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Refined quadratic estimations of Shafer's inequality

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Abstract

We establish an inequality by quadratic estimations; the double inequality

$$\frac{\pi^2 x}{4+\sqrt{(\pi^2-4)^2+(2\pi x)^2}} < \arctan x < \frac{\pi^2 x}{4+\sqrt{32+(2\pi x)^2}}$$

holds for $x > 0$, where the constants $(\pi^2 - 4)^2$ and 32 are the best possible.

MSC: Primary 26D15; 42A10

Keywords: Shafer's inequality; an upper bound for arctangent; a lower bound for arctangent

1 Introduction

Shafer [1–3] showed that the inequality

$$\arctan x > \frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} \tag{1.1}$$

holds for $x > 0$. Various Shafer-type inequalities are known, and they have been applied, extended and refined, see [4–8] and [9–12]. Especially, Zhu [12] showed an upper bound for inequality (1.1) and proved that the following double inequality

$$\frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} < \arctan x < \frac{8x}{3+\sqrt{25+\frac{256}{\pi^2}x^2}} \tag{1.2}$$

holds for $x > 0$, where the constants $80/3$ and $256/\pi^2$ are the best possible. Recently, in [8], Sun and Chen proved that the following inequality

$$\arctan x < \frac{8x+\frac{32}{4725}x^7}{3+\sqrt{25+\frac{80}{3}x^2}} \tag{1.3}$$

holds for $x > 0$; moreover, they showed that the inequality

$$\frac{8x+\frac{32}{4725}x^7}{3+\sqrt{25+\frac{80}{3}x^2}} < \frac{8x}{3+\sqrt{25+\frac{256}{\pi^2}x^2}} \tag{1.4}$$

holds for $0 < x < x_0 \cong 1.4243$. In this paper, we shall establish the refinements of inequalities (1.2) and (1.3).

2 Results and discussion

Motivated by (1.2), (1.3) and (1.4), in this paper, we give inequalities involving arctangent. The following are our main results.

Theorem 2.1 *For $x > 0$, we have*

$$\frac{\pi^2 x}{4 + \sqrt{(\pi^2 - 4)^2 + (2\pi x)^2}} < \arctan x < \frac{\pi^2 x}{4 + \sqrt{32 + (2\pi x)^2}}, \tag{2.1}$$

where the constants $(\pi^2 - 4)^2$ and 32 are the best possible.

Theorem 2.2 *For $x > \alpha$, we have*

$$\frac{\pi^2 x}{4 + \sqrt{(\pi^2 - 4)^2 + (2\pi x)^2}} > \frac{8x}{3 + \sqrt{25 + \frac{80}{3}x^2}}, \tag{2.2}$$

where the constant $\alpha = \sqrt{\frac{9600 - 1860\pi^2 + 90\pi^4}{2304 - 480\pi^2 + 25\pi^4}} \cong 2.26883$ is the best possible.

Theorem 2.3 *For $x > \beta$, we have*

$$\frac{8x}{3 + \sqrt{25 + \frac{256}{\pi^2}x^2}} > \frac{\pi^2 x}{4 + \sqrt{32 + (2\pi x)^2}}, \tag{2.3}$$

where the constant $\beta = \sqrt{\frac{4096 + 1536\pi^2 - 528\pi^4 + 24\pi^6 + \pi^8}{4096\pi^2 - 768\pi^4 + 36\pi^6}} \cong 1.30697$ is the best possible.

Theorem 2.4 *For $x > \gamma$, we have*

$$\frac{8x + \frac{32}{4725}x^7}{3 + \sqrt{25 + \frac{80}{3}x^2}} > \frac{\pi^2 x}{4 + \sqrt{32 + (2\pi x)^2}}, \tag{2.4}$$

where the constant $\gamma \cong 1.38918$ is the best possible and satisfies the equation

$$\begin{aligned} &151200 - 14175\pi^2 + 128\gamma^6 - 1575\sqrt{15}\pi^2\sqrt{15 + 16\gamma^2} \\ &\quad + 75600\sqrt{8 + \pi^2\gamma^2} + 64\gamma^6\sqrt{8 + \pi^2\gamma^2} = 0. \end{aligned}$$

From Theorems 2.1, 2.2, 2.3 and 2.4, we can get the following proposition, immediately.

