n \geq t_1^2 + 9t_1 + 29$. The inequality $f(uu') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 8, \\ t_1 = 7 \quad \text{and} \quad n \geq 15, \\ t_1 = 6 \quad \text{and} \quad n \geq 14, \\ t_1 = 5 \quad \text{and} \quad n \geq 13, \\ t_1 = 4 \quad \text{and} \quad n \geq 12, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- vv':

$$f(vv') = \frac{(n - t_1 - 4)^2 + 3n - 2t_1 - 13 + 9 + n - t_1 + 2}{n - t_1 - 1}.$$

Then $f(vv') \leq n - 4$ if and only if $(t_1 - 1)n \geq t_1^2 + t_1 + 10$. The inequality $f(vv') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- ww':

$$f(ww') = \frac{n - t_1 + 40}{7}.$$

Then $f(ww') \leq n - 4$ if and only if $6n + t_1 \geq 68$. The inequality $f(ww') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- $u'v'$:

$$f(u'v') = \frac{n + 27}{6}.$$

Then $f(u'v') \leq n - 4$ if and only if $5n \geq 51$. The inequality $f(u'v') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- $v'w'$:

$$f(v'w') = \frac{n - t_1 + 30}{6}.$$

Then $f(v'w') \leq n - 4$ if and only if $5n + t_1 \geq 54$. The inequality $f(v'w') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- $u'w'$:

$$f(u'w') = \frac{t_1 + 35}{6}.$$

Then $f(u'w') \leq n - 4$ if and only if $6n - t_1 \geq 59$. The inequality $f(u'w') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- $u'u'$:

$$f(u'u') = \frac{9 + t_1 + 7}{3}.$$

Then $f(u'u') \leq n - 4$ if and only if $3n - t_1 \geq 28$. The inequality $f(u'u') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- $v'v'$:

$$f(v'v') = \frac{9 + n - t_1 + 2}{3}.$$

Then $f(v'v') \leq n - 4$ if and only if $2n + t_1 \geq 23$. The inequality $f(v'v') \leq n - 4$ holds with

$$t_1 \geq 3.$$

- $w'w'$:

$$f(w'w') = \frac{19}{3}.$$

Then $f(w'w') \leq n - 4$ if and only if $3n \geq 31$. The inequality $f(v'v') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

We summarize what has been discussed above as follows.
- If $t_1 \geq 9$, then $\max\{f(e)|e \in E(G)\} \leq n - 4$. Moreover, since G is not a bipartite graph, it follows from Lemma 2.5 that $\mu(G) < n - 4$.
- If $t_1 = 8$, then $n \geq 15$. When $n \geq 16$, we have $\max\{f(e)|e \in E(G)\} \leq n - 4$. Hence $\mu(G) < n - 4$. When $n = 15$, that is, $G = H(8, 3, 1, 0)$. Note that $\mu(H(8, 3, 1, 0)) \approx 10.0680 < n - 4$. Thus $\mu(G) < n - 4$ when $t_1 = 8$.
- If $t_1 = 7$, then $n \geq 14$. When $n \geq 15$, we infer that $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 14$, then $G = H(7, 3, 1, 0)$. Since $\mu(H(7, 3, 1, 0)) \approx 9.0913 < n - 4$, it follows that $\mu(G) < n - 4$ when $t_1 = 7$.

- If $t_1 = 6$, then $n \geq 13$. When $n \geq 14$, we have $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 13$, then $G = H(6, 3, 1, 0)$. By the fact that $\mu(H(6, 3, 1, 0)) \approx 8.1298 < n - 4$, it follows that $\mu(G) < n - 4$ when $t_1 = 6$.
- If $t_1 = 5$, then $n \geq 12$. When $n \geq 13$, we infer that $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 12$, then $G = H(5, 3, 1, 0)$. Since $\mu(H(5, 3, 1, 0)) \approx 7.2022 < n - 4$, it follows that $\mu(G) < n - 4$ when $t_1 = 5$.
- If $t_1 = 4$, then $n \geq 11$. When $n \geq 12$, we infer that $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 11$, then $G = H(4, 3, 1, 0)$. Now that $\mu(H(4, 3, 1, 0)) \approx 6.3694 < n - 4$, it follows that $\mu(G) < n - 4$ when $t_1 = 4$.
- If $t_1 = 3$, then $n \geq 10$. When $n \geq 12$, we infer that $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 11$, then $G = H(3, 3, 2, 0)$ or $H(3, 3, 1, 1)$. If $n = 10$, then $G = H(3, 3, 1, 0)$. Note that $\mu(H(3, 3, 2, 0)) \approx 6.1116 < n - 4$, $\mu(H(3, 3, 1, 1)) \approx 6.4142 < n - 4$ and $\mu(H(3, 3, 1, 0)) \approx 5.8577 < n - 4$. Therefore $\mu(G) < n - 4$ in this case.

