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Anisotropic Picone identities and anisotropic Hardy inequalities

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Abstract

In this paper, we derive an anisotropic Picone identity for the anisotropic Laplacian, which contains some known Picone identities. As applications, a Sturmian comparison principle to the anisotropic elliptic equation and an anisotropic Hardy type inequality are shown.

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Keywords: anisotropic Picone identity; anisotropic Hardy type inequality; anisotropic elliptic equation; Sturmian comparison principle

1 Introduction and main results

In recent years, the anisotropic Laplacian

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right), \quad p_i > 1, \tag{1.1}$$

has been considerably concerned. Note that if $p_i = 2$ ($i = 1, \ldots, n$), then (1.1) becomes the classical Laplacian; if $p_i = p = \text{const}$, then (1.1) is the pseudo-p-Laplacian (see [1])

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right).$$

The anisotropic Laplacian has not only the widespread practical background in the natural science, but also the important theoretical value in the mathematics. For example, it reflects anisotropic physical properties of some reinforced materials (Lions [2] and Tang [3]), and describes the dynamics of fluids in the anisotropic media when the conductivities of the media are different in each direction [4, 5]. The equations associated with (1.1) are also deduced in the image processing [6]. Existence, integrability, boundedness, and continuity of solutions to anisotropic elliptic equations have received much attention; see [7–15] and the references therein. In this paper, we prove an anisotropic Picone identity for the anisotropic Laplacian, which contains some known Picone identities. As applications, a Sturmian comparison principle to the anisotropic elliptic equation and an anisotropic Hardy type inequality are given. Before giving the main results of this paper, we briefly recall the existing results for the isotropic case.

Picone [16] considered the homogeneous linear second order differential system

$$\begin{cases} (a_1(x)u')' + b_1(x)u = 0, \\ (a_2(x)v')' + b_2(x)v = 0, \end{cases}$$

where u and v are differentiable functions in x, and proved the identity that, for the differentiable function $v(x) \neq 0$,

$$\left(\frac{u}{v}\bigl(a_1u'v - a_2uv'\bigr)\right)' = (b_2 - b_1)u^2 + (a_1 - a_2)u'^2 + a_2\left(u' - v'\frac{u}{v}\right)^2; \tag{1.2}$$

then a Sturmian comparison principle and the oscillation theory of solutions were obtained via (1.2). Picone [17] (see also Allegretto [18]) generalized (1.2) to a Laplacian that, for differentiable functions $v > 0$ and $u \geq 0$,

$$\begin{aligned} \left(\nabla u - \frac{u}{v}\nabla v\right)^2 &= |\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla v \cdot \nabla u \\ &= |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v. \end{aligned} \tag{1.3}$$

Allegretto and Huang [19], Dunninger [20] independently extended (1.3) to a p-Laplacian, for differentiable functions $v > 0$ and $u \geq 0$,

$$\begin{aligned} &|\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \cdot \nabla u \\ &\quad = |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v, \end{aligned} \tag{1.4}$$

and applied (1.4) to derive a Sturmian comparison principle, Liouville's theorem, the Hardy inequality, and some profound results for p-Laplace equations and systems. For other generalizations of the Picone identities and applications, see Bal [21], Dwivedi [22], Dwivedi and Tyagi [23], Niu, Zhang and Wang [24], Tyagi [25]. These results indicate that Picone identities are seemingly simple in form, but extremely useful in the study of partial differential equations, and they have become an important tool in the analysis.

Our main results are as follows.

Theorem 1.1 (Anisotropic Picone identity) *Let $v > 0$ and $u \geq 0$ be two differentiable functions in the set $\Omega \subset R^n$, and denote*

$$R(u,v) = \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^{p_i} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{u^{p_i}}{v^{p_i-1}}\right)\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}, \tag{1.5}$$

$$\begin{aligned} L(u,v) = {} & \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^{p_i} - \sum_{i=1}^n p_i\frac{u^{p_i-1}}{v^{p_i-1}}\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i} \\ & + \sum_{i=1}^n (p_i - 1)\frac{u^{p_i}}{v^{p_i}}\left|\frac{\partial v}{\partial x_i}\right|^{p_i}, \end{aligned} \tag{1.6}$$

where $p_i > 1$ $(i = 1, \dots, n)$. *Then*

$$R(u,v) = L(u,v). \tag{1.7}$$

Moreover, we have

$$L(u,v) \geq 0;$$

furthermore, $L(u,v) = 0$ *a.e. in* Ω *if and only if* $u = cv$ *a.e. in* Ω, c *is a positive constant*.

Remark 1.2 If $p_i = 2$ $(i = 1, \dots, n)$ in (1.5) and (1.6), we have (1.3) from (1.7). If $p_i = p =$ const $(i = 1, \dots, n)$ in (1.5) and (1.6), the result in [26] follows. Moreover, the identity in Theorem 1.1 is different from the one in [26].

