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A new kind of Bernstein-Schurer-Stancu-Kantorovich-type operators based on q-integers

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Abstract

Agrawal *et al.* (Boll. Unione Mat. Ital. 8:169-180, 2015) introduced a Stancu-type Kantorovich modification of the operators proposed by Ren and Zeng (Bull. Korean Math. Soc. 50(4):1145-1156, 2013) and studied a basic convergence theorem by using the Bohman-Korovokin criterion, the rate of convergence involving the modulus of continuity, and the Lipschitz function. The concern of this paper is to obtain Voronoskaja-type asymptotic result by calculating an estimate of fourth order central moment for these operators and discuss the rate of convergence for the bivariate case by using the complete and partial moduli of continuity and the degree of approximation by means of a Lipschitz-type function and the Peetre K-functional. Also, we consider the associated GBS (generalized Boolean sum) operators and estimate the rate of convergence for these operators with the help of a mixed modulus of smoothness. Furthermore, we show the rate of convergence of these operators (univariate case) to certain functions with the help of the illustrations using Maple algorithms and in the bivariate case, the rate of convergence of these operators is compared with the associated GBS operators by illustrative graphics.

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Keywords: q-Bernstein-Schurer-Kantorovich; rate of convergence; modulus of continuity; GBS operators; mixed modulus of continuity

1 Introduction

Following [3], for any fixed real number $q > 0$, satisfying the condition $0 < q < 1$, the q-integer $[k]_q$, for $k \in \mathbb{N}$ and q-factorial $[k]_q!$ are defined as

$$[k]_q = \begin{cases} \frac{(1-q^k)}{(1-q)}, & \text{if } q \neq 1, \\ k, & \text{if } q = 1, \end{cases}$$

and

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots 1, & \text{if } k \geq 1, \\ 1, & \text{if } k = 0, \end{cases}$$

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respectively. For any integers n, k satisfying $0 \leq k \leq n$, the q-binomial coefficient is given by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

The q-analogue of $(1-x)^n$ is given by

$$(1-x)_q^n = \begin{cases} \prod_{j=0}^{n-1}(1-q^j x), & n = 1,2,\ldots, \\ 1, & n = 0. \end{cases}$$

The q-integration in the interval $[0,a]$ is defined by

$$\int_0^a f(t)\,d_q t = a(1-q)\sum_{n=0}^{\infty} f\bigl(aq^n\bigr)q^n, \quad 0 < q < 1,$$

provided the series converges.

Let $I = [0, 1+p]$ and $p \in \mathbb{N} \cup \{0\}$. For $f \in C(I)$, the space of all continuous functions on I endowed with the norm $\|f\| = \sup_{x\in[0,1+p]} |f(x)|$ and $0 < q < 1$, Ren and Zeng [2] defined the following new version of the q-Bernstein-Schurer operator which preserves the linear functions:

$$\overline{B}_n^p\bigl(f(t);q,x\bigr) = \sum_{k=0}^{n+p} \tilde{p}_{n,k}^*(q,x) f\biggl(\frac{[k]_q}{[n]_q}\biggr), \tag{1.1}$$

where

$$\tilde{p}_{n,k}^*(q,x) = \frac{[n]_q^{n+p}}{[n+p]_q^{n+p}} \binom{n+p}{k}_q x^k \biggl(\frac{[n+p]_q}{[n]_q} - x\biggr)_q^{n+p-k}.$$

Later, Acu [4] proposed a q-Durrmeyer modification of the operators (1.1) as

$$D_{n,p}(f;q,x) = \frac{[n+p+1]_q[n]_q}{[n+p]_q} \sum_{k=0}^{n+p} \tilde{p}_{n,k}^*(q,x) \int_0^{\frac{[n+p]_q}{[n]_q}} f(t)\tilde{b}_{n,k}^p(q,qt)\,d_q t \tag{1.2}$$

and discussed the rate of convergence in terms of the modulus of continuity, a Lipschitz class function, and a Voronovskaja-type result. Subsequently, for $\alpha, \beta \in \mathbb{R}$ such that $0 \leq \alpha \leq \beta$ and $f \in C(I)$, Agrawal *et al.* [1] introduced a Stancu-type Kantorovich modification of the operators (1.1), defined as

$$\mathcal{K}_{n,p}^{(\alpha,\beta)}(f;q,x) = \sum_{k=0}^{n+p} \tilde{p}_{n,k}^*(q,x) \int_0^1 f\biggl(\frac{[k]_q + q^k t + \alpha}{[n+1]_q + \beta}\biggr)\,d_q t, \tag{1.3}$$

and discussed the basic convergence theorem, the rate of convergence involving modulus of continuity and Lipschitz function. Significant contributions have been made by researchers in this area of approximation theory (*cf.* [5] and the references their in).

The purpose of this paper is to discuss the Voronoskaja asymptotic result by calculating an estimate of the fourth order central moment for the operators (1.3) and construct the bivariate case of these operators. We obtain the rate of approximation of the bivariate operators by using the complete and partial moduli of continuity and the degree of approximation with the aid of a Lipschitz-type space and the Peetre K-functional. Lastly, we consider the associated GBS (generalized Boolean sum) operators and study the approximation of Bögel continuous and Bögel differentiable functions by means of the mixed modulus of smoothness.

Lemma 1 ([1]) *For the operators given by* (1.3), *the following equalities hold*:

(i) $\mathcal{K}_{n,p}^{(\alpha,\beta)}(1;q,x)=1$;

(ii) $\mathcal{K}_{n,p}^{(\alpha,\beta)}(t;q,x)=\frac{\alpha}{[n+1]_q+\beta}+\frac{2q[n]_qx+1}{[2]_q([n+1]_q+\beta)}$;

(iii) $\mathcal{K}_{n,p}^{(\alpha,\beta)}(t^2;q,x)=\frac{1}{[2]_q[3]_q([n+1]_q+\beta)^2}\{\frac{[n]_q^2[n+p-1]_q}{[n+p]_q}([3]_qq^2+3q^4)x^2+\{(4\alpha+3)q[3]_q+q^2(1+[2]_q)\}[n]_qx+[4]_q\alpha^2+2\alpha[3]_q+(1+q\alpha^2)[2]_q\}$.

Lemma 2 ([1]) *For $m\in\mathbb{N}\cup\{0\}$, the mth order central moment of $\mathcal{K}_{n,p}^{(\alpha,\beta)}(f;q,x)$ defined as $\mu_{n,m,q}^*(x)=\mathcal{K}_{n,p}^{\alpha,\beta}((t-x)^m;q,x)$, we have*

(i) $\mu_{n,1,q}^*(x)=\frac{(2-[2]_q)q[n]_qx-(\beta+1)[2]_qx+1}{[2]_q([n+1]_q+\beta)}+\frac{\alpha}{[n+1]_q+\beta}$;

(ii) $\mu_{n,2,q}^*(x)=\{\frac{[n]_q^2[n+p-1]_q([3]_qq^2+3q^4)}{[n+p]_q([n+1]_q+\beta)^2[2]_q[3]_q}-\frac{4q[n]_q}{[2]_q[n+1]_q+\beta}+1\}x^2+\{\frac{\{(4\alpha+3)[3]_qq+q^2(1+[2]_q)\}[n]_q}{([n+1]_q+\beta)^2[2]_q[3]_q}-\frac{2\alpha}{[n+1]_q+\beta}-\frac{2}{[2]_q[n+1]_q+\beta}\}x+\frac{[4]_q\alpha^2+2\alpha[3]_q+(1+q\alpha^2)[2]_q}{([n+1]_q+\beta)^2[2]_q[3]_q}$.

In the following we obtain an estimate of the fourth order central moment of the operators defined by (1.3).

By the definition of the Jackson integral and the inequality $(a+b)^4\leq 8(a^4+b^4)$, where $a>0$, $b>0$, and Lemma 2.4 *in* [2], *we have*

$$\begin{aligned}
\mathcal{K}_{n,p}^{(\alpha,\beta)}\big((t-x)^4;q_n,x\big) &= [n+1]_{q_n}\sum_{k=0}^{n+p}\tilde{p}_{n,k}^*(q_n,x)\int_0^1\left(\frac{[k]_{q_n}+q_n^kt+\alpha}{[n+1]_{q_n}+\beta}-x\right)^4 d_{q_n}t\\
&= \sum_{k=0}^{n+p}\tilde{p}_{n,k}^*(q_n,x)(1-q_n)\sum_{j=0}^{\infty}\left(\frac{[k]_{q_n}+q_n^kq_n^j+\alpha}{[n+1]_{q_n}+\beta}-x\right)^4\times q_n^j\\
&\leq 8\sum_{k=0}^{n+p}\tilde{p}_{n,k}^*(q_n,x)\left(\frac{[k]_{q_n}+\alpha}{[n+1]_{q_n}+\beta}-x\right)^4\\
&\quad+8(1-q_n)\sum_{k=0}^{n+p}\tilde{p}_{n,k}^*(q_n,x)\sum_{j=0}^{\infty}\left(\frac{q_n^k}{[n+1]_{q_n}+\beta}\right)^4q_n^{5j}\\
&\leq 64\sum_{k=0}^{n+p}p_{n,k}^*(q_n,x)\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}\right)^4\left(\frac{q_n^n}{[n+1]_{q_n}+\beta}\right)^4\\
&\quad+64\sum_{k=0}^{n+p}\tilde{p}_{n,k}^*(q_n,x)\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}-x\right)^4\\
&\quad+\frac{8}{1+q_n+q_n^2+q_n^3+q_n^4}\sum_{k=0}^{n+p}\tilde{p}_{n,k}^*(q_n,x)\left(\frac{q_n^k}{[n+1]_{q_n}}\right)^4\\
&= 64\sum_{k=0}^{n+p}p_{n,k}^*(q_n,x)\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}\right)^4\left(\frac{q_n^n}{[n+1]_{q_n}+\beta}\right)^4
\end{aligned}$$

$$+ 64\overline{B}_n^p\left(\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}-x\right)^4;q,x\right)$$
$$+ \frac{8}{1+q_n+q_n^2+q_n^3+q_n^4}\sum_{k=0}^{n+p}\tilde{p}_{n,k}^*(q_n,x)\left(\frac{q_n^k}{[n+1]_{q_n}}\right)^4$$
$$\leq 64\frac{1}{[n]_{q_n}^2}+64M_2\frac{1/4}{[n]_{q_n}^2}+\frac{8}{[n]_{q_n}^2}=\frac{64+16M_2+8}{[n]_{q_n}^2}. \tag{1.4}$$

In the following, let $(q_n)_n$, $0 < q_n < 1$ be a sequence satisfying $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} q_n^n = a$ $(0 \leq a < 1)$.