Thus we have derived that $\mu(G) < n - 4$ when $t_2 = 3$. This completes the proof.   □

Lemma 2.9 *Let $G = H(t_1, t_2, t_3, t_4)$ be a Halin graph with $t_1 \geq t_2 = 2$.*
 (1) *If $t_1 = 2$ or $n - 6$, then $\mu(G) > n - 4$.*
 (2) *If $3 \leq t_1 \leq n - 7$ and $G \neq H(3, 2, 1, 1)$, then $\mu(G) < n - 4$.*

Proof For $t_2 = 2$. If $t_1 = 2$ or $n - 6$, then $\Delta(G) = n - 5$. According to Lemma 2.1, it follows that $\mu(G) > \Delta + 1 = n - 4$. Therefore (1) holds.

Suppose $3 \leq t_1 \leq n - 7$. Obviously, $n \geq t_1 + 7$. We also use the bound in Lemma 2.5 to prove the result. Let $u' \in N(u) \cap V(C)$, $v' \in N(v) \cap V(C)$ and $w' \in N(w) \cap V(C)$. Note that

$$\begin{cases} d(u) = t_1 + 1, \\ d(v) = n - t_1 - 3, \\ d(w) = 3, \\ d(u') = d(v') = d(w') = 3. \end{cases}$$

Then the 2-degree of each vertex is as follows:

$$\begin{cases} d(u)m(u) = n + 2t_1 - 3, \\ d(v)m(v) = 3n - 2t_1 - 11, \\ d(w)m(w) = n - t_1 + 3, \\ d(u')m(u') = t_1 + 7, \\ d(v')m(v') = n - t_1 + 3, \\ d(w')m(w') = 9. \end{cases}$$

For all types of edges in G, consider the index in Lemma 2.5. Let $e = xy$ be any one edge of G. Put

$$f(e) = f(xy) = \frac{d(x)(d(x) + m(x)) + d(y)(d(y) + m(y))}{d(x) + d(y)}.$$

For simplicity, we use $u'u'$ to denote the edges $u'u'' \in E(G)$ where $u', u'' \in N(u) \cap V(C)$. Similarly, we define the symbol $v'v'$ and $w'w'$. Then we will prove that the inequality $f(e) \leq$

$n - 4$ holds. Note that every edge of G belongs to the one below, and the types $u'w'$ and $v'v'$ exist in some circumstances.

- uv:

$$f(uv) = \frac{(t_1 + 1)^2 + n + 2t_1 - 3 + (n - t_1 - 3)^2 + 3n - 2t_1 - 11}{n - 2}.$$

Then $f(uv) \leq n - 4$ if and only if $(t_1 - 2)n \geq t_1^2 + 4t_1 - 6$. The inequality $f(uv) \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 8, \\ t_1 = 7 \quad \text{and} \quad n \geq 15, \\ t_1 = 6 \quad \text{and} \quad n \geq 14, \\ t_1 = 5 \quad \text{and} \quad n \geq 13, \\ t_1 = 4 \quad \text{and} \quad n \geq 13, \\ t_1 = 3 \quad \text{and} \quad n \geq 15. \end{cases}$$

- vw:

$$f(vw) = \frac{(n - t_1 - 3)^2 + 3n - 2t_1 - 11 + 9 + n - t_1 + 3}{n - t_1}.$$

Then $f(vw) \leq n - 4$ if and only if $(t_1 - 2)n \geq t_1^2 - t_1 + 10$. The inequality $f(vw) \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 16. \end{cases}$$