Theorem 1.3 (Anisotropic Hardy type inequality) *Let* $u \in C_0^1(A)$, $1 < p_i < n$, $i = 1, \dots, n$, $A = \{x \in R^n | x_i \neq 0, i = 1, \dots, n\}$. *Then we have*

$$\sum_{i=1}^{n} \int_A \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \geq \sum_{i=1}^{n} \left(\frac{p_i - 1}{p_i} \right)^{p_i} \int_A \frac{|u|^{p_i}}{|x_i|^{p_i}}. \tag{1.8}$$

This paper is organized as follows: The proofs of Theorem 1.1 and a Sturmian comparison principle to the anisotropic elliptic equation are given in Section 2; Section 3 is devoted to the proof of Theorem 1.3 in which a key ingredient is to choose a suitable auxiliary function (see (3.3) below) for the anisotropic case. Two corollaries are also furnished.

2 Proof of Theorem 1.1

Proof of Theorem 1.1 One derives easily that

$$\begin{aligned}
R(u,v) &= \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{u^{p_i}}{v^{p_i-1}} \right) \left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i} \\
&= \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} - \sum_{i=1}^{n} \frac{p_i u^{p_i-1} \frac{\partial u}{\partial x_i} v^{p_i-1} - u^{p_i}(p_i-1) v^{p_i-2} \frac{\partial v}{\partial x_i}}{[v^{p_i-1}]^2} \left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i} \\
&= \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} - \sum_{i=1}^{n} p_i \frac{u^{p_i-1}}{v^{p_i-1}} \left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} \\
&= L(u,v),
\end{aligned}$$

which is (1.7). To check $L(u,v) \geq 0$, we rewrite $L(u,v)$ by

$$\begin{aligned}
L(u,v) = {} & \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} - \sum_{i=1}^{n} p_i \frac{u^{p_i-1}}{v^{p_i-1}} \left| \frac{\partial v}{\partial x_i} \right|^{p_i-1} \left| \frac{\partial u}{\partial x_i} \right| + \sum_{i=1}^{n} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} \\
& + \sum_{i=1}^{n} p_i \frac{u^{p_i-1}}{v^{p_i-1}} \left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \left\{ \left| \frac{\partial v}{\partial x_i} \right| \left| \frac{\partial u}{\partial x_i} \right| - \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} \right\} \\
:= {} & I + II,
\end{aligned} \tag{2.1}$$

where

$$I = \sum_{i=1}^{n} p_i \left[\frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} + \frac{p_i - 1}{p_i} \left(\left(\frac{u}{v} \left| \frac{\partial v}{\partial x_i} \right| \right)^{p_i - 1} \right)^{\frac{p_i}{p_i - 1}} \right] - \sum_{i=1}^{n} p_i \frac{u^{p_i - 1}}{v^{p_i - 1}} \left| \frac{\partial v}{\partial x_i} \right|^{p_i - 1} \left| \frac{\partial u}{\partial x_i} \right|,$$

$$II = \sum_{i=1}^{n} p_i \frac{u^{p_i - 1}}{v^{p_i - 1}} \left| \frac{\partial v}{\partial x_i} \right|^{p_i - 2} \left\{ \left| \frac{\partial v}{\partial x_i} \right| \left| \frac{\partial u}{\partial x_i} \right| - \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} \right\}.$$

Recall Young's inequality: for $a \geq 0$ and $b \geq 0$,

$$ab \leq \frac{a^{p_i}}{p} + \frac{b^{q_i}}{q}, \tag{2.2}$$

where $p_i > 1$, $q_i > 1$ $(i = 1, \dots, n)$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$; the equality holds if and only if $a^{p_i} = b^{q_i}$, namely, $a = b^{\frac{1}{p_i - 1}}$. We take $a = |\frac{\partial u}{\partial x_i}|$ and $b = (\frac{u}{v} |\frac{\partial v}{\partial x_i}|)^{p_i - 1}$ in (2.2) to obtain

$$p_i \left| \frac{\partial u}{\partial x_i} \right| \left(\frac{u}{v} \left| \frac{\partial v}{\partial x_i} \right| \right)^{p_i - 1} \leq p_i \left[\frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} + \frac{p_i - 1}{p_i} \left(\left(\frac{u}{v} \left| \frac{\partial v}{\partial x_i} \right| \right)^{p_i - 1} \right)^{\frac{p_i}{p_i - 1}} \right], \tag{2.3}$$

and so $I \geq 0$ from (2.3). Clearly, $II \geq 0$ in virtue of $|\frac{\partial v}{\partial x_i}||\frac{\partial u}{\partial x_i}| - \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i} \geq 0$. Hence $L(u, v) \geq 0$ from (2.1).