Proposition 2.5 *The double inequality* (2.1) *is sharper than* (1.2) *for $x > \alpha$. Moreover, the right-hand side of* (2.1) *is sharper than* (1.3) *for $x > \gamma$.*

2.1 Proof of Theorem 2.1

Becker-Stark's inequality is known as the inequality

$$\frac{8}{\pi^2-4x^2}<\frac{\tan x}{x}<\frac{\pi^2}{\pi^2-4x^2} \tag{2.5}$$

which holds for $0<x<\pi/2$. Also, Becker-Stark's inequality (2.5) has various applications, extensions and refinements, see [13–16] and [17–19]. Especially, Zhu [19] gave the following refinement of (2.5): The inequality

$$\frac{8}{\pi^2-4x^2}+\frac{2}{\pi^2}-\lambda\left(\pi^2-4x^2\right)<\frac{\tan x}{x}<\frac{8}{\pi^2-4x^2}+\frac{2}{\pi^2}-\mu\left(\pi^2-4x^2\right) \tag{2.6}$$

holds for $0<x<\pi/2$, where the constants $\lambda=(\pi^2-9)/(6\pi^4)$ and $\mu=(10-\pi^2)/\pi^4$ are the best possible. In this paper, the result of Zhu (2.6) plays an important role in the proof of Theorem 2.1.

Proof of Theorem 2.1 The equation

$$\arctan x=\frac{\pi^2x}{4+\sqrt{c+(2\pi x)^2}}$$

is equivalent to

$$c=\frac{\pi^4x^2-8\pi^2x\arctan x+16\arctan^2x-4\pi^2x^2\arctan^2x}{\arctan^2x}.$$

We set $t=\arctan x$, then

$$\begin{aligned}c&=\frac{\pi^4\tan^2t}{t^2}-\frac{8\pi^2\tan t}{t}+16-4\pi^2\tan^2t\\&=16+F_1(t).\end{aligned}$$

First, we assume that $0<t\le 1/2$. Here, the derivative of $F_1(t)$ is

$$\begin{aligned}F_1'(t)&=-\frac{8\pi^2\sec^2t}{t}+\frac{8\pi^2\tan t}{t^2}-8\pi^2\sec^2t\tan t+\frac{2\pi^4\sec^2t\tan t}{t^2}-\frac{2\pi^4\tan^2t}{t^3}\\&=\frac{\sin t}{\cos^2t}\left(-\frac{8\pi^2}{t\sin t}+\frac{8\pi^2\cos t}{t^2}-\frac{8\pi^2}{\cos t}+\frac{2\pi^4}{t^2\cos t}-\frac{2\pi^4\sin t}{t^3}\right)\\&=\frac{\sin t}{\cos^2t}F_2(t).\end{aligned}$$

Since we have

$$t-\frac{t^3}{6}<\sin t<t-\frac{t^3}{6}+\frac{t^5}{120}$$

and

$$1-\frac{t^2}{2}+\frac{t^4}{24}-\frac{t^6}{720}<\cos t<1-\frac{t^2}{2}+\frac{t^4}{24}$$

for $0 < t < \pi/2$, the following inequality holds:

$$\begin{aligned}
F_2(t) &< -\frac{8\pi^2}{t(t-\frac{t^3}{6}+\frac{t^5}{120})} + \frac{8\pi^2(1-\frac{t^2}{2}+\frac{t^4}{24})}{t^2} \\
&\quad - \frac{8\pi^2}{(1-\frac{t^2}{2}+\frac{t^4}{24})} + \frac{2\pi^4}{t^2(1-\frac{t^2}{2}+\frac{t^4}{24}-\frac{t^6}{720})} - \frac{2\pi^4(t-\frac{t^3}{6})}{t^3} \\
&= \frac{\pi^2 F_3(t)}{3(120-20t^2+t^4)(24-12t^2+t^4)(-720+360t^2-30t^4+t^6)},
\end{aligned}$$

where $F_3(t) = 82944000 - 8294400\pi^2 - 72990720t^2 + 7084800\pi^2 t^2 + 24883200t^4 - 2246400\pi^2 t^4 - 4832640t^6 + 371520\pi^2 t^6 + 596736t^8 - 35904\pi^2 t^8 - 48192t^{10} + 2076\pi^2 t^{10} + 2472t^{12} - 68\pi^2 t^{12} - 74t^{14} + \pi^2 t^{14} + t^{16}$. We set $s = t^2$, then