- uu':

$$f(uu') = \frac{(t_1 + 1)^2 + n + 2t_1 - 3 + 9 + t_1 + 7}{t_1 + 4}.$$

Then $f(uu') \leq n - 4$ if and only if $(t_1 + 3)n \geq t_1^2 + 9t_1 + 30$. The inequality $f(uu') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 9, \\ t_1 = 8 \quad \text{and} \quad n \geq 16, \\ t_1 = 7 \quad \text{and} \quad n \geq 15, \\ t_1 = 6 \quad \text{and} \quad n \geq 14, \\ t_1 = 5 \quad \text{and} \quad n \geq 13, \\ t_1 = 4 \quad \text{and} \quad n \geq 12, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- vv':

$$f(vv') = \frac{(n - t_1 - 3)^2 + 3n - 2t_1 - 11 + 9 + n - t_1 + 3}{n - t_1}.$$

Then $f(vv') \le n - 4$ if and only if $(t_1 - 2)n \ge t_1^2 - t_1 + 10$. The inequality $f(vv') \le n - 4$ holds with

$$\begin{cases} t_1 \ge 4, \\ t_1 = 3 \quad \text{and} \quad n \ge 16. \end{cases}$$

- ww':

$$f(ww') = \frac{n - t_1 + 30}{6}.$$

Then $f(ww') \le n - 4$ if and only if $5n + t_1 \ge 54$. The inequality $f(ww') \le n - 4$ holds with

$$\begin{cases} t_1 \ge 4, \\ t_1 = 3 \quad \text{and} \quad n \ge 11. \end{cases}$$

- $u'v'$:

$$f(u'v') = \frac{n + 28}{6}.$$

Then $f(u'v') \le n - 4$ if and only if $5n \ge 52$. The inequality $f(u'v') \le n - 4$ holds with

$$\begin{cases} t_1 \ge 4, \\ t_1 = 3 \quad \text{and} \quad n \ge 11. \end{cases}$$

- $v'w'$:

$$f(v'w') = \frac{n - t_1 + 30}{6}.$$

Then $f(v'w') \le n - 4$ if and only if $5n + t_1 \ge 54$. The inequality $f(v'w') \le n - 4$ holds with

$$\begin{cases} t_1 \ge 4, \\ t_1 = 3 \quad \text{and} \quad n \ge 11. \end{cases}$$

- $u'w'$:

$$f(u'w') = \frac{t_1 + 34}{6}.$$

Then $f(u'w') \le n - 4$ if and only if $6n - t_1 \ge 58$. The inequality $f(u'w') \le n - 4$ holds with

$$\begin{cases} t_1 \ge 4, \\ t_1 = 3 \quad \text{and} \quad n \ge 11. \end{cases}$$

- $u'u'$:

$$f(u'u') = \frac{9 + t_1 + 7}{3}.$$

Then $f(u'u') \leq n - 4$ if and only if $3n - t_1 \geq 28$. The inequality $f(u'u') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- $v'v'$:

$$f(v'v') = \frac{9 + n - t_1 + 3}{3}.$$

Then $f(v'v') \leq n - 4$ if and only if $2n + t_1 \geq 24$. The inequality $f(v'v') \leq n - 4$ holds with

$$\begin{cases} t_1 \geq 4, \\ t_1 = 3 \quad \text{and} \quad n \geq 11. \end{cases}$$

- $w'w'$:

$$f(w'w') = 6.$$

Since $n \geq t_1 + 7 \geq 10$, we have $f(w'w') \leq n - 4$.

So we have the following conclusions.