2 Voronovskaja-type theorem

Let $C^2[0,1+p]$ denote the space of twice continuously differentiable functions on $[0,1+p]$.

Theorem 1 *For any* $f \in C^2[0,1+p]$,

$$\lim_{n\to\infty}[n]_{q_n}\big(\mathcal{K}_{n,p}^{(\alpha,\beta)}(f;q_n,x)-f(x)\big)=\left(\frac{-x(a+1+2\beta)}{2}+\alpha+\frac{1}{2}\right)f'(x)-x^2f''(x)$$

uniformly in $[0,1]$.

Proof Using Taylor's expansion for f, we obtain

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)(t-x)^2}{2}+\xi(t,x)(t-x)^2, \tag{2.1}$$

where the function $\xi(t,x)$ is the Peano form of the remainder, $\xi(t,x) \in C[0,1+p]$, and $\lim_{t\to x}\xi(t,x) = 0$.

By linearity of the operators $\mathcal{K}_{n,p}^{(\alpha,\beta)}(;q_n,x)$ and using Lemma 2, we get

$$\begin{aligned}\lim_{n\to\infty}[n]_{q_n}\big(\mathcal{K}_{n,p}^{(\alpha,\beta)}(f;q_n,x)-f(x)\big)&=\left(\frac{-x(a+1+2\beta)}{2}+\alpha+\frac{1}{2}\right)f'(x)-x^2f''(x)\\&\quad+\lim_{n\to\infty}[n]_{q_n}\mathcal{K}_{n,p}^{(\alpha,\beta)}\big(\xi(t,x)(t-x)^2;q_n,x\big)\end{aligned} \tag{2.2}$$

uniformly in $[0,1]$.

For the last term of the right side, using the Cauchy-Schwarz inequality, we are led to

$$[n]_{q_n}\mathcal{K}_{n,p}^{(\alpha,\beta)}\big(\xi(t,x)(t-x)^2;q_n,x\big)\leq[n]_{q_n}\sqrt{\mathcal{K}_{n,p}^{(\alpha,\beta)}\big(\xi^2(t,x);q_n,x\big)}\sqrt{\mathcal{K}_{n,p}^{(\alpha,\beta)}\big((t-x)^4;q_n,x\big)}.$$

We observe that $\xi^2(t,x) \in C[0,1+p]$ and $\xi^2(x,x) = 0$, hence, by Theorem 1

$$\lim_{n\to\infty}\mathcal{K}_{n,p}^{(\alpha,\beta)}\big(\xi^2(t,x);q_n,x\big)=\xi^2(x,x)=0,\quad \text{uniformly with respect to } x\in[0,1].$$

Further using (1.4), $\lim_{n\to\infty}[n]_{q_n}\sqrt{\mathcal{K}_{n,p}^{(\alpha,\beta)}((t-x)^4;q_n,x)}$ is finite.

Hence,

$$\lim_{n\to\infty}[n]_{q_n}\mathcal{K}_{n,p}^{(\alpha,\beta)}\big(\xi(t,x)(t-x)^2;q_n,x\big)=0 \tag{2.3}$$

uniformly in $x \in [0,1]$. Finally, consideration of (2.2) and (2.3) completes the proof. □

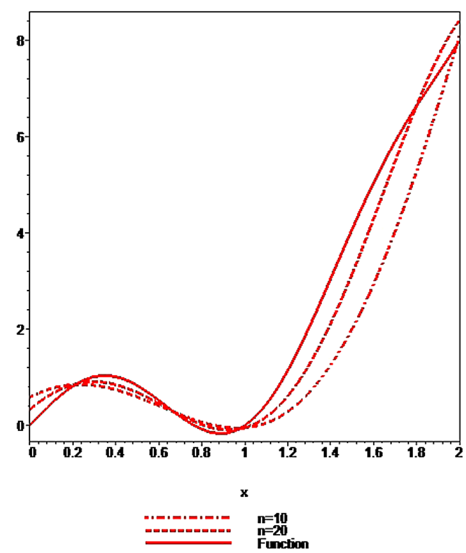


Figure 1 The convergence of $K_{n,p}^{(\alpha,\beta)}(f;q,x)$ to $f(x)$.

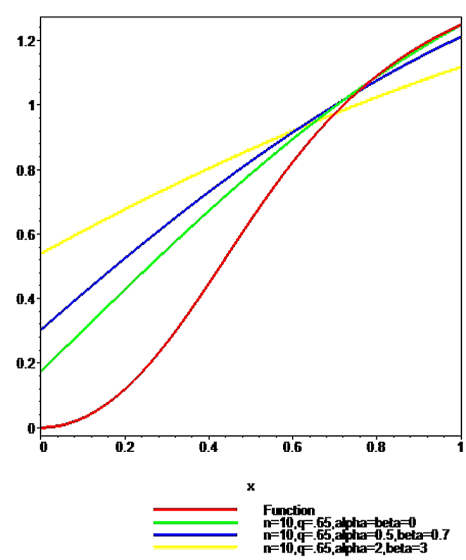


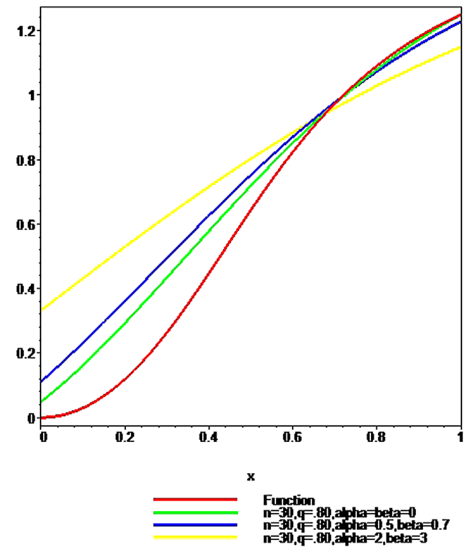
Figure 2 The convergence of $K_{n,p}^{(\alpha,\beta)}(f;q,x)$ to $f(x)$.

In the following examples, we illustrate the rate of convergence of the operators given by (1.3) to certain functions.

Example 1 Let $q_n = (n-1)/n$. For $\alpha = 0.5$, $\beta = 0.7$, $p = 1$ with $n = 10$ and 20, the convergence of $\mathcal{K}_{n,p}^{(\alpha,\beta)}(f;q,x)$ given by (1.3) to $f(x) = x^3 + \sin(3\pi x/2)$ is shown in Figure 1. It is observed that the approximation becomes better on increasing the value of n.

Example 2 Let $f(x) = \arctan(3x^2)$, $p = 0.80$, $n = 10$, $q = 0.65$ and $n = 30$, $q = 0.80$. For $\alpha = \beta = 0$, $\alpha = 0.5$, $\beta = 0.7$ and $\alpha = 2$, $\beta = 3$ the convergence of $\mathcal{K}_{n,p}^{(\alpha,\beta)}(f;q,x)$ to $f(x)$ is shown in Figures 2 and 3 respectively. It is observed that the approximation becomes better when the values of $\alpha, \beta \in [0,1)$ and the convergence is better in a small interval for larger values of α, β.

Figure 3 **The convergence of $K_{n,p}^{(\alpha,\beta)}(f;q,x)$ to $f(x)$.**



3 Construction of the bivariate operators

Let $C(I_1 \times I_2)$, where $I_1 = [0, 1 + p_1]$ and $I_2 = [0, 1 + p_2]$, denote the space of all real valued continuous functions on $I_1 \times I_2$ endowed with the norm

$$\|f\|_{C(I_1\times I_2)} = \sup_{(x,y)\in I_1\times I_2} \bigl|f(x,y)\bigr|.$$

For $f \in C(I_1 \times I_2)$, $0 < q_1, q_2 < 1$ and $J = [0,1]$, the bivariate generalization of the operators given by (1.3) is defined as

$$\begin{aligned}
&\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(f(t,s);q_1,q_2,x,y\bigr)\\
&\quad= \sum_{k_1=0}^{n_1+p_1}\sum_{k_2=0}^{n_2+p_2} \tilde{p}^*_{n_1,n_2,k_1,k_2}(q_1,q_2;x,y)\\
&\qquad\times \int_0^1\int_0^1 f\bigl(\Psi_{n_1,k_1,q_1}^{\alpha_1,\beta_1}(t), \Psi_{n_2,k_2,q_2}^{\alpha_2,\beta_2}(s)\bigr)\,d_{q_1}t\,d_{q_2}s,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&\tilde{p}^*_{n_1,n_2,k_1,k_2}(q_1,q_2,x,y)\\
&\quad= \frac{[n_1]_{q_1}^{n_1+p_1}}{[n_1+p_1]_{q_1}^{n_1+p_1}}\begin{bmatrix} n_1+p_1\\ k_1\end{bmatrix}_{q_1} x^{k_1}\left(\frac{[n_1+p_1]_{q_1}}{[n_1]_{q_1}} - x\right)_{q_1}^{n_1+p_1-k_1}\\
&\qquad\times \frac{[n_2]_{q_2}^{n_2+p_2}}{[n_2+p_2]_{q_2}^{n_2+p_2}}\begin{bmatrix} n_2+p_2\\ k_2\end{bmatrix}_{q_2} y^{k_2}\left(\frac{[n_2+p_2]_{q_2}}{[n_2]_{q_2}} - y\right)_{q_2}^{n_2+p_2-k_2}, \quad x,y\in J \quad\text{and}\\
&\Psi_{n_1,k_1,q_1}^{\alpha_1,\beta_1}(t) = \frac{[k_1]_{q_1} + q_1^{k_1}t + \alpha_1}{[n_1+1]+\beta_1}, \qquad \Psi_{n_2,k_2,q_2}^{\alpha_2,\beta_2}(s) = \frac{[k_2]_{q_2} + q_2^{k_2}s + \alpha_2}{[n_2+1]+\beta_2}.
\end{aligned}$$