$$\begin{aligned}
F_3(t) &> 82944000 - 8294400\left(\frac{315}{100}\right)^2 - 72990720s + 7084800\left(\frac{314}{100}\right)^2 s \\
&\quad + 24883200s^2 - 2246400\left(\frac{315}{100}\right)^2 s^2 - 4832640s^3 + 371520\left(\frac{314}{100}\right)^2 s^3 \\
&\quad + 596736s^4 - 35904\left(\frac{315}{100}\right)^2 s^4 - 48192s^5 + 2076\left(\frac{314}{100}\right)^2 s^5 \\
&\quad + 2472s^6 - 68\left(\frac{315}{100}\right)^2 s^6 - 74s^7 + \left(\frac{314}{100}\right)^2 s^7 + s^8 \\
&= 642816 - \frac{78435648s}{25} + 2593296s^2 - \frac{146200176s^3}{125} + \frac{6011964s^4}{25} \\
&\quad - \frac{17327169s^5}{625} + \frac{179727s^6}{100} - \frac{160351s^7}{2500} + s^8 \\
&= \frac{1}{2500}\bigl(1607040000 - 7843564800s + 6483240000s^2 - 2924003520s^3 \\
&\quad + 601196400s^4 - 69308676s^5 + 4493175s^6 - 160351s^7 + 2500s^8\bigr) \\
&= \frac{1}{2500}\Biggl(1607040000 - 7843564800s + \left(\frac{7}{8}\right)6483240000s^2 \\
&\quad + s^2\left(\left(\frac{1}{8}\right)6483240000 - 2924003520s + 601196400s^2 - 69308676s^3\right) \\
&\quad + s^6\bigl(4493175 - 160351s + 2500s^2\bigr)\Biggr) \\
&= \frac{1}{2500}\bigl(F_4(s) + s^2F_5(s) + s^6F_6(s)\bigr).
\end{aligned}$$

We shall show that the functions $F_4(s) > 0$, $F_5(s) > 0$ and $F_6(s) > 0$. Here,

$$\begin{aligned}
F_4(s) &= 5400\bigl(297600 - 1452512s + 1050525s^2\bigr) \\
&= 5400F_7(t).
\end{aligned}$$

The derivative of $F_7(t)$ is

$$\begin{aligned} F_7'(s) &= 2(-726256 + 1050525s) \\ &\leq 2\left(-726256 + 1050525\left(\frac{1}{4}\right)\right) \\ &= -\frac{1854499}{2}. \end{aligned}$$

Since $F_7(s)$ is strictly decreasing for $0 < s < 1/4$ and $F_7(1/4) = 2077/16$, we have $F_4(s) > 0$.

$$\begin{aligned} F_5(s) &= 36\left(22511250 - 81222320s + 16699900s^2 - 1925241s^3\right) \\ &> 36\left(22511250 - 81222320s - 1925241s^3\right) \\ &\geq 36\left(22511250 - 81222320\left(\frac{1}{4}\right) - 1925241\left(\frac{1}{4}\right)^3\right) \\ &= \frac{1854335151}{16} \end{aligned}$$

and

$$\begin{aligned} F_6(s) &> 4493175 - 160351\left(\frac{1}{4}\right) \\ &= \frac{17812349}{4}. \end{aligned}$$

Therefore, we can get $F_3(t) > 0$. By $120 - 20t^2 + t^4 > 0$, $24 - 12t^2 + t^4 > 0$ and $-720 + 360t^2 - 30t^4 + t^6 < 0$, thus $F_2(t) < 0$ and $F_1(t)$ is strictly decreasing for $0 < t < 1/2$. From $F_1(0+) = (\pi^2 - 4)^2 - 16$, we can get

$$F_1\left(\frac{1}{2}\right) \leq F_1(t) < \left(\pi^2 - 4\right)^2 - 16$$

for $0 < t \leq 1/2$. Next, we assume that $1/2 < t < \pi/2$. From inequality (2.6), we have