- If $t_1 \geq 9$, then $\max\{f(e)|e \in E(G)\} \leq n - 4$. According to Lemma 2.5, it follows that $\mu(G) < n - 4$.
- If $t_1 = 8$, then $n \geq 15$. When $n \geq 16$, we have $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 15$, then $G = H(8, 2, 2, 0)$ or $H(8, 2, 1, 1)$. Note that $\mu(H(8, 2, 2, 0)) \approx 10.0928 < n - 4$ and $\mu(H(8, 2, 1, 1)) \approx 10.1016 < n - 4$. Thus $\mu(G) < n - 4$ when $t_1 = 8$.
- If $t_1 = 7$, then $n \geq 14$. When $n \geq 15$, we have $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 14$, then $G = H(7, 2, 2, 0)$ or $H(7, 2, 1, 1)$. Note that $\mu(H(7, 2, 2, 0)) \approx 9.1261 < n - 4$ and $\mu(H(7, 2, 1, 1)) \approx 9.1414 < n - 4$. Hence $\mu(G) < n - 4$ when $t_1 = 7$.
- If $t_1 = 6$, then $n \geq 13$. When $n \geq 14$, we have $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 13$, then $G = H(6, 2, 2, 0)$ or $H(6, 2, 1, 1)$. Note that $\mu(H(6, 2, 2, 0)) \approx 8.1820 < n - 4$ and $\mu(H(6, 2, 1, 1)) \approx 8.2113 < n - 4$. Hence $\mu(G) < n - 4$ when $t_1 = 6$.
- If $t_1 = 5$, then $n \geq 12$. When $n \geq 13$, we have $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 12$, then $G = H(5, 2, 2, 0)$ or $H(5, 2, 1, 1)$. Note that $\mu(H(5, 2, 2, 0)) \approx 7.2861 < n - 4$ and $\mu(H(5, 2, 1, 1)) \approx 7.3502 < n - 4$. Hence $\mu(G) < n - 4$ when $t_1 = 5$.
- If $t_1 = 4$, then $n \geq 11$. When $n \geq 13$, we have $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 12$, then $G = H(4, 2, 3, 0)$ or $H(4, 2, 2, 1)$. If $n = 11$, then $G = H(4, 2, 2, 0)$ or $H(4, 2, 1, 1)$. Note that $\mu(H(4, 2, 3, 0)) \approx 6.8985 < n - 4$, $\mu(H(4, 2, 2, 1)) \approx 7.0131 < n - 4$, $\mu(H(4, 2, 2, 0)) \approx 6.5037 < n - 4$ and $\mu(H(4, 2, 1, 1)) \approx 6.6518 < n - 4$. Therefore $\mu(G) < n - 4$ when $t_1 = 4$.
- If $t_1 = 3$, then $n \geq 10$. When $n \geq 15$, we have $\max\{f(e)|e \in E(G)\} \leq n - 4$. If $n = 10, 11, 12, 13, 14$, then $G \in \mathbb{H} = \{H(3, 2, 2, 0), H(3, 2, 1, 1), H(3, 2, 3, 0), H(3, 2, 2, 1), H(3, 2, 4, 0), H(3, 2, 3, 1), H(3, 2, 2, 2), H(3, 2, 5, 0), H(3, 2, 4, 1), H(3, 2, 3, 2), H(3, 2, 6, 0), H(3, 2, 5, 1), H(3, 2, 4, 2), H(3, 2, 3, 3)\}$.

  Note that $\mu(H(3, 2, 1, 1)) \approx 6.2470 > n - 4$ and if $G \in \mathbb{H}\setminus\{H(3, 2, 1, 1)\}$ then $\mu(G) < n - 4$ (see Table 1). This implies that if $t_1 = 3$ and $G \neq H(3, 2, 1, 1)$, then $\mu(G) < n - 4$.

Consequently, we infer that (2) holds. This completes the proof.　　　　　　　□

Table 1 The Laplacian spectral radii of some Halin graphs with three interior vertices

	G	$\mu(G)$		G	$\mu(G)$
$n=8$	$H(2,2,1,0)$	5.4142	$n=13$	$H(6,3,1,0)$	8.1298
$n=9$	$H(2,2,2,0)$	5.6996		$H(3,2,5,0)$	8.2480
	$H(2,2,1,1)$	6.0000		$H(3,2,4,1)$	8.3198
	$H(3,2,1,0)$	5.7480		$H(3,2,3,2)$	8.3052
$n=10$	$H(3,3,1,0)$	5.8577		$H(6,2,2,0)$	8.1820
	$H(2,2,3,0)$	6.4423		$H(6,2,1,1)$	8.2113
	$H(2,2,2,1)$	6.5846	$n=14$	$H(7,3,1,0)$	9.0913
	$H(4,2,1,0)$	6.3500		$H(3,2,6,0)$	9.1708
	$H(3,2,2,0)$	5.9709		$H(3,2,5,1)$	9.2272
	$H(3,2,1,1)$	6.2470		$H(3,2,4,2)$	9.2180
$n=11$	$H(3,3,2,0)$	6.1116		$H(3,2,3,3)$	9.2198
	$H(3,3,1,1)$	6.4142		$H(7,2,2,0)$	9.1261
	$H(4,3,1,0)$	6.3694		$H(7,2,1,1)$	9.1414
	$H(3,2,3,0)$	6.5894	$n=15$	$H(8,3,1,0)$	10.0680
	$H(3,2,2,1)$	6.7387		$H(8,2,2,0)$	10.0928
	$H(4,2,2,0)$	6.5037		$H(8,2,1,1)$	10.1016
	$H(4,2,1,1)$	6.6518			
$n=12$	$H(5,3,1,0)$	7.2022			
	$H(3,2,4,0)$	7.3612			
	$H(3,2,3,1)$	7.4771			
	$H(3,2,2,2)$	7.4446			
	$H(4,2,3,0)$	6.8985			
	$H(4,2,2,1)$	7.0131			
	$H(5,2,2,0)$	7.2861			
	$H(5,2,1,1)$	7.3502			