Lemma 3 *let $e_{ij}(t,s) = t^i s^j$, $(t,s) \in (I_1 \times I_2)$, $(i,j) \in N^0 \times N^0$ with $i + j \le 2$ be the two dimensional test functions. Then the following equalities hold for the operators* (3.1):

(i) $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(e_{00};q_1,q_2,x,y) = 1$;
(ii) $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(e_{10};q_1,q_2,x,y) = \frac{\alpha_1}{[n_1+1]_{q_1}+\beta_1} + \frac{2q_1[n_1]_{q_1}x+1}{[2]_{q_1}([n_1+1]_{q_1}+\beta_1)}$;
(iii) $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(e_{01};q_1,q_2,x,y) = \frac{\alpha_2}{[n_2+1]_{q_2}+\beta_2} + \frac{2q_2[n_2]_{q_2}y+1}{[2]_{q_2}([n_2+1]_{q_2}+\beta_2)}$;
(iv) $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(e_{20};q_1,q_2,x,y) = \frac{1}{[2]_{q_1}[3]_{q_1}([n_1+1]_{q_1}+\beta_1)^2}\{\frac{[n_1]_{q_1}^2[n_1+p_1-1]_{q_1}}{[n_1+p_1]_{q_1}}([3]_{q_1}q_1^2+3q_1^4)x^2 + \{(4\alpha_1+3)q_1[3]_{q_1}+q_1^2(1+[2]_{q_1})\}[n_1]_{q_1}x + [4]_{q_1}\alpha_1^2 + 2\alpha_1[3]_{q_1} + (1+q_1\alpha_1^2)[2]_{q_1}\}$.
(v) $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(e_{02};q_1,q_2,x,y) = \frac{1}{[2]_{q_2}[3]_{q_2}([n_2+1]_{q_2}+\beta_2)^2}\{\frac{[n_2]_{q_2}^2[n_2+p_2-1]_{q_2}}{[n_2+p_2]_{q_2}}([3]_{q_2}q_2^2+3q_2^4)y^2 + \{(4\alpha_2+3)q_2[3]_{q_2}+q_2^2(1+[2]_{q_2})\}[n_2]_{q_2}y + [4]_{q_2}\alpha_2^2 + 2\alpha_2[3]_{q_2} + (1+q_2\alpha_2^2)[2]_{q_2}\}$.

Proof We have $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(t^is^j;q_1,q_2,x,y) = \mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(t^i;q_1,x)\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}(s^j;q_2,y)$, for $0 \le i,j \le 2$.
By using Lemma 1, the proof of the lemma is straightforward. Hence the details are omitted. □

For $f \in C(I_1 \times I_2)$ and $\delta > 0$, the first order complete modulus of continuity for the bivariate case is defined as follows:

$$\omega(f;\delta_1,\delta_2) = \sup\{|f(t,s)-f(x,y)| : |t-x| \le \delta_1, |s-y| \le \delta_2\},$$

where $\delta_1,\delta_2 > 0$. Further $\omega(f;\delta_1,\delta_2)$ satisfies the following properties:
(a) $\omega(f;\delta_1,\delta_2) \to 0$ if $\delta_1 \to 0$ and $\delta_2 \to 0$,
(b) $|f(t,s)-f(x,y)| \le \omega(f;\delta_1,\delta_2)(1+\frac{|t-x|}{\delta_1})(1+\frac{|s-y|}{\delta_2})$.

Now, we give an estimate of the rate of convergence of the bivariate operators. In the following, let $0 < q_{n_i} < 1$ be sequences in $(0,1)$ such that $q_{n_i} \to 1$ and $q_{n_i}^{n_i} \to a_i$ $(0 \le a_i < 1)$, as $n_i \to \infty$ for $i = 1,2$. Further, let $\delta_{n_1}(x) = \mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}((t-x)^2;q_{n_1},x)$ and $\delta_{n_2}(y) = \mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}((s-y)^2;q_{n_2},y)$.

Theorem 2 *For $f \in C(I_1 \times I_2)$ and all $(x,y) \in J^2$, we have*

$$\left|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y) - f(x,y)\right| \le 4\omega\left(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(y)}\right).$$

Proof Since $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)$ is a linear positive operator, by the property (b) of bivariate modulus of continuity, Lemma 1, and the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(f(t,s);q_{n_1},q_{n_2},x,y\big) - f(x,y)\right| \\
&\quad\le \left(\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\left|f(t,s)-f(x,y)\right|;q_{n_1},q_{n_2},x,y\right) \\
&\quad\le \omega\left(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(y)}\right)\left(\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(1;q_{n_1},x) + \frac{1}{\sqrt{\delta_{n_1}(x)}}\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\big(|t-x|;q_{n_1},x\big)\right) \\
&\qquad\times\left(\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}(1;q_{n_2},y) + \frac{1}{\sqrt{\delta_{n_2}(y)}}\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big(|s-y|;q_{n_2},y\big)\right) \\
&\quad\le \omega\left(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(y)}\right)\left(1 + \frac{1}{\sqrt{\delta_{n_1}(x)}}\sqrt{\left(\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(t-x)^2;q_{n_1},x\right)}\right) \\
&\qquad\times\left(1 + \frac{1}{\sqrt{\delta_{n_2}(y)}}\sqrt{\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big((s-y)^2;q_{n_2},y\big)}\right),
\end{aligned}$$

we get the desired result. □

Theorem 3 *If $f(x,y)$ has continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, then the inequality*

$$\begin{aligned}
&\bigl|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\bigr|\\
&\quad\le M_1\lambda_{n_1}(x)+\omega\bigl(f_x',\delta_{n_1}(x)\bigr)\bigl(1+\sqrt{\delta_{n_1}(x)}\bigr)\\
&\qquad+M_2\lambda_{n_2}(y)+\omega\bigl(f_y',\delta_{n_2}(y)\bigr)\bigl(1+\sqrt{\delta_{n_2}(y)}\bigr),
\end{aligned}$$

where M_1, M_2 are the positive constants such that

$$\left|\frac{\partial f}{\partial x}\right|\le M_1,\qquad\left|\frac{\partial f}{\partial y}\right|\le M_2\quad(0\le x\le a,0\le y\le b)$$

and

$$\begin{aligned}
\lambda_{n_1}(x)&=\left|\frac{(2-[2]_{q_{n_1}})q_{n_1}[n_1]_{q_{n_1}}-(\beta_1+1)[2]_{q_{n_1}}}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}\right|x+\frac{(1+[2]_{q_{n_1}}\alpha_1)}{[n_1+1]_{q_{n_1}}+\beta_1};\\
\lambda_{n_2}(y)&=\left|\frac{(2-[2]_{q_{n_2}})q_{n_2}[n_2]_{q_{n_2}}-(\beta_2+1)[2]_{q_{n_2}}}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)}\right|y+\frac{(1+[2]_{q_{n_2}}\alpha_2)}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)}.
\end{aligned}$$

Proof From the mean value theorem we have

$$\begin{aligned}
f(t,s)-f(x,y)&=f(t,y)-f(x,y)+f(t,s)-f(t,y)\\
&=(t-x)\frac{\partial f(\xi_1,y)}{\partial x}+(s-y)\frac{\partial f(x,\xi_2)}{\partial y}\\
&=(t-x)\frac{\partial f(x,y)}{\partial x}+(t-x)\left(\frac{\partial f(\xi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right)+(s-y)\frac{\partial f(x,y)}{\partial y}\\
&\quad+(s-y)\left(\frac{\partial f(x,\xi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right),
\end{aligned}\tag{3.2}$$

where $x<\xi<t$ and $y<\xi_2<s$. Since

$$\begin{aligned}
&\left|\frac{\partial f(\xi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right|\le\omega\bigl(f_x';|t-x|\bigr)\le\left(1+\frac{|t-x|}{\delta_{n_1}}\right)\omega\bigl(f_x',\delta_{n_1}\bigr)\quad\text{and}\\
&\left|\frac{\partial f(x,\xi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right|\le\omega\bigl(f_y';|s-y|\bigr)\le\left(1+\frac{|s-y|}{\delta_{n_2}}\right)\omega\bigl(f_y',\delta_{n_2}\bigr)
\end{aligned}$$

for some $\delta_{n_1},\delta_{n_2}>0$, on applying the operator $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot;q_{n_1},q_{n_2},x,y)$ on both sides of (3.2), we have

$$\begin{aligned}
&\bigl|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\bigr|\\
&\quad\le M_1\bigl|\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(e_{10}-x;q_{n_1},x)\bigr|\\
&\qquad+\sum_{k_1=0}^{n_1+p_1}\sum_{k_2=0}^{n_2+p_2}\tilde{p}_{n_1,n_2,k_1,k_2}^{*}(q_{n_1},q_{n_2};x,y)\\
&\qquad\times\int_0^1\int_0^1\bigl|\Psi_{n_1,k_1,q_{n_1}}^{\alpha_1,\beta_1}(t)-x\bigr|\omega\bigl(f_x',\delta_{n_1}\bigr)\left(\frac{|\Psi_{n_1,k_1,q_{n_1}}^{\alpha_1,\beta_1}(t)-x|}{\delta_{n_1}}+1\right)d_{q_{n_1}}t\,d_{q_{n_2}}s
\end{aligned}$$

$$
\begin{aligned}
&+ M_2\left|\mathcal{K}_{n_2,p_2,q_{n_2}}^{(\alpha_2,\beta_2)}(e_{01}-y;y)\right| \\
&+ \sum_{k_1=0}^{n_1+p_1}\sum_{k_2=0}^{n_2+p_2}\tilde{p}^*_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2};x,y) \\
&\times \int_0^1\int_0^1 \left|\Psi_{n_2,k_2,q_{n_2}}^{\alpha_2,\beta_2}(s)-y\right|\omega\left(f'_y,\delta_{n_2}\right)\left(\frac{|\Psi_{n_2,k_2,q_{n_2}}^{\alpha_2,\beta_2}(s)-y|}{\delta_{n_2}}+1\right)d_{q_{n_1}}t\,d_{q_{n_2}}s.
\end{aligned}
$$