If $u = cv$, c is a positive constant, then clearly $L(u, v) = 0$. Now let us conclude that $L(u, v) = 0$ implies $u = cv$. In fact, if $L(u, v)(x_0) = 0$, $x_0 \in \Omega$, then we consider the two cases $u(x_0) \neq 0$ and $u(x_0) = 0$, respectively.

(a) If $u(x_0) \neq 0$, then $I = 0$ and $II = 0$. One shows by $I = 0$ that

$$\left| \frac{\partial u}{\partial x_i} \right| = \frac{u}{v} \left| \frac{\partial v}{\partial x_i} \right|. \tag{2.4}$$

Using $II = 0$, it implies

$$\frac{\partial u}{\partial x_i} = c \frac{\partial v}{\partial x_i}. \tag{2.5}$$

Putting (2.5) into (2.4) yields $u = cv$.

(b) If $u(x_0) = 0$, then we denote $S = \{x \in \Omega | u(x) = 0\}$ and $\frac{\partial u}{\partial x_i} = 0$ a.e. in S. Thus

$$\frac{\partial}{\partial x_i} \left(\frac{u}{v} \right) = \frac{v \frac{\partial u}{\partial x_i} - u \frac{\partial v}{\partial x_i}}{v^2} = 0,$$

which shows $u = cv$. The proof of Theorem 1.1 is completed. □

Let us address anisotropic Sobolev spaces; see Adams [27], Lu [28], Troisi [29] *etc.* Given a domain $\Omega \subset R^n$, $p_i > 1$, $i = 1, 2, \dots, n$. We define two anisotropic Sobolev spaces by

$$W^{1,(p_i)}(\Omega) = \left\{ u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, \dots, n \right\}$$

and

$$W_0^{1,(p_i)}(\Omega) = \left\{ u \in W_0^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, \dots, n \right\},$$

with the norms

$$\|u\|_{W^{1,(p_i)}(\Omega)} = \int_\Omega |u|\, dx + \sum_{i=1}^n \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}}$$

and

$$\|u\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^n \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}},$$

respectively. Note that $W_0^{1,(p_i)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,(p_i)}(\Omega)$. It is well known that $W^{1,(p_i)}(\Omega)$ and $W_0^{1,(p_i)}(\Omega)$ are both separable and reflexive Banach spaces.

We will show a Sturmian comparison principle to the anisotropic elliptic equation by Theorem 1.1.

Proposition 2.1 *Let $f_1(x)$ and $f_2(x)$ be two continuous functions with $f_1(x) < f_2(x)$ in the bounded domain Ω. Assume that there exists a positive function $u \in W_0^{1,(p_i)}(\Omega)$ satisfying*

$$\begin{cases} -\sum_{i=1}^n \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}) = \sum_{i=1}^n f_1(x)u^{p_i-1}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.6}$$

Then any nontrivial solution v to the following anisotropic elliptic equation:

$$-\sum_{i=1}^n \frac{u^{p_i}}{v^{p_i-1}} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i} \right) = \sum_{i=1}^n f_2(x) u^{p_i}, \quad x \in \Omega, \tag{2.7}$$

must change sign.

Proof Suppose that v to (2.7) does not change sign, without loss of generality, let $v > 0$ in Ω. By (2.6), (2.7), and (1.7), we observe

$$\begin{aligned} 0 &\le \int_\Omega L(u,v)\, dx = \int_\Omega R(u,v)\, dx \\ &= \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i} \left(\frac{u^{p_i}}{v^{p_i-1}} \right) \left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i}\, dx \end{aligned}$$

$$= \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx + \sum_{i=1}^{n} \int_{\Omega} \frac{u^{p_i}}{v^{p_i-1}} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i} \right) dx$$
$$= \sum_{i=1}^{n} \int_{\Omega} \big(f_1(x) - f_2(x)\big) u^{p_i}\, dx$$
$$< 0,$$

which is a contradiction. This completes the proof. □

3 Proof of Theorem 1.3

To prove Theorem 1.3, we need a lemma from Theorem 1.1.