For Halin graphs with three interior vertices. From the proof of the above lemmas, we see that only $H(3,2,1,1)$, $H(2,2,t_3,t_4)$ and $H(n-6,2,1,0)$ have the Laplacian spectral radii greater than $n-4$. Clearly, $n-4 < \mu(H(3,2,1,1)) < n-3$ (see Table 1).

Lemma 2.10 *Let* $G \in \{H(2,2,t_3,t_4), H(n-6,2,1,0)\}$, *where* $t_3 + t_4 \geq 2$, *then* $\mu(G) \leq n-3$. *If* $G = H(2,2,1,0)$, *then* $n-3 < \mu(G) < n-2$.

Proof It is clear that $H(2,2,t_3,t_4)$ and $H(n-6,2,1,0)$ have the same degree sequence:

$$(d_1, d_2, \ldots, d_n) = (n-5, 3, 3, \ldots, 3).$$

Let $G \in \{H(2,2,t_3,t_4), H(n-6,2,1,0)\}$. By Lemmas 2.2 and 2.3, we have

$$\mu(G) < \min_{1 \leq i \leq n} \left\{ \frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_1 + 1)^2 + 8(i-1)(d_1 - d_i)}}{2} \right\}$$

$$\leq \frac{n-5+6-1+\sqrt{(6-(n-5)+1)^2 + 8(n-5-3)}}{2}$$

$$= \frac{n + \sqrt{(n-12)^2 + 8(n-8)}}{2}.$$

If $n \geq 11$, it is easy to check that

$$n-3 \geq \frac{n + \sqrt{(n-12)^2 + 8(n-8)}}{2}.$$

Therefore $\mu(G) < n - 3$. If $8 \leq n \leq 10$, then $G \in \{H(2,2,1,0), H(2,2,2,0), H(2,2,1,1),$ $H(3,2,1,0), H(2,2,3,0), H(2,2,2,1), H(4,2,1,0)\}$. If $G = H(2,2,1,0)$, then $n - 3 < \mu(G) <$ $n - 2$. Otherwise, $\mu(G) \leq n - 3$ (see Table 1). This lemma follows. □

Now we consider the Halin graphs with two interior vertices. Let $G = H(t_1, t_2)$ and $t_1 \geq t_2 \geq 2$. Note that $t_1 = n - t_2 - 2 \geq t_2$, then $n \geq 2t_2 + 2$.

Lemma 2.11 *Let $G = H(t_1, t_2)$ be a Halin graph with $t_1 \geq t_2 \geq 5$. Then $\mu(G) < n - 4$.*

Proof Suppose u and v are the two interior vertices. Let $u' \in N(u) \cap V(C)$ and $v' \in N(v) \cap V(C)$. Note that

$$\begin{cases} d(u) = n - t_2 - 1, \\ d(v) = t_2 + 1, \\ d(u') = d(v') = 3. \end{cases}$$

Then the 2-degree of each vertex is as follows:

$$\begin{cases} d(u)m(u) = 3n - 2t_2 - 5, \\ d(v)m(v) = n + 2t_2 - 1, \\ d(u')m(u') = n - t_2 + 5, \\ d(v')m(v') = t_2 + 7. \end{cases}$$

For all types of edges in G, consider the index in Lemma 2.5. Let $e = xy$ be any one edge of G. We may take

$$f(e) = f(xy) = \frac{d(x)(d(x) + m(x)) + d(y)(d(y) + m(y))}{d(x) + d(y)}.$$

For simplicity, we use $u'u'$ to denote the edges $u'u'' \in E(G)$ where $u', u'' \in N(u) \cap V(C)$. Similarly, we define the symbol $v'v'$. It is clear that every edge of G belongs to the one below.