Now applying the Cauchy-Schwarz inequality

$$
\begin{aligned}
&\left|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\right| \\
&\quad\le M_1\left|\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(e_{10};q_{n_1},x)\right| \\
&\qquad+ \omega\left(f'_x,\delta_{n_1}\right)\left\{\sum_{k_1=0}^{n_1+p_1}\sum_{k_2=0}^{n_2+p_2}\tilde{p}^*_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2};x,y)\right. \\
&\qquad\left.\times\int_0^1\int_0^1\left(\Psi_{n_1,k_1,q_{n_1}}^{\alpha_1,\beta_1}(t)-x\right)^2 d_{q_{n_1}}t\,d_{q_{n_2}}s\right\}^{\frac{1}{2}} \\
&\qquad+ \frac{\omega(f'_x,\delta_{n_1})}{\delta_{n_1}}\sum_{k_1=0}^{n_1+p_1}\sum_{k_2=0}^{n_2+p_2}\tilde{p}^*_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2};x,y) \\
&\qquad\times\int_0^1\int_0^1\left(\Psi_{n_1,k_1,q_{n_1}}^{\alpha_1,\beta_1}(t)-x\right)^2 d_{q_{n_1}}t\,d_{q_{n_2}}s \\
&\qquad+ M_2\left|\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}(e_{01};q_{n_2},y)\right| \\
&\qquad+ \omega\left(f'_y,\delta_{n_2}\right)\left\{\sum_{k_1=0}^{n_1+p_1}\sum_{k_2=0}^{n_2+p_2}\tilde{p}^*_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2};x,y)\right. \\
&\qquad\left.\times\int_0^1\int_0^1\left(\Psi_{n_2,k_2,q_{n_2}}^{\alpha_2,\beta_2}(s)-y\right)^2 d_{q_{n_1}}t\,d_{q_{n_2}}s\right\}^{\frac{1}{2}} \\
&\qquad+ \frac{\omega(f'_y,\delta_{n_2})}{\delta_{n_2}}\sum_{k_1=0}^{n_1+p_1}\sum_{k_2=0}^{n_2+p_2}\tilde{p}^*_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2};x,y) \\
&\qquad\times\int_0^1\int_0^1\left(\Psi_{n_2,k_2,q_{n_2}}^{\alpha_2,\beta_2}(s)-y\right)^2 d_{q_{n_1}}t\,d_{q_{n_2}}s \\
&\quad= M_1\lambda_{n_1}(x)+\omega\left(f'_x,\delta_{n_1}\right)(1+\sqrt{\delta_{n_1}})+M_2\lambda_{n_2}(y)+\omega\left(f'_y,\delta_{n_2}\right)(1+\sqrt{\delta_{n_2}}),
\end{aligned}
$$

on choosing $\delta_{n_1}=\delta_{n_1}(x)$ and $\delta_{n_2}=\delta_{n_2}(y)$, we obtain the required result. □

3.1 Degree of approximation

In our next result, we study the degree of approximation for the bivariate operators by means of the Lipschitz class.

For $0<\xi_1\le 1$ and $0<\xi_2\le 1$, we define the Lipschitz class $\mathrm{Lip}_M(\xi_1,\xi_2)$ for the bivariate case as follows:

$$
\left|f(t,s)-f(x,y)\right|\le M|t-x|^{\xi_1}|s-y|^{\xi_2},
$$

where $(t,s),(x,y)\in(I_1\times I_2)$ are arbitrary.

Theorem 4 *Let* $f \in \mathrm{Lip}_M(\xi_1, \xi_2)$. *Then, for all* $(x,y) \in J^2$, *we have*

$$\left|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y) - f(x,y)\right| \le M\big(\delta_{n_1}(x)\big)^{\frac{\xi_1}{2}}\big(\delta_{n_2}(y)\big)^{\frac{\xi_2}{2}}.$$

Proof By our hypothesis, we can write

$$\begin{aligned}
&\left|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y) - f(x,y)\right| \\
&\quad\le \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(\left|f(t,s) - f(x,y)\right|;q_{n_1},q_{n_2},x,y\big) \\
&\quad\le M\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(|t-x|^{\xi_1}|s-y|^{\xi_2};q_{n_1},q_{n_2},x,y\big) \\
&\quad= M\big(\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}|t-x|^{\xi_1};q_{n_1},x\big)\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big(|s-y|^{\xi_2};q_{n_2},y\big).
\end{aligned}$$

Now, applying the Hölder's inequality with $u_1 = \frac{2}{\xi_1}$, $v_1 = \frac{2}{2-\xi_1}$ and $u_2 = \frac{2}{\xi_2}$ and $v_2 = \frac{2}{2-\xi_2}$, respectively, we have

$$\begin{aligned}
&\left|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y) - f(x)\right| \\
&\quad\le M\big(\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(t-x)^2;q_{n_1},x\big)^{\frac{\xi_1}{2}}\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(1;q_{n_1},x)^{\frac{2-\xi_1}{2}} \\
&\qquad\times \mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big((s-y)^2;q_{n_2},y\big)^{\frac{\xi_2}{2}}\mathcal{K}_{n_2,p_2}^{\alpha_2,\beta_2}(1;q_{n_2},y)^{\frac{2-\xi_2}{2}} \\
&\quad= M\big(\delta_{n_1}(x)\big)^{\frac{\xi_1}{2}}\big(\delta_{n_2}(y)\big)^{\frac{\xi_2}{2}}.
\end{aligned}$$

Hence, the proof is completed. □

Let $C^1(I_1 \times I_2)$ denote the space of all continuous functions on $I_1 \times I_2$ such that their first partial derivatives are continuous on $I_1 \times I_2$.

Theorem 5 *For* $f \in C^1(I_1 \times I_2)$ *and* $(x,y) \in J^2$ *we have*

$$\left|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y) - f(x,y)\right| \le \|f'_x\|_{C(I_1\times I_2)}\sqrt{\delta_{n_1}(x)} + \|f'_y\|_{C(I_1\times I_2)}\sqrt{\delta_{n_2}(y)}.$$

Proof Let $(x,y) \in J^2$ be a fixed point. Then by our hypothesis

$$f(t,s) - f(x,y) = \int_x^t f'_u(u,s)\,d_qu + \int_y^s f'_v(x,v)\,d_qv.$$

Now, operating by $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot;q_{n_1},q_{n_2},x,y)$ on both sides of the above equation, we are led to

$$\begin{aligned}
&\left|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y) - f(x,y)\right| \\
&\quad\le \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\left(\left|\int_t^x \left|f'_u(u,s)\right|\,d_qu\right|;q_{n_1},q_{n_2},x,y\right) \\
&\qquad+ \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\left(\left|\int_y^s \left|f'_v(x,v)\right|\,d_qv\right|;q_{n_1},q_{n_2},x,y\right).
\end{aligned}$$

Since $|\int_t^x |f'_u(u,s)|\,d_qu| \leq \|f'_x\|_{C(I_1\times I_2)}|t-x|$ and $|\int_y^s |f'_v(x,v)|\,d_qv| \leq \|f'_y\|_{C(I_1\times I_2)}|s-y|$, we have

$$\begin{aligned}&\big|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\big|\\&\quad\leq \|f'_x\|_{C(I_1\times I_2)}\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\big(|t-x|;q_{n_1},x\big)+\|f'_y\|_{C(I_1\times I_2)}\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big(|s-y|;q_{n_2},y\big).\end{aligned}$$

Applying the Cauchy-Schwarz inequality and Lemma 1, we have

$$\begin{aligned}&\big|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\big|\\&\quad\leq \|f'_x\|_{C(I_1\times I_2)}\sqrt{\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\big((t-x)^2;q_{n_1},x\big)}\sqrt{\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(1;q_{n_1},x)}\\&\qquad+\|f'_y\|_{C(I_1\times I_2)}\sqrt{\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big((s-y)^2;q_{n_2},y\big)}\sqrt{\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}(1;q_{n_2},y)}\\&\quad=\|f'_x\|_{C(I_1\times I_2)}\sqrt{\delta_{n_1}(x)}+\|f'_y\|_{C(I_1\times I_2)}\sqrt{\delta_{n_2}(y)}.\end{aligned}$$

This completes the proof of the theorem. □

For $f\in C(I_1\times I_2)$ and $\delta>0$, the partial moduli of continuity with respect to x and y are given by

$$\bar{\omega}_1(f;\delta)=\sup\big\{\big|f(x_1,y)-f(x_2,y)\big|:y\in I_2\text{ and }|x_1-x_2|\leq\delta\big\}$$

and

$$\bar{\omega}_2(f;\delta)=\sup\big\{\big|f(x,y_1)-f(x,y_2)\big|:x\in I_1\text{ and }|y_1-y_2|\leq\delta\big\}.$$

Clearly, both moduli of continuity satisfy the properties of the usual modulus of continuity.