$$\begin{aligned} &-8\pi^2\left\{\frac{2}{\pi^2} + \frac{8}{\pi^2 - 4t^2} - \frac{(10 - \pi^2)(\pi^2 - 4t^2)}{\pi^4}\right\} \\ &\quad + \pi^2(\pi - 2t)(\pi + 2t)\left\{\frac{2}{\pi^2} + \frac{8}{\pi^2 - 4t^2} - \frac{(-9 + \pi^2)(\pi^2 - 4t^2)}{6\pi^4}\right\}^2 \\ &< F_1(t) \\ &< -8\pi^2\left\{\frac{2}{\pi^2} + \frac{8}{\pi^2 - 4t^2} - \frac{(-9 + \pi^2)(\pi^2 - 4t^2)}{6\pi^4}\right\} \\ &\quad + \pi^2(\pi - 2t)(\pi + 2t)\left\{\frac{2}{\pi^2} + \frac{8}{\pi^2 - 4t^2} - \frac{(10 - \pi^2)(\pi^2 - 4t^2)}{\pi^4}\right\}^2 \end{aligned}$$

and

$$\frac{G_1(t)}{36\pi^6} < F_1(t) < \frac{G_2(t)}{3\pi^6},$$

where

$$\begin{aligned} G_1(t) &= 4761\pi^6 - 426\pi^8 + \pi^{10} - 18252\pi^4 t^2 + 1944\pi^6 t^2 - 12\pi^8 t^2 \\ &\quad + 7344\pi^2 t^4 - 1248\pi^4 t^4 + 48\pi^6 t^4 - 5184 t^6 + 1152\pi^2 t^6 - 64\pi^4 t^6 \end{aligned}$$

and

$$\begin{aligned} G_2(t) &= -276\pi^6 + 4\pi^8 + 3\pi^{10} - 624\pi^4 t^2 + 416\pi^6 t^2 - 36\pi^8 t^2 \\ &\quad + 12480\pi^2 t^4 - 2688\pi^4 t^4 + 144\pi^6 t^4 - 19200 t^6 + 3840\pi^2 t^6 - 192\pi^4 t^6. \end{aligned}$$

We set $s = t^2$, then

$$\begin{aligned} G_1(t) &= 4761\pi^6 - 426\pi^8 + \pi^{10} - 12\pi^4\left(1521 - 162\pi^2 + \pi^4\right)s \\ &\quad + 48(-3+\pi)\pi^2(3+\pi)\left(-17+\pi^2\right)s^2 - 64(-3+\pi)^2(3+\pi)^2 s^3 \\ &= G_3(s) \end{aligned}$$

and

$$\begin{aligned} G_2(t) &= -276\pi^6 + 4\pi^8 + 3\pi^{10} - 4\pi^4\left(156 - 104\pi^2 + 9\pi^4\right)s \\ &\quad + 48\pi^2\left(\pi^2 - 10\right)\left(-26 + 3\pi^2\right)s^2 - 192\left(\pi^2 - 10\right)^2 s^3 \\ &= G_4(s). \end{aligned}$$

The derivatives of $G_3(s)$ are

$$\begin{aligned} G_3'(s) &= 12\left(-1521\pi^4 + 162\pi^6 - \pi^8 + 1224\pi^2 s - 208\pi^4 s \right. \\ &\quad \left. + 8\pi^6 s - 1296 s^2 + 288\pi^2 s^2 - 16\pi^4 s^2\right) \end{aligned}$$

and

$$G_3''(t) = 96(-3+\pi)(3+\pi)\left(-17\pi^2 + \pi^4 + 36s - 4\pi^2 s\right).$$

From the inequality

$$\begin{aligned} -17\pi^2 + \pi^4 + \left(36 - 4\pi^2\right)s &< -17\pi^2 + \pi^4 + \left(36 - 4\pi^2\right)\left(\frac{1}{4}\right) \\ &= 9 - 18\pi^2 + \pi^4 \\ &\cong -71.2438, \end{aligned}$$

$G_3''(s) < 0$ and $G_3'(s)$ is strictly decreasing for $1/4 < s < \pi^2/4$. Since $G_3'(1/4) = 12(-81 + 324\pi^2 - 1574\pi^4 + 164\pi^6 - \pi^8) \cong -24310.3$, $G_3'(s) < 0$ and $G_3(s)$ is strictly decreasing for $1/4 < s < \pi^2/4$. Therefore, we have $G_1(t) > G_3(\pi^2/4) = 576\pi^6$ for $1/2 < t < \pi/2$. Next, the derivatives of $G_4(s)$ are

$$\begin{aligned}G_4'(s) = 4\big(&-156\pi^4 + 104\pi^6 - 9\pi^8 + 6240\pi^2 s - 1344\pi^4 s + 72\pi^6 s\\ &- 14400s^2 + 2880\pi^2 s^2 - 144\pi^4 s^2\big)\end{aligned}$$

and

$$G_4''(s) = 96\big(10 - \pi^2\big)\big(26\pi^2 - 3\pi^4 - 120s + 12\pi^2 s\big).$$