- uv:

$$f(uv) = \frac{(n - t_2 - 1)^2 + 3n - 2t_2 - 5 + (t_2 + 1)^2 + n + 2t_2 - 1}{n}.$$

Then $f(uv) \leq n - 4$ if and only if $(t_2 - 3)n \geq t_2^2 + 2t_2 - 2$. The inequality $f(uu') \leq n - 4$ holds with

$$\begin{cases} t_2 \geq 7, \\ t_2 = 6 \quad \text{and} \quad n \geq 16, \\ t_2 = 5 \quad \text{and} \quad n \geq 17. \end{cases}$$

- uu':

$$f(uu') = \frac{(n - t_2 - 1)^2 + 3n - 2t_2 - 5 + 9 + n - t_2 + 5}{n - t_2 + 2}.$$

Then $f(uu') \le n - 4$ if and only if $(t_2 - 4)n \ge t_2^2 - 5t_2 + 18$. The inequality $f(uu') \le n - 4$ holds with

$$
\begin{cases}
t_2 \ge 6, \\
t_2 = 5 \quad \text{and} \quad n \ge 18.
\end{cases}
$$

- vv':

$$
f(vv') = \frac{(t_2 + 1)^2 + n + 2t_2 - 1 + 9 + t_2 + 7}{t_2 + 4}.
$$

Then $f(vv') \le n - 4$ if and only if $(t_2 + 3)n \ge t_2^2 + 9t_2 + 32$. The inequality $f(vv') \le n - 4$ holds with

$$
\begin{cases}
t_2 \ge 6, \\
t_2 = 5 \quad \text{and} \quad n \ge 13.
\end{cases}
$$

- $u'v'$:

$$
f(u'v') = \frac{n + 30}{6}.
$$

Then $f(u'v') \le n - 4$ if and only if $5n \ge 54$. If $t_2 \ge 5$, $f(u'v') \le n - 4$.
- $u'u'$:

$$
f(u'u') = \frac{18 + 2(n - t_2 + 5)}{6}.
$$

Then $f(u'u') \le n - 4$ if and only if $2n + t_2 \ge 26$. If $t_2 \ge 5$, $f(u'u') \le n - 4$.
- $v'v'$:

$$
f(v'v') = \frac{18 + 2(t_2 + 7)}{6}.
$$

Then $f(v'v') \le n - 4$ if and only if $3n - t_2 \ge 28$. If $t_2 \ge 5$, $f(v'v') \le n - 4$.
   Thus we infer that $\max\{f(e) | e \in E(G)\} \le n - 4$ if $t_2 \ge 7$, $\begin{cases} t_2 = 6, \\ n \ge 16, \end{cases}$ or $\begin{cases} t_2 = 5, \\ n \ge 18. \end{cases}$ According to Lemma 2.5, it follows that $\mu(G) < n - 4$. Otherwise,

$$
G \in \{H(5,5), H(6,5), H(6,6), H(7,5), H(7,6), H(8,5), H(9,5), H(10,5)\}.
$$

It is easy to see that $\mu(G) < n - 4$ in this case (see Table 2). This completes the proof. $\square$

Lemma 2.12 *Let $G = H(n - 6, 4)$ be a Halin graph with $n \ge 10$ vertices. Then $n - 4 < \mu(G) < n - 3$.*

Proof Since $\Delta(G) = n - 5$, it follows from Lemma 2.1 that $\mu(G) > n - 4$. The degree sequence of G is $(d_1, d_2, \ldots, d_n) = (n - 5, 5, 3, \ldots, 3)$. From Lemmas 2.2 and 2.3, we have

$$
\begin{aligned}
\mu(G) &< \min_{1 \le i \le n} \left\{ \frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_1 + 1)^2 + 8(i-1)(d_1 - d_i)}}{2} \right\} \\
&\le \frac{d_1 + 2d_2 - 1 + \sqrt{(2d_2 - d_1 + 1)^2 + 8(d_1 - d_2)}}{2} \\
&= \frac{n + 4 + \sqrt{(n-16)^2 + 8(n-10)}}{2}.
\end{aligned}
$$