Theorem 6 *If $f\in C(I_1\times I_2)$ and $(x,y)\in J^2$, then we have*

$$|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\leq 2\big\{\big(\bar{\omega}_1\big(f;\sqrt{\delta_{n_1}(x)}\big)\big)+\big(\bar{\omega}_2\big(f;\sqrt{\delta_{n_2}(y)}\big)\big)\big\}.$$

Proof Using the definition of partial moduli of continuity, Lemma 1, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\big|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\big|\\&\quad\leq\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(\big|f(t,s)-f(x,y)\big|;q_{n_1},q_{n_2},x,y\big)\\&\quad\leq\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(\big|f(t,s)-f(t,y)\big|;q_{n_1},q_{n_2},x,y\big)\\&\qquad+\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(\big|f(t,y)-f(x,y)\big|;q_{n_1},q_{n_2},x,y\big)\\&\quad\leq\bar{\omega}_1\big(f;\sqrt{\delta_{n_1}(x)}\big)\bigg(\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(1;q_{n_1},x)+\frac{1}{\sqrt{\delta_{n_1}(x)}}\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\big(|t-x|;q_{n_1},x\big)\bigg)\\&\qquad+\bar{\omega}_2\big(f;\sqrt{\delta_{n_2}(y)}\big)\bigg(\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}(1;q_{n_2},y)+\frac{1}{\sqrt{\delta_{n_2}(y)}}\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big(|s-y|;q_{n_2},y\big)\bigg)\end{aligned}$$

$$\leq \bar{\omega}_1\bigl(f;\sqrt{\delta_{n_1}(x)}\bigr)\left(1+\frac{1}{\sqrt{\delta_{n_1}(x)}}\sqrt{\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\bigl((t-x)^2;q_{n_1},x\bigr)}\right)$$
$$+\bar{\omega}_2\bigl(f;\sqrt{\delta_{n_2}(y)}\bigr)\left(1+\frac{1}{\sqrt{\delta_{n_2}(y)}}\sqrt{\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\bigl((s-y)^2;q_{n_2},y\bigr)}\right),$$

from which the required result is straightforward. □

Let $C^2(I_1\times I_2)$ be the space of all functions $f\in C(I_1\times I_2)$ such that second partial derivatives of f belong to $C(I_1\times I_2)$. The norm on the space $C^2(I_1\times I_2)$ is defined as

$$\|f\|_{C^2(I_1\times I_2)}=\|f\|+\sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|+\left\|\frac{\partial^i f}{\partial y^i}\right\|\right).$$

The Peetre K-functional of the function $f\in C(I_1\times I_2)$ is defined as

$$\mathcal{K}(f;\delta)=\inf_{g\in C^2(I_1\times I_2)}\bigl\{\|f-g\|_{C(I_1\times I_2)}+\delta\|g\|_{C^2(I_1\times I_2)}\bigr\},\quad \delta>0.$$

Also by [6], it follows that

$$\mathcal{K}(f;\delta)\leq M\bigl\{\tilde{\omega}_2(f;\sqrt{\delta})+\min(1,\delta)\|f\|_{C(I_1\times I_2)}\bigr\}\tag{3.3}$$

holds for all $\delta>0$.

The constant M in the above inequality is independent of δ and f and $\tilde{\omega}_2(f;\sqrt{\delta})$ is the second order modulus of continuity.

Theorem 7 *For the function $f\in C(I_1\times I_2)$, we have the following inequality*:

$$\bigl|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\bigr|$$
$$\leq M\Bigl\{\tilde{\omega}_2\Bigl(f;\sqrt{A_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)}\Bigr)+\min\bigl\{1,A_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)\bigr\}\|f\|_{C(I_1\times I_2)}\Bigr\}$$
$$+\omega\Bigl(f;\sqrt{B_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)}\Bigr),$$

where

$$A_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)$$
$$=\left\{\delta_{n_1}^2(x)+\delta_{n_2}^2(y)+\left(\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1}+\frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}-x\right)^2\right.$$
$$\left.+\left(\frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2}+\frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)}-y\right)^2\right\}$$

and

$$B_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)=\left(\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1}+\frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}-x\right)^2$$
$$+\left(\frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2}+\frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)}-y\right)^2,$$

and the constant M (>0), is independent of f and $A_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)$.

Proof We define the auxiliary operators as follows:

$$\begin{aligned}
&\mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)\\
&= \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)\\
&\quad -f\biggl(\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1}+\frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)},\\
&\quad \frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2}+\frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)}\biggr)+f(x,y).
\end{aligned} \tag{3.4}$$

Considering Lemma 3, one has $\mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(1;q_{n_1},q_{n_2},x,y)=1$, $\mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}((t-x);q_{n_1},q_{n_2},x,y)=0$, and $\mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}((s-y);q_{n_1},q_{n_2},x,y)=0$.

Let $g\in C^2(I_1\times I_2)$ and $(x,y)\in J^2$. Using Taylor's theorem, we may write

$$\begin{aligned}
g(t,s)-g(x,y) &= g(t,y)-g(x,y)+g(t,s)-g(t,y)\\
&= \frac{\partial g(x,y)}{\partial x}(t-x)+\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}\,du\\
&\quad +\frac{\partial g(x,y)}{\partial y}(s-y)+\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}\,dv.
\end{aligned}$$

Applying the operator $\mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot;q_{n_1},q_{n_2},x,y)$ on the above equation and using (3.4), we are led to

$$\begin{aligned}
&\mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(g;q_{n_1},q_{n_2},x,y)-g(x,y)\\
&= \mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\biggl(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}\,du;q_{n_1},q_{n_2},x,y\biggr)\\
&\quad +\mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\biggl(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}\,dv;q_{n_1},q_{n_2},x,y\biggr)\\
&= \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\biggl(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}\,du;q_{n_1},q_{n_2},x,y\biggr)\\
&\quad -\int_x^{\frac{\alpha_1}{[n_1+1]_{q_1}+\beta_1}+\frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}}\biggl(\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1}+\frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}-u\biggr)\\
&\quad \times\frac{\partial^2 g(u,y)}{\partial u^2}\,du+\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\biggl(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}\,dv;q_{n_1},q_{n_2},x,y\biggr)\\
&\quad -\int_y^{\frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2}+\frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)}}\biggl(\frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2}+\frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)}-v\biggr)\\
&\quad \times\frac{\partial^2 g(x,v)}{\partial v^2}\,dv.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigl|\mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(g;q_{n_1},q_{n_2},x,y)-g(x,y)\bigr|\\
&\le \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\biggl(\biggl|\int_x^t \bigl|(t-u)\bigr|\biggl|\frac{\partial^2 g(u,y)}{\partial u^2}\biggr|\,du\biggr|;q_{n_1},q_{n_2},x,y\biggr)
\end{aligned}$$

$$+ \left| \int_x^{\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1} + \frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}} \left| \left(\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1} + \frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)} - u \right) \right| \right.$$

$$\left. \times \left| \frac{\partial^2 g(u,y)}{\partial u^2} \right| du \right| + \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)} \left(\left| \int_y^s \left| (s-v) \right| \left| \frac{\partial^2 g(x,v)}{\partial v^2} \right| dv \right| ; q_{n_1}, q_{n_2}, x, y \right)$$

$$+ \left| \int_y^{\frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2} + \frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)}} \left| \left(\frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2} + \frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)} - v \right) \right| \right.$$

$$\left. \times \left| \frac{\partial^2 g(x,v)}{\partial v^2} \right| dv \right|$$

$$= A_{n_1,n_2}^{p_1,p_2}(q_{n_1}, q_{n_2}, x, y) \|g\|_{C^2(I_1 \times I_2)}. \tag{3.5}$$

Also,

$$\left| \mathcal{L}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f; q_{n_1}, q_{n_2}, x, y) \right|$$

$$\leq \left| \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f; q_{n_1}, q_{n_2}, x, y) \right| + \left| f\left(\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1} + \frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}, \right. \right.$$

$$\left. \left. \frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2} + \frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)} \right) \right| + \left| f(x,y) \right|$$

$$\leq 3\|f\|_{C(I_1 \times I_2)}. \tag{3.6}$$

Hence, considering (3.4), (3.6), and (3.5) (in that order),

$$\left| \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y) \right|$$

$$= \left| \mathcal{L}^{n_1,n_2,p_1,p_2}_{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y) + f\left(\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1} + \frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}, \right. \right.$$

$$\left. \left. \frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2} + \frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)} \right) - f(x,y) \right|$$

$$\leq \left| \mathcal{L}^{n_1,n_2,p_1,p_2}_{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f-g; q_{n_1}, q_{n_2}, x, y) \right| + \left| \mathcal{L}^{n_1,n_2,p_1,p_2}_{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(g; q_{n_1}, q_{n_2}, x, y) - g(x,y) \right|$$

$$+ \left| g(x,y) - f(x,y) \right|$$

$$+ \left| f\left(\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1} + \frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}, \right. \right.$$

$$\left. \left. \frac{\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2} + \frac{2q_{n_2}[n_2]_{q_{n_2}}y+1}{[2]_{q_{n_2}}([n_2+1]_{q_{n_2}}+\beta_2)} \right) - f(x,y) \right|$$

$$\leq 4\|f-g\|_{C(I_1 \times I_2)} + \left| \mathcal{K}^{n_1,n_2,p_1,p_2}_{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(g; q_{n_1}, q_{n_2}, x, y) - g(x,y) \right|$$

$$+ \left| f\left(\frac{\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1} + \frac{2q_{n_1}[n_1]_{q_{n_1}}x+1}{[2]_{q_{n_1}}([n_1+1]_{q_{n_1}}+\beta_1)}, \right. \right.$$

$$\left. \left. \frac{\alpha_2}{[n_2+1]_{q_2}+\beta_2} + \frac{2q_2[n_2]_{q_2}y+1}{[2]_{q_2}([n_2+1]_{q_2}+\beta_2)} \right) - f(x,y) \right|$$

$$\leq \left(4\|f-g\|_{C(I_1 \times I_2)} + A_{n_1,n_2}^{p_1,p_2}(q_{n_1}, q_{n_2}, x, y) \|g\|_{C^2(I_1 \times I_2)} \right)$$

$$+ \omega\left(f; \sqrt{B_{n_1,n_2}^{(p_1,p_2)}(q_{n_1}, q_{n_2}, x, y)} \right).$$

Now, taking the infimum on the right hand side all over $g \in C^2(I_1 \times I_2)$ and using (3.3)

$$\begin{aligned}
&\left|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\right| \\
&\quad\le 4\mathcal{K}\big(f;A_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)\big)+\omega\Big(f;\sqrt{B_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)}\Big) \\
&\quad\le M\Big\{\tilde{\omega}_2\Big(f;\sqrt{A_{n_1,n_2}^{p_1,p_2}(q_{n_1},q_{n_2},x,y)}\Big)+\min\big\{1,A_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)\big\}\|f\|_{C(I_1\times I_2)}\Big\} \\
&\qquad+\omega\Big(f;\sqrt{B_{n_1,n_2}^{(p_1,p_2)}(q_{n_1},q_{n_2},x,y)}\Big).
\end{aligned}$$

Thus, we get the desired result. □

Theorem 8 *Let $f \in C^2(I_1 \times I_2)$. Then for every $(x,y) \in J^2$,*

$$\begin{aligned}
&\lim_{[n]_{q_n}\to\infty}[n]_{q_n}\big\{\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(f(t,s);q_n,x,y\big)-f(x,y)\big\} \\
&\quad=f_x(x,y)\left(\frac{-x(a+1+2\beta_1)}{2}+\alpha_1+\frac{1}{2}\right)+f_y(x,y)\left(\frac{-y(a+1+2\beta_2)}{2}+\alpha_2+\frac{1}{2}\right) \\
&\qquad+\frac{1}{2}\left\{f_{xx}(x,y)\frac{x(1-x)}{2}+f_{yy}(x,y)\frac{y(1-y)}{2}\right\}
\end{aligned}$$

uniformly in $(x,y) \in J^2$.