From the inequality

$$\begin{aligned}26\pi^2 - 3\pi^4 - 120s + 12\pi^2 s &< 26\pi^2 - 3\pi^4 + \big(-120 + 12\pi^2\big)\left(\frac{1}{4}\right)\\ &= -30 + 29\pi^2 - 3\pi^4\\ &\cong -36.0087,\end{aligned}$$

$G_4''(s) < 0$ and $G_4'(s)$ is strictly decreasing for $1/4 < s < \pi^2/4$. Since $G_4'(1/4) = 4(-900 + 1740\pi^2 - 501\pi^4 + 122\pi^6 - 9\pi^8) \cong -2544.56$, $G_4'(s) < 0$ and $G_4(s)$ is strictly decreasing for $1/4 < s < \pi^2/4$. Therefore, we have $G_2(t) > G_4(\pi^2/4) = 48\pi^6$ for $1/2 < t < \pi/2$. By the squeeze theorem, $F_1(t) > 16$ for $1/2 < t < \pi/2$. Also, we have

$$F_1(t) < \frac{G_2(\frac{1}{2})}{3\pi^6}$$

for $1/2 < t < \pi/2$ and

$$\begin{aligned}F_1(0+) - \frac{G_2(\frac{1}{2})}{3\pi^6} &= \big(\pi^2 - 4\big)^2 - 16 - \frac{G_2(\frac{1}{2})}{3\pi^6}\\ &= \big(\pi^2 - 4\big)^2 - 16 - \frac{-300 + 840\pi^2 - 327\pi^4 - 163\pi^6 - 5\pi^8 + 3\pi^{10}}{3\pi^6}\\ &= \frac{300 - 840\pi^2 + 327\pi^4 + 163\pi^6 - 19\pi^8}{3\pi^6}.\end{aligned}$$

By $300 - 840\pi^2 + 327\pi^4 + 163\pi^6 - 19\pi^8 \cong 286.654$, we have

$$F_1(0+) > \frac{G_2(\frac{1}{2})}{3\pi^6}.$$

Thus, we can get $16 < F_1(t) < F_1(0+)$ for $0 < t < \pi/2$. The proof of Theorem 2.1 is complete. □

2.2 Proof of Theorem 2.2

Proof of Theorem 2.2 We have

$$\begin{aligned}F_1(x) &= \frac{\pi^2 x}{4+\sqrt{(\pi^2-4)^2+(2\pi x)^2}} - \frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} \\ &= \frac{x(-96+9\pi^2+\sqrt{15}\pi^2\sqrt{15+16x^2}-24\sqrt{16-8\pi^2+\pi^4+4\pi^2x^2})}{(9+\sqrt{15}\sqrt{15+16x^2})(4+\sqrt{16-8\pi^2+\pi^4+4\pi^2x^2})} \\ &= \frac{xF_2(x)}{(9+\sqrt{15}\sqrt{15+16x^2})(4+\sqrt{16-8\pi^2+\pi^4+4\pi^2x^2})}.\end{aligned}$$

The derivative of $F_2(x)$ is

$$\begin{aligned}F_2'(x) &= \frac{16\pi^2 x(-6\sqrt{15+16x^2}+\sqrt{15}\sqrt{16-8\pi^2+\pi^4+4\pi^2x^2})}{\sqrt{15+16x^2}\sqrt{16-8\pi^2+\pi^4+4\pi^2x^2}} \\ &= \frac{16\pi^2 xF_3(x)}{\sqrt{15+16x^2}\sqrt{16-8\pi^2+\pi^4+4\pi^2x^2}}.\end{aligned}$$