If $n \ge 19$, then we get

$$
n - 3 \ge \frac{n + 4 + \sqrt{(n-16)^2 + 8(n-10)}}{2}.
$$

Therefore $\mu(G) < n - 3$ when $n \ge 19$. If $n \le 18$, then $G = H(t_1, 4)$ where $t_1 = 4, 5, 6, \ldots, 12$. It is easy to check that $\mu(G) < n - 3$ (see Table 2). Thus we complete the proof. $\qquad\square$

Lemma 2.13 *Let $G = H(n - 5, 3)$ be a Halin graph with $n \ge 8$ vertices. Then $n - 3 < \mu(G) \le n - 2$. Moreover, the right equality holds if and only $G = H(3, 3)$.*

Proof The degree sequence of G is $(d_1, d_2, \ldots, d_n) = (n - 4, 4, 3, \ldots, 3)$. It follows from Lemma 2.1 that $\mu(G) > \Delta(G) + 1 = n - 3$. From Lemmas 2.2 and 2.3, we have

$$
\begin{aligned}
\mu(G) &< \min_{1 \le i \le n} \left\{ \frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_1 + 1)^2 + 8(i-1)(d_1 - d_i)}}{2} \right\} \\
&\le \frac{d_1 + 2d_2 - 1 + \sqrt{(2d_2 - d_1 + 1)^2 + 8(d_1 - d_2)}}{2} \\
&= \frac{n + 3 + \sqrt{(n-13)^2 + 8(n-8)}}{2}.
\end{aligned}
$$

If $n \ge 14$, then

$$
n - 2 \ge \frac{n + 3 + \sqrt{(n-13)^2 + 8(n-8)}}{2}.
$$

Therefore $\mu(G) < n - 3$. If $n \le 13$, then $G = H(t_1, 3)$ where $t_1 = 3, 4, 5, \ldots, 8$. It is clear that $\mu(H(3,3)) = n - 2$ and $\mu(H(t_1, 3)) < n - 2$ where $t_1 = 4, 5, \ldots, 8$ (see Table 2). Thus we complete the proof. $\qquad\square$

Lemma 2.14 *Let $G = H(n - 4, 2)$ be a Halin graph with $n \ge 6$ vertices. Then $n - 2 < \mu(G) \le n - 1$. Moreover, the right equality holds if and only $G = H(2, 2)$.*

Table 2 The Laplacian spectral radii of some Halin graphs with two interior vertices

	G	$\mu(G)$			G	$\mu(G)$
$n = 6$	$H(2,2)$	5.0000	$n = 14$		$H(6,6)$	8.8562
$n = 7$	$H(3,2)$	5.6180			$H(7,5)$	9.2930
					$H(8,4)$	10.1358
$n = 8$	$H(3,3)$	6.0000	$n = 15$		$H(7,6)$	9.4835
	$H(4,2)$	6.3234			$H(8,5)$	10.1954
$n = 9$	$H(4,3)$	6.5315			$H(9,4)$	11.1012
$n = 10$	$H(4,4)$	6.8820	$n = 16$		$H(9,5)$	11.1399
	$H(5,3)$	7.3058			$H(10,4)$	12.0785
$n = 11$	$H(5,4)$	7.4911	$n = 17$		$H(10,5)$	12.1055
	$H(6,3)$	8.1938			$H(11,4)$	13.0627
$n = 12$	$H(5,5)$	7.8605	$n = 18$		$H(12,4)$	14.0513
	$H(6,4)$	8.2932				
	$H(7,3)$	9.1335				
$n = 13$	$H(6,5)$	8.4827				
	$H(7,4)$	9.1923				
	$H(8,3)$	10.0976				