Proof By Taylor's formula for f, we have

$$\begin{aligned}
f(t,s)&=f(x,y)+f_x(x,y)(t-x)+f_y(x,y)(s-y) \\
&\quad+\frac{1}{2}\big\{f_{xx}(x,y)(t-x)^2+2f_{xy}(x,y)(t-x)(s-y)+f_{yy}(x,y)(s-y)^2\big\} \\
&\quad+\xi(t,s,x,y)\sqrt{(t-x)^4+(s-y)^4},
\end{aligned}$$

where $\xi(t,s,x,y)\to 0$ as $(t,s)\to(x,y)$ and $\xi(t,s,x,y)\in C^2(I_1\times I_2)$. Now, applying the operator $\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot;q_n,x,y)$ on the above equation, we get

$$\begin{aligned}
&\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(f(t,s);q_n,x,y\big) \\
&\quad=f(x,y)+f_x(x,y)\mathcal{K}_{n,p_1}^{(\alpha_1,\beta_1)}\big((t-x);q_n,x\big)+f_y(x,y)\mathcal{K}_{n,p_2}^{(\alpha_2,\beta_2)}\big((s-y);q_n,y\big) \\
&\qquad+\frac{1}{2}\big\{f_{xx}(x,y)\mathcal{K}_{n,p_1}^{(\alpha_1,\beta_1)}\big((t-x)^2;q_n,x\big)+2f_{xy}(x,y)\mathcal{K}_{n,p_1}^{(\alpha_1,\beta_1)}\big((t-x);q_n,x\big) \\
&\qquad\times\mathcal{K}_{n,p_2}^{\alpha_2,\beta_2}\big((s-y);q_n,y\big)+f_{yy}\mathcal{K}_{n,p_2}^{\alpha_2,\beta_2}\big((s-y)^2;q_n,y\big)\big\} \\
&\qquad+\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(\xi(t,s,x,y)\sqrt{(t-x)^4+(s-y)^4};q_n,x,y\big).
\end{aligned}$$

Hence, using Lemma 2,

$$\begin{aligned}
&\lim_{[n]_{q_n}\to\infty}[n]_{q_n}\big\{\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(f(t,s);q_n,x,y\big)-f(x,y)\big\} \\
&\quad=f_x(x,y)\left(\frac{-x(a+1+2\beta_1)}{2}+\alpha_1+\frac{1}{2}\right)+f_y(x,y)\left(\frac{-y(a+1+2\beta_2)}{2}+\alpha_2+\frac{1}{2}\right)
\end{aligned}$$

$$+\frac{1}{2}\left\{f_{xx}(x,y)\frac{x(1-x)}{2}+f_{yy}(x,y)\frac{y(1-y)}{2}\right\}$$
$$+\lim_{[n]_{q_n}\to\infty}[n]_{q_n}\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(\xi(t,s,x,y)\sqrt{(t-x)^4+(s-y)^4};x,y\bigr)$$

uniformly in $(x,y)\in J^2$.

Applying the Cauchy-Schwarz inequality

$$\bigl|\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(\xi(t,s)\sqrt{(t-x)^4+(s-y)^4};q_n,x,y\bigr)\bigr|$$
$$\leq\sqrt{\mathcal{K}_{n,n,p_1,p_2}^{\alpha_1,\alpha_2,\beta_1,\beta_2}\bigl(\xi^2(t,s);q_n,x,y\bigr)}\sqrt{\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl((t-x)^4+(s-y)^4;q_n,x,y\bigr)}.$$

Since, by Theorem 2 and in view of Lemma 2,

$$\lim_{[n]_{q_n}\to\infty}\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(\xi^2(t,s);x,y\bigr)=\xi^2(x,y)=0,$$
$$\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl((t-x)^4;q_n,x\bigr)=O\left(\frac{1}{[n]_{q_n}^2}\right),\quad\text{and}$$
$$\mathcal{K}_{n,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl((s-y)^4;q_n,y\bigr)=O\left(\frac{1}{[n]_{q_n}^2}\right)$$

uniformly in $(x,y)\in J^2$, it follows that

$$\lim_{n\to\infty}[n]_{q_n}\bigl\{\mathcal{K}_{n,n,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(\xi(t,s)\sqrt{(t-x)^4+(s-y)^4};q_n,x,y\bigr)\bigr\}=0$$

uniformly in $(x,y)\in J^2$, the desired result is obtained. □

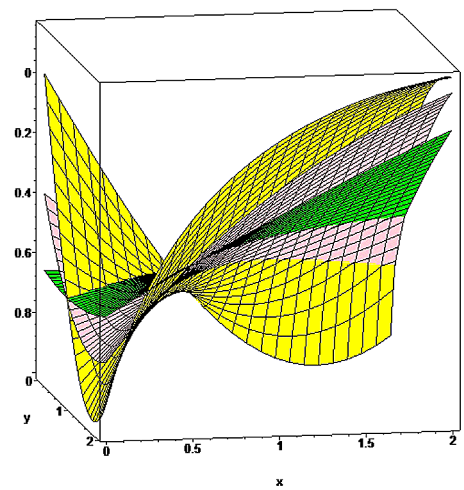
In the following example, the rate of convergence of the bivariate operators given by (3.1) to a certain function is shown by illustrative graphics. We observe that when the values of q_1 and q_2 increase, the approximation of f by the operator $\mathcal{K}_{n_1,n_2,p_1p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ becomes better.

Example 3 Let $n_1=n_2=5$, $\alpha_1=0.5$, $\beta_1=0.6$, $\alpha_2=0.7$, $\beta_2=0.8$, $p_1=p_2=1$. For $q_1=0.45$, $q_2=0.50$ (green) and $q_1=0.85$, $q_2=0.90$ (pink), the convergence of the operators $\mathcal{K}_{n_1,n_2,p_1p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ given by (3.1) to $f(x,y)=\sin(x+y)/(1+xy)$ (yellow) is illustrated in Figure 4.

4 Construction of GBS operator of *q*-Bernstein-Schurer-Kantorovich type

In [7] and [8], Bögel proposed the concepts of *B*-continuous and *B*-differentiable functions. Later, Dobrescu and Matei [9] discussed the approximation of *B*-continuous functions on a bounded interval by a generalized Boolean sum of bivariate generalization of Bernstein polynomials. Subsequently, Badea and Cottin [10] established Korovkin theorems for GBS operators. Pop [11] studied the GBS operators associated to a certain class of linear and positive operators defined by an infinite sum and discussed the approximation of *B*-continuous and *B*-differentiable functions by these operators. Recently, Sidharth *et al.* [12] proposed the GBS operators of *q*-Bernstein-Schurer-Kantorovich type and studied the rate of convergence of these operators by means of the mixed modulus of smoothness. Agrawal and Ispir [13] introduced the bivariate generalization of Chlodowsky-Szasz-Charlier-type operators and obtained the degree of approximation for the associated GBS

Figure 4 The convergence of $K_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ to $f(x,y)$.



operators. In this section, we give some basic definitions and notations, for further details, one can see [14].

Let X and Y be compact subsets of $\mathbb{R}$. A function $f : X \times Y \longrightarrow \mathbb{R}$ is called a B-continuous (Bögel continuous) function at $(x_0, y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f\big[(x_0,y_0);(x,y)\big] = 0,$$

where $\Delta f[(x_0,y_0);(x,y)]$ denotes the mixed difference defined by

$$\Delta f\big[(x_0,y_0);(x,y)\big] = f(x,y) - f(x,y_0) - f(x_0,y) + f(x_0,y_0). \tag{4.1}$$

The function $f : X \times Y \to \mathbb{R}$ is called B-bounded on $X \times Y$ if there exists $M > 0$ such that $|\Delta f[(t,s);(x,y)]| \leq M$, for every $(x,y),(t,s) \in (X \times Y)$. Since $X \times Y$ is a compact subset of $\mathbb{R}^2$, each B-continuous function is a B-bounded function on $X \times Y \to \mathbb{R}$.

Throughout this paper, $B_b(X \times Y)$ denotes all B-bounded functions on $X \times Y \to \mathbb{R}$, equipped with the norm $\|f\|_B = \sup_{(x,y),(t,s)\in X\times Y} |\Delta f[(t,s);(x,y)]|$. We denote by $C_b(X \times Y)$, the space of all B-continuous functions on $X \times Y$. $B(X \times Y), C(X \times Y)$ denote the space of all bounded functions and the space of all continuous (in the usual sense) functions on $X \times Y$ endowed with the sup-norm $\|\cdot\|_\infty$. It is well known that $C(X \times Y) \subset C_b(X \times Y)$ ([14], p.52).

A function $f : X \times Y \longrightarrow \mathbb{R}$ is called a B-differentiable (Bögel differentiable) function at $(x_0, y_0) \in X \times Y$ if the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f[(x_0,y_0);(x,y)]}{(x-x_0)(y-y_0)}$$

exists and is finite.

The limit is said to be the B-differential of f at the point (x_0, y_0) and is denoted by $D_B(f; x_0, y_0)$ and the space of all B-differentiable functions is denoted by $D_b(X \times Y)$. The mixed modulus of smoothness of $f \in C_b(I_1 \times I_2)$ is defined as

$$\omega_{\text{mixed}}(f;\delta_1,\delta_2) := \sup\big\{\big|\Delta f\big[(t,s);(x,y)\big]\big| : |x-t| < \delta_1, |y-s| < \delta_2\big\}$$

for all $(x,y),(t,s) \in (I_1 \times I_2)$ and for any $(\delta_1,\delta_2) \in (0,\infty) \times (0,\infty)$ with $\omega_{\text{mixed}} : [0,\infty) \times [0,\infty) \to \mathbb{R}$. The basic properties of ω_{mixed} were obtained by Badea *et al.* in [15] and [16], which are similar to the properties of the usual modulus of continuity.