Here, we have $15(16-8\pi^2+\pi^4+4\pi^2x^2)-36(15+16x^2) = 3(-100-40\pi^2+5\pi^4-192x^2+20\pi^2x^2)$. Since $-192+20\pi^2>0$ and $-100-40\pi^2+5\pi^4-192x^2+20\pi^2x^2=0$ for $x=\sqrt{\frac{100+40\pi^2-5\pi^4}{20\pi^2-192}} \cong 1.198$, we have $F_3(x)<0$ for $0<x<\sqrt{\frac{100+40\pi^2-5\pi^4}{20\pi^2-192}}$ and $F_3(x)>0$ for $x>\sqrt{\frac{100+40\pi^2-5\pi^4}{20\pi^2-192}}$. Therefore, $F_2(x)$ is strictly decreasing for $0<x<\sqrt{\frac{100+40\pi^2-5\pi^4}{20\pi^2-192}}$ and strictly increasing for $x>\sqrt{\frac{100+40\pi^2-5\pi^4}{20\pi^2-192}}$. From $F_2(0+)=0$ and

$$\begin{aligned}F_2(\alpha) &= -96+9\pi^2+\sqrt{15}\pi^2\sqrt{15+16\alpha^2}-24\sqrt{16-8\pi^2+\pi^4+4\pi^2\alpha^2} \\ &= -96+9\pi^2+\sqrt{15}\pi^2\left(\frac{\sqrt{15}(112-11\pi^2)}{-48+5\pi^2}\right)-24\left(\frac{192+32\pi^2-5\pi^4}{-48+5\pi^2}\right) \\ &= 0,\end{aligned}$$

we can get $F_2(x)>0$ for $x>\alpha$ and α is the best possible. The proof of Theorem 2.2 is complete. □

2.3 Proof of Theorem 2.3

Proof of Theorem 2.3 We have

$$\begin{aligned}F_1(x) &= \frac{8x}{3+\sqrt{25+\frac{256}{\pi^2}x^2}} - \frac{\pi^2 x}{4+\sqrt{32+(2\pi x)^2}} \\ &= \frac{\pi x(32-3\pi^2-\pi\sqrt{25\pi^2+256x^2}+16\sqrt{8+\pi^2x^2})}{2(3\pi+\sqrt{25\pi^2+256x^2})(2+\sqrt{8+\pi^2x^2})} \\ &= \frac{\pi xF_2(x)}{2(3\pi+\sqrt{25\pi^2+256x^2})(2+\sqrt{8+\pi^2x^2})}.\end{aligned}$$

The derivative of $F_2(x)$ is

$$F_2'(x) = \frac{16\pi x(\pi\sqrt{25\pi^2 + 256x^2} - 16\sqrt{8 + \pi^2x^2})}{\sqrt{25\pi^2 + 256x^2}\sqrt{8 + \pi^2x^2}} = \frac{16\pi x F_3(x)}{\sqrt{25\pi^2 + 256x^2}\sqrt{8 + \pi^2x^2}}.$$

Since $\pi^2(25\pi^2 + 256x^2) - 16^2(8 + \pi^2x^2) = -2048 + 25\pi^4 \cong 387.227$, we can get $\pi^2(25\pi^2 + 256x^2) > 16^2(8 + \pi^2x^2)$ for $x > 0$. Therefore, $F_3(x) > 0$ and $F_2'(x) > 0$ for $x > 0$. Since $F_2(x)$ is strictly increasing for $x > 0$ and

$$\begin{aligned} F_2(\beta) &= 32 - 3\pi^2 - \pi\sqrt{25\pi^2 + 256\beta^2} + 16\sqrt{8 + \pi^2\beta^2} \\ &= 32 - 3\pi^2 - \pi\left(\frac{512 + 96\pi^2 - 17\pi^4}{\pi(-32 + 3\pi^2)}\right) + 16\left(\frac{192 - 12\pi^2 - \pi^4}{2(-32 + 3\pi^2)}\right) \\ &= 0, \end{aligned}$$

we can get $F_2(x) > 0$ for $x > \beta$ and β is the best possible. The proof of Theorem 2.3 is complete. □

2.4 Proof of Theorem 2.4

Lemma 2.6 *For $x > 0$, we have*

$$\frac{75600\pi^2x}{\sqrt{8 + \pi^2x^2}} + \frac{64\pi^2x^7}{\sqrt{8 + \pi^2x^2}} > \frac{25200\sqrt{15}\pi^2x}{\sqrt{15 + 16x^2}}.$$