Proof The degree sequence of G is $(d_1, d_2, \ldots, d_n) = (n - 3, 3, 3, \ldots, 3)$. It follows from Lemma 2.1 that $\mu(G) > \Delta(G) + 1 = n - 2$. From Lemmas 2.2 and 2.3, we have

$$\mu(G) < \min_{1 \le i \le n} \left\{ \frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_1 + 1)^2 + 8(i-1)(d_1 - d_i)}}{2} \right\}$$

$$\le \frac{d_1 + 2d_2 - 1 + \sqrt{(2d_2 - d_1 + 1)^2 + 8(d_1 - d_2)}}{2}$$

$$= \frac{n + 2 + \sqrt{(n-10)^2 + 8(n-6)}}{2}.$$

If $n \ge 9$, then

$$n - 1 \ge \frac{n + 2 + \sqrt{(n-10)^2 + 8(n-6)}}{2}.$$

Therefore $\mu(G) < n - 1$. If $n \le 8$, then $G = H(2,2)$, $H(3,2)$ or $H(4,2)$. It is clear that $\mu(H(2,2)) = n - 1$, $\mu(H(3,2)) < n - 1$ and $\mu(H(4,2)) < n - 1$ (see Table 2). Thus we complete the proof. □

Now we are ready to present the proof of Theorem 1.1. In fact, from the previous lemmas, it is easy to obtain the main result. For the sake of completeness, we provide a brief proof.

3 Proof of Theorem 1.1

Let G be a Halin graph. We make a summary of Lemmas 2.6-2.14.

If G has $k \ge 4$ interior vertices, then $\mu(G) < n - 4$.

If G has three interior vertices, then $\mu(G) < n - 4$ when $G \notin \{H(2,2,t_3,t_4), H(n - 6, 2, 1, 0), H(3, 2, 1, 1)\}$, where $t_3 + t_4 \ge 1$; if $G \in \{H(2, 2, t_3, t_4), H(n - 6, 2, 1, 0), H(3, 2, 1, 1)\}$, where $t_3 + t_4 \ge 2$, then $n - 4 < \mu(G) \le n - 3$; if $G = H(2, 2, 1, 0)$, then $n - 3 < \mu(G) < n - 2$.

If G has two interior vertices, then $\mu(G) < n - 4$ when

$$G \notin \left\{ H(n - 6, 4), H(n - 5, 3), H(n - 4, 2) \right\}.$$

On the other hand, we have $n - 4 < \mu(H(n - 6, 4)) < n - 3$, $n - 3 < \mu(H(n - 5, 3)) \leq n - 2$, $n - 2 < \mu(H(n - 4, 2)) \leq n - 1$ and $\mu(H(n - 5, 3)) > \mu(H(2, 2, 1, 0))$.

If G has one interior vertex, then $G = W_n$ and $\mu(W_n) = n$.

It is now obvious that the theorem holds.

Remark 3.1 From the proof, we see that there is no graph with $n - 1 < \mu(G) < n$. If $\mu(G) = n - 1$ iff $G = H(2, 2)$. If $\mu(G) = n - 2$ iff $G = H(3, 3)$. There is no graph with $\mu(G) = n - 4$.

Remark 3.2 Let $H(n - t - 2, t)$ be a Halin with n vertices and $n \geq 2t + 2$. Then $\Delta = n - t - 1$, so $\mu(H(n - t - 2, t)) > n - t$. The degree sequence is $(n - t - 1, t + 1, 3, \ldots, 3)$, then if $n \geq 5t - 1$, we have

$$\mu\big(H(n - t - 2, t)\big) \leq \frac{d_1 + 2d_2 - 1 + \sqrt{(2d_2 - d_1 + 1)^2 + 8(d_1 - d_2)}}{2} \leq n - t + 1.$$

That is, for an integer k, when n is sufficiently large, then $n - t < \mu(H(n - t - 2, t)) \leq n - t + 1$. From this we propose the following conjecture.

Conjecture 3.1 *Let $H(t_1, t_2)$ be a Halin graph with two interior vertices and order n, where $n = t_1 + t_2 + 2$ and $t_1 \geq t_2$. Then*
  (1) $n - t_2 < \mu(H(t_1, t_2)) \leq n - t_2 + 1$;
  (2) $\mu(H(t_1, t_2)) < \mu(H(t_1 + 1, t_2 - 1))$.

4 Conclusions

We determine all the Halin graphs with $\mu(G) \geq n - 4$. Moreover, we also obtain the graphs with the first three largest Laplacian spectral radius among all the Halin graphs on n vertices. Considering the further order of the Laplacian spectral radius of Halin graphs is still an interesting and important problem.

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