We define the GBS operator of the operator $\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}$ given by (1.3), for any $f \in C_b(I_1 \times I_2)$ and $m,n \in \mathbb{N}$, by

$$\begin{aligned} &T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(f(t,s);q_{n_1},q_{n_2},x,y\bigr) \\ &\quad := \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(f(t,y)+f(x,s)-f(t,s);q_{n_1},q_{n_2},x,y\bigr) \end{aligned} \tag{4.2}$$

for all $(x,y) \in J^2$.

Hence for any $f \in C_b(I_1 \times I_2)$, the GBS operator of the q-Bernstein-Schurer-Kantorovich type is

$$T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)} : C_b(I_1 \times I_2) \longrightarrow C(I_1 \times I_2)$$

given by

$$\begin{aligned} &T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y) \\ &\quad = \sum_{k_1=0}^{n_1+p_1}\sum_{k_2=0}^{n_2+p_2} \tilde{p}^*_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2};x,y) \\ &\qquad \times \int_0^1\int_0^1 f\biggl\{\biggl(\frac{[k_1]_{q_{n_1}}+q_{n_1}^{k_1}t+\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1},y\biggr)+f\biggl(x,\frac{[k_2]_{q_{n_2}}+q_{n_2}^{k_2}s+\alpha_2}{[n_2+1]_{q_{n_2}}+\beta_2}\biggr) \\ &\qquad -f\biggl(\frac{[k_1]_{qn_1}+q_{n_1}^{k_1}t+\alpha_1}{[n_1+1]_{q_{n_1}}+\beta_1},\frac{[k_2]_{qn_2}+q_{n_2}^{k_2}s+\alpha_2}{[n_2+1]_{qn_2}+\beta_2}\biggr)\biggr\}\,d_{q_{n_1}}t\,d_{q_{n_2}}s, \end{aligned}$$

where

$$\begin{aligned} \tilde{p}^*_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2},x,y) = {}& \frac{[n_1]_{q_{n_1}}^{n_1+p_1}}{[n_1+p_1]_{q_{n_1}}^{n_1+p_1}}\begin{bmatrix} n_1+p_1 \\ k_1 \end{bmatrix}_{q_{n_1}} x^{k_1}\biggl(\frac{[n_1+p_1]_{q_{n_1}}}{[n_1]_{q_{n_1}}}-x\biggr)_{q_{n_1}}^{n_1+p_1-k_1} \\ &\times \frac{[n_2]_{q_{n_2}}^{n_2+p_2}}{[n_2+p_2]_{q_{n_2}}^{n_2+p_2}}\begin{bmatrix} n_2+p_2 \\ k_2 \end{bmatrix}_{q_{n_2}} y^{k_2}\biggl(\frac{[n_2+p_2]_{q_{n_2}}}{[n_2]_{q_{n_2}}}-y\biggr)_{q_{n_2}}^{n_2+p_2-k_2}. \end{aligned}$$

Clearly, the operator $T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}$ is linear and preserves linear functions.

Theorem 9 *For every $f \in C_b(I_1 \times I_2)$, at each point $(x,y) \in J^2$, the operator* (4.2) *verifies the following inequality*:

$$\bigl|T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\bigr| \le 4\omega_{\text{mixed}}\bigl(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(y)}\bigr).$$

Proof By the property

$$\omega_{\text{mixed}}(f;\lambda_1\delta_1,\lambda_2\delta_2) \le (1+\lambda_1)(1+\lambda_2)\omega_{\text{mixed}}(f,\delta_1,\delta_2); \quad \lambda_1,\lambda_2 > 0,$$

we can write

$$\begin{aligned}\big|\Delta f\big[(t,s);(x,y)\big]\big| &\le \omega_{\mathrm{mixed}}\big(f;|t-x|,|s-y|\big)\\ &\le \left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)\omega_{\mathrm{mixed}}(f;\delta_1,\delta_2) \qquad (4.3)\end{aligned}$$

for every $(t,s)\in(I_1\times I_2)$, $(x,y)\in J^2$ and any $\delta_1,\delta_2>0$. From (4.2) and the definition of the mixed difference $\Delta f[(t,s);(x,y)]$, on applying Lemma 3 and the inequality (4.3), we get

$$\begin{aligned}&\big|T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\big|\\ &\quad\le \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\big(\big|\Delta f\big[(t,s);(x,y)\big]\big|;q_{n_1},q_{n_2},x,y\big)\\ &\quad\le\Bigg(\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(1;q_{n_1},x)+\frac{1}{\sqrt{\delta_{n_1}}}\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\big(|t-x|;q_{n_1},x\big)\\ &\qquad+\frac{1}{\sqrt{\delta_{n_2}}}\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big(|s-y|;q_{n_2},y\big)+\frac{1}{\sqrt{\delta_{n_1}}\sqrt{\delta_{n_2}}}\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\big(|t-x|;q_{n_1},x\big)\\ &\qquad\times\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big(|s-y|;q_{n_2},y\big)\Bigg)\omega_{\mathrm{mixed}}\big(f;\sqrt{\delta_{n_1}},\sqrt{\delta_{n_2}}\big).\end{aligned}$$

Now, applying the Cauchy-Schwarz inequality

$$\begin{aligned}&\big|T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\big|\\ &\quad\le\Bigg(\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(e_{00};q_{n_1},x)+\frac{1}{\sqrt{\delta_{n_1}}}\sqrt{\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\big((t-x)^2;q_{n_1},x\big)}\\ &\qquad+\frac{1}{\sqrt{\delta_{n_2}}}\sqrt{\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big((s-y)^2;q_{n_2},y\big)}+\frac{1}{\sqrt{\delta_{n_1}}\sqrt{\delta_{n_2}}}\sqrt{\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\big((t-x)^2;q_{n_1},x\big)}\\ &\qquad\times\sqrt{\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\big((s-y)^2;q_{n_2},y\big)}\Bigg)\omega_{\mathrm{mixed}}(f,\delta_{n_1},\delta_{n_2})\\ &\quad=4\omega_{\mathrm{mixed}}\big(f;\sqrt{\delta_{n_1}},\sqrt{\delta_{n_2}}\big),\end{aligned}$$

on choosing $\delta_{n_1}=\delta_{n_1}(x)$ and $\delta_{n_2}=\delta_{n_2}(y)$. This completes the proof. □

Next, let us define the Lipschitz class for B-continuous functions. For $f\in C_b(I_1\times I_2)$, the Lipschitz class $\mathrm{Lip}_M(\xi,\eta)$ with $\xi,\eta\in(0,1]$ is defined by

$$\begin{aligned}\mathrm{Lip}_M(\xi,\eta)=\big\{&f\in C_b(I_1\times I_2):\big|\Delta f\big[(t,s);(x,y)\big]\big|\le M|t-x|^{\xi}|s-y|^{\eta},\\ &\text{for }(t,s),(x,y)\in I_1\times I_2\big\}.\end{aligned}$$

In our next result, we determine the degree of approximation for the operators $T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ by means of the class $\mathrm{Lip}_M(\xi,\eta)$ of the class of Bögel continuous functions.

Theorem 10 *For $f\in\mathrm{Lip}_M(\xi,\eta)$, we have*

$$\big|T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\big|\le M\big(\delta_{n_1}(x)\big)^{\frac{\xi}{2}}\big(\delta_{n_2}(y)\big)^{\frac{\eta}{2}}$$

for $M>0$, $\xi,\eta\in(0,1]$.

Proof From (4.2), (4.1), and by our hypothesis, we may write

$$\begin{aligned}
&\bigl|T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\bigr| \\
&\quad\le \mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(\bigl|\Delta f\bigl[(t,s);(x,y)\bigr]\bigr|;x,y\bigr) \\
&\quad\le M\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(|t-x|^{\xi}|s-y|^{\eta};x,y\bigr) \\
&\quad= M\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\bigl(|t-x|^{\xi};x\bigr)\overline{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\bigl(|s-y|^{\eta};y\bigr).
\end{aligned}$$

Applying Hölder's inequality with $p_1=2/\xi$, $q_1=2/(2-\xi)$ and $p_2=2/\eta$, $q_2=2/(2-\eta)$, we are led to

$$\begin{aligned}
&\bigl|T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\bigr| \\
&\quad\le M\bigl(\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}\bigl((t-x)^2;x\bigr)\bigr)^{\xi/2}\mathcal{K}_{n_1,p_1}^{(\alpha_1,\beta_1)}(e_0;x)^{(2-\xi)/2} \\
&\qquad\times\bigl(\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}\bigl((s-y)^2;y\bigr)\bigr)^{\eta/2}\mathcal{K}_{n_2,p_2}^{(\alpha_2,\beta_2)}(e_0;y)^{(2-\eta)/2}.
\end{aligned}$$

In view of Lemma 1, the desired result is immediate. □

Theorem 11 *For $f\in D_b(I_1\times I_2)$ with $D_Bf\in B(I_1\times I_2)$ and each $(x,y)\in J^2$, we have*

$$\begin{aligned}
&\bigl|T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\bigr| \\
&\quad\le\frac{M}{[n_1]_{q_{n_1}}^{1/2}[n_2]_{q_{n_2}}^{1/2}}\bigl(\|D_Bf\|_\infty+\omega_{\text{mixed}}\bigl(D_Bf;[n_1]_{q_{n_1}}^{-1/2},[n_2]_{q_{n_2}}^{-1/2}\bigr)\bigr).
\end{aligned}$$