Proof We have

$$\left(\frac{75600\pi^2x}{\sqrt{8 + \pi^2x^2}} + \frac{64\pi^2x^7}{\sqrt{8 + \pi^2x^2}}\right)^2 - \left(\frac{25200\sqrt{15}\pi^2x}{\sqrt{15 + 16x^2}}\right)^2 = \frac{256\pi^4x^2F_1(x)}{(15 + 16x^2)(8 + \pi^2x^2)},$$

where $F_1(x) = 37209375 + 357210000x^2 - 37209375\pi^2x^2 + 567000x^6 + 604800x^8 + 240x^{12} + 256x^{14}$. Here, we have

$$\begin{aligned} F_1(x) &> 37209375 + 357210000x^2 - 37209375\pi^2x^2 + 567000x^6 \\ &= 70875(525 + 5040x^2 - 525\pi^2x^2 + 8x^6). \end{aligned}$$

We set $t = x^2$ and $F_2(t) = 525 + 5040t - 525\pi^2t + 8t^3$, then the derivative of $F_2(t)$ is $F_2'(t) = 5040 - 525\pi^2 + 24t^2$. Since $F_2'(t) = 0$ for $t = \frac{1}{2}\sqrt{\frac{1}{2}(-1680 + 175\pi^2)} \cong 2.4285$, we have $F_2'(t) < 0$ for $0 < t < \frac{1}{2}\sqrt{\frac{1}{2}(-1680 + 175\pi^2)}$ and $F_2'(t) > 0$ for $t > \frac{1}{2}\sqrt{\frac{1}{2}(-1680 + 175\pi^2)}$. Hence,

$$\begin{aligned} F_2(t) &\geq F_2\left(\frac{1}{2}\sqrt{\frac{1}{2}(-1680 + 175\pi^2)}\right) \\ &= \frac{35}{2}\left(30 + 48\sqrt{70(-48 + 5\pi^2)} - 5\pi^2\sqrt{70(-48 + 5\pi^2)}\right) \\ &\cong 295.843 \end{aligned}$$

for $t > 0$. Therefore, $F_1(x) > 0$ and the proof of Lemma 2.6 is complete. □

Proof of Theorem 2.4 We have

$$F_1(x) = \frac{8x + \frac{32x^7}{4725}}{3 + \sqrt{25 + \frac{80}{3}x^2}} - \frac{\pi^2 x}{4 + \sqrt{32 + (2\pi x)^2}}$$

$$= \frac{xF_2(x)}{3150(9 + \sqrt{15}\sqrt{15 + 16x^2})(2 + \sqrt{8 + \pi^2 x^2})},$$

where $F_2(x) = 151200 - 14175\pi^2 + 128x^6 - 1575\sqrt{15}\pi^2\sqrt{15 + 16x^2} + 75600\sqrt{8 + \pi^2 x^2} + 64x^6\sqrt{8 + \pi^2 x^2}$. The derivative of $F_2(x)$ is

$$F_2'(x) = 768x^5 - \frac{25200\sqrt{15}\pi^2 x}{\sqrt{15 + 16x^2}} + \frac{75600\pi^2 x}{\sqrt{8 + \pi^2 x^2}} + \frac{64\pi^2 x^7}{\sqrt{8 + \pi^2 x^2}} + 384x^5\sqrt{\pi^2 x^2 + 8}$$

$$> -\frac{25200\sqrt{15}\pi^2 x}{\sqrt{15 + 16x^2}} + \frac{75600\pi^2 x}{\sqrt{8 + \pi^2 x^2}} + \frac{64\pi^2 x^7}{\sqrt{8 + \pi^2 x^2}}.$$

By Lemma 2.6, we have $F_2'(x) > 0$ and $F_2(x)$ is strictly increasing for $x > 0$. From $F_2(0+) = 37800(4 + 4\sqrt{2} - \pi^2) \cong -8041.96$, $F_2(\gamma) = 0$ and $F(\infty) = \infty$, we can get $F_2(x) > 0$ for $x > \gamma$. The proof of Theorem 2.4 is complete. □

3 Conclusions

In this paper, we established some inequalities involving arctangent. The double inequality in Theorem 2.1 provides sharper quadratic estimations than (1.2) and (1.3) for a location away from zero. By Theorems 2.2, 2.3 and 2.4, we obtained Proposition 2.5 immediately.

Competing interests
The author declares that he has no competing interests.

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