Proof By our hypothesis, using the relations

$$\Delta f\bigl[(t,s);(x,y)\bigr]=(t-x)(s-y)D_Bf(\xi,\eta),\quad\text{where } x<\xi<t;y<\eta<s,$$

and

$$D_Bf(\xi,\eta)=\Delta D_Bf(\xi,\eta)+D_Bf(\xi,y)+D_Bf(x,\eta)-D_Bf(x,y),$$

we obtain

$$\begin{aligned}
&\bigl|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(\Delta f\bigl[(t,s);(x,y)\bigr];q_{n_1},q_{n_2},x,y\bigr)\bigr| \\
&\quad=\bigl|\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl((t-x)(s-y)D_Bf(\xi,\eta);q_{n_1},q_{n_2},x,y\bigr)\bigr| \\
&\quad\le\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(|t-x||s-y|\bigl|\Delta D_Bf(\xi,\eta)\bigr|;x,y\bigr) \\
&\qquad+\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(|t-x||s-y|\bigl(\bigl|D_Bf(\xi,y)\bigr| \\
&\qquad+\bigl|D_Bf(x,\eta)\bigr|+\bigl|D_Bf(x,y)\bigr|\bigr);q_{n_1},q_{n_2},x,y\bigr) \\
&\quad\le\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(|t-x||s-y|\omega_{\text{mixed}}\bigl(D_Bf;|\xi-x|,|\eta-y|\bigr);x,y\bigr) \\
&\qquad+3\|D_Bf\|_\infty\mathcal{K}_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}\bigl(|t-x||s-y|;q_{n_1},q_{n_2},x,y\bigr).
\end{aligned}$$

Hence taking into account and applying the Cauchy-Schwarz inequality we obtain

$$\omega_{\text{mixed}}\big(D_B f; |\xi - x|, |\eta - y|\big) \le \left(1 + \frac{|t-x|}{\delta_{n_1}}\right)\left(1 + \frac{|s-y|}{\delta_{n_2}}\right)\omega_{\text{mixed}}(D_B f; \delta_{n_1}, \delta_{n_2}).$$

We have

$$\begin{aligned}
&\big|T^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y)\big| \\
&\quad\le 3\|D_B f\|_\infty \sqrt{\mathcal{K}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}\big((t-x)^2(s-y)^2; q_{n_1}, q_{n_2}, x, y\big)} \\
&\qquad + \Big(\sqrt{\mathcal{K}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}\big((t-x)^2(s-y)^2; q_{n_1}, q_{n_2}, x, y\big)} \\
&\qquad + \delta_{n_1}^{-1}\sqrt{\mathcal{K}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}\big((t-x)^4(s-y)^2; q_{n_1}, q_{n_2}, x, y\big)} \\
&\qquad + \delta_{n_2}^{-1}\sqrt{\mathcal{K}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}\big((t-x)^2(s-y)^4; q_{n_1}, q_{n_2}, x, y\big)} \\
&\qquad + \delta_{n_1}^{-1}\delta_{n_2}^{-1}\mathcal{K}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}\big((t-x)^2(s-y)^2; q_{n_1}, q_{n_2}, x, y\big)\Big)\omega_{\text{mixed}}(D_B f; \delta_{n_1}, \delta_{n_2}).
\end{aligned} \tag{4.4}$$

From Lemma 2, we observe that for $(t,s) \in (I_1 \times I_2)$, $(x,y) \in J^2$ and $i,j = 1,2$,

$$\begin{aligned}
&K^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}\big((t-x)^{2i}(s-y)^{2j}; q_{n_1}, q_{n_2}, x, y\big) \\
&\quad = K^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}\big((t-x)^{2i}; q_{n_1}, x, y\big) K^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}\big((s-y)^{2j}; q_{n_2}, x, y\big). \\
&\quad = K^{(\alpha_1,\beta_1)}_{n_1,p_1}\big((t-x)^{2i}; q_{n_1}, x\big) K^{(\alpha_2,\beta_2)}_{n_2,p_2}\big((s-y)^{2j}; q_{n_2}, y\big) \\
&\quad \le \frac{M_1}{[n_1]^i_{q_{n_1}}}\frac{M_2}{[n_2]^j_{q_{n_2}}}
\end{aligned}$$

for some constants $M_1, M_2 > 0$.

Now, let $\delta_{n_1} = \frac{1}{[n_1]^{1/2}_{q_{n_1}}}$ and $\delta_{n_2} = \frac{1}{[n_2]^{1/2}_{q_{n_2}}}$,

$$\begin{aligned}
&\big|T^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y)\big| \\
&\quad = 3\|D_B\|_\infty O\left(\frac{1}{[n_1]^{1/2}_{q_{n_1}}}\right) O\left(\frac{1}{[n_2]^{1/2}_{q_{n_2}}}\right) \\
&\qquad + O\left(\frac{1}{[n_1]^{1/2}_{q_{n_1}}}\right) O\left(\frac{1}{[n_2]^{1/2}_{q_{n_2}}}\right)\omega_{\text{mixed}}\big(D_B f; [n_1]^{-1/2}_{q_{n_1}}, [n_2]^{-1/2}_{q_{n_2}}\big).
\end{aligned} \tag{4.5}$$

Thus, we obtain the required result. □

Now, we illustrate the rate of convergence of the GBS operators (4.2) to certain functions by graphics. It is observed that when the values of q_1 and q_2 increase, the convergence of the GBS operator $T^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1 p_2}(f; q_1, q_2, x, y)$ to the function $f(x,y)$ becomes better.

Example 4 Let $n_1 = n_2 = 5$, $\alpha_1 = 5$, $\beta_1 = 6$, $\alpha_2 = 7$, $\beta_2 = 8$, $p_1 = p_2 = 1$. For $q_1 = 0.45$, $q_2 = 0.50$ and $q_1 = 0.85$, $q_2 = 0.90$, the convergence of the GBS operators $T^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1 p_2}(f; q_1, q_2, x, y)$ (turquoise, orange) to $f(x,y) = \cos(x^2)/(1+y)$ (yellow) is shown in Figure 5.

Figure 5 Convergence of $T^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)$ to $f(x,y)$.

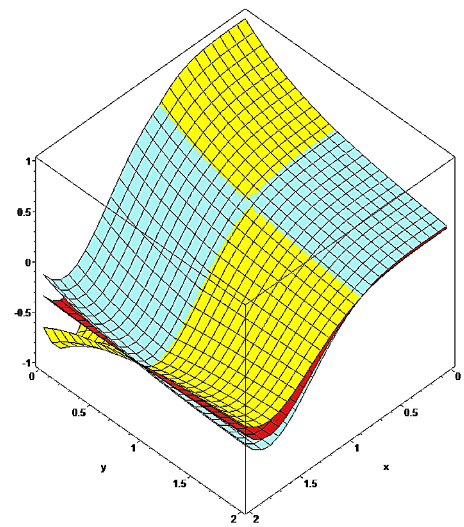


Figure 6 The comparison of rate of convergence of $K^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)$ and $T^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)$ to $f(x,y)$.

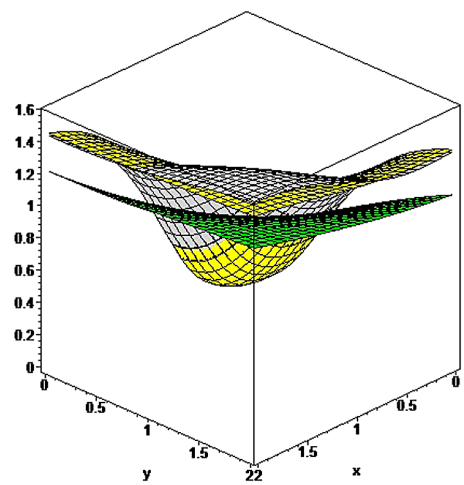
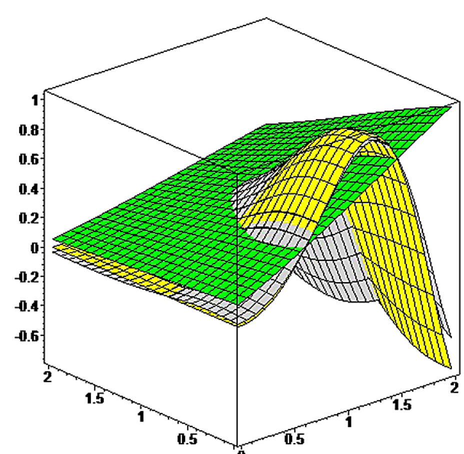


Figure 7 The comparison of rate of convergence of $K^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)$ and $T^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)$ to $f(x,y)$.



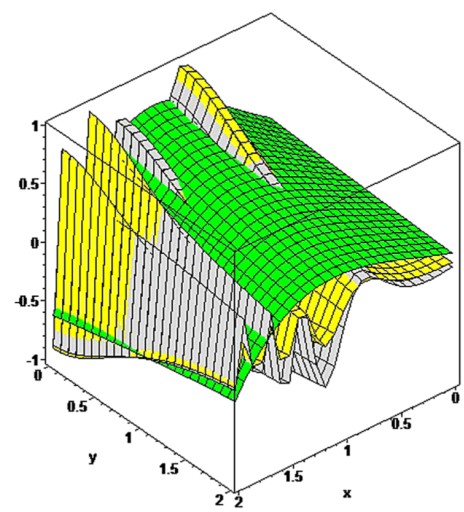


Figure 8 The comparison of rate of convergence of $K_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ and $T_{n_1,n_2,p_1,p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ to $f(x,y)$.

Lastly, we compare the convergence of the operators $\mathcal{K}_{n_1,n_2,p_1p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ given by (3.1) and its GBS operators $T_{n_1,n_2,p_1p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ to some functions.

Example 5 For $n_1,n_2=5$, $\alpha_1=2$, $\beta_1=3$, $\alpha_2=4$, $\beta_2=5$, $p_1=p_2=1$ and $q_1=0.75$, $q_2=0.80$, the comparison of convergence of the operators $\mathcal{K}_{n_1,n_2,p_1p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ (green) and $T_{n_1,n_2,p_1p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ (gray) to the functions $f(x,y)=\arctan(x^3+y^2)$, $f(x,y)=\sin(x^2)/(1+y^3)$, $f(x,y)=\sin(3x^3)/(1+y^2)$ is illustrated, respectively, in Figures 6, 7, and 8. We observe that the rate of convergence of $T_{n_1,n_2,p_1p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$ is better than the operator $\mathcal{K}_{n_1,n_2,p_1p_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f;q_1,q_2,x,y)$.

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
All authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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