Lemma 3.1 *If there exist a constant $k_i > 0$ and a function $h_i(x)$, $i = 1, \ldots, n$, such that a differentiable function $v > 0$ in the set Ω satisfies*

$$-\frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i} \right) \geq k_i h_i(x) v^{p_i-1}, \tag{3.1}$$

then, for any $0 \leq u \in C_0^1(\Omega)$, we have

$$\sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \geq \sum_{i=1}^{n} k_i \int_{\Omega} h_i(x) u^{p_i}\, dx. \tag{3.2}$$

Proof By (3.1) and (1.7), we see

$$0 \leq \int_{\Omega} L(u,v)\, dx = \int_{\Omega} R(u,v)\, dx$$
$$= \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - \sum_{i=1}^{n} \int_{\Omega} \frac{\partial}{\partial x_i} \left(\frac{u^{p_i}}{v^{p_i-1}} \right) \left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i}\, dx$$
$$= \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx + \sum_{i=1}^{n} \int_{\Omega} \frac{u^{p_i}}{v^{p_i-1}} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i} \right) dx$$
$$\leq \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - \sum_{i=1}^{n} k_i \int_{\Omega} h_i(x) u^{p_i}\, dx,$$

which implies (3.2). □

Proof of Theorem 1.3 Without loss of generality, we let $0 \leq u \in C_0^\infty$. To use Lemma 3.1, we introduce the auxiliary function

$$v = \prod_{j=1}^{n} |x_j|^{\beta_j} := |x_i|^{\beta_i} \overline{v}_i, \tag{3.3}$$

where $\beta_j = \frac{p_j-1}{p_j}$ and $\overline{v}_i = \prod_{j=1, j\neq i}^{n} |x_j|^{\beta_j}$, hence

$$\frac{\partial v}{\partial x_i} = \beta_i \bar{v}_i |x_i|^{\beta_i-2} x_i,$$

$$\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2} = \beta_i^{p_i-2}\bar{v}_i^{p_i-2}|x_i|^{\beta_i p_i-2\beta_i-p_i+2},$$

$$\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i} = \beta_i^{p_i-1}\bar{v}_i^{p_i-1}|x_i|^{\beta_i p_i-\beta_i-p_i}x_i,$$

and

$$-\frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\right) = \left(\frac{p_i-1}{p_i}\right)^{p_i}\frac{v^{p_i-1}}{|x_i|^{p_i}}. \tag{3.4}$$

Taking $k_i = (\frac{p_i-1}{p_i})^{p_i}$ and $h_i(x) = \frac{1}{|x_i|^{p_i}}$, and using Lemma 3.1, we obtain (1.8). □

Corollary 3.2 *For $u \in C_0^1(A)$, it follows that*

$$\int_A |\nabla u|^2\,dx \geq \frac{n^2}{4}\int_A \frac{|u|^2}{|x|^2}\,dx. \tag{3.5}$$

Proof Letting $p_i = 2$ ($i = 1,\ldots,n$) in (1.8) and noting the elementary inequality

$$n\left(\sum_{i=1}^n \frac{1}{a_i}\right)^{-1} \leq \frac{1}{n}\left(\sum_{i=1}^n a_i\right) \quad \text{for } a_i \geq 0, i = 1,\ldots,n, \tag{3.6}$$

we have by taking $a_i = |x_i|^2$,

$$\begin{aligned}\int_A |\nabla u|^2\,dx &= \sum_{i=1}^n \int_A \left|\frac{\partial u}{\partial x_i}\right|^2 dx\\ &\geq \frac{1}{4}\int_A |u|^2\left(\sum_{i=1}^n \frac{1}{|x_i|^2}\right)dx\\ &\geq \frac{1}{4}\int_A |u|^2\left(\frac{n^2}{\sum_{i=1}^n |x_i|^2}\right)dx\\ &= \frac{n^2}{4}\int_A \frac{|u|^2}{|x|^2}\,dx.\end{aligned}$$

□

Corollary 3.3 *If $p > 2$, then, for $u \in C_0^1(A)$, it follows that*

$$\int_A |\nabla u|^p\,dx \geq \left(\frac{p-1}{p}\right)^p n^{\frac{p+2}{2}}\int_A \frac{|u|^p}{|x|^p}\,dx. \tag{3.7}$$

Proof Let $p_i = p > 2$ ($i = 1,\ldots,n$) in (1.8). Recall the inequality

$$\sum_{i=1}^n a_i^2 \leq \left(\sum_{i=1}^n a_i^p\right)^{\frac{2}{p}} n^{\frac{p-2}{p}} \quad \text{for } a_i \geq 0, i = 1,\ldots,n,$$

which gives

$$\sum_{i=1}^n a_i^p \geq n^{-\frac{p-2}{2}}\left(\sum_{i=1}^n a_i^2\right)^{\frac{p}{2}}. \tag{3.8}$$

Taking $a_i = \frac{1}{|x_i|}$ in (3.8), it implies by (3.6) that

$$\sum_{i=1}^{n} \frac{1}{|x_i|^p} \geq n^{-\frac{p-2}{2}} \left(\sum_{i=1}^{n} \frac{1}{|x_i|^2} \right)^{\frac{p}{2}} \geq n^{-\frac{p-2}{2}} \left(\frac{n^2}{\sum_{i=1}^{n} |x_i|^2} \right)^{\frac{p}{2}} = n^{\frac{p+2}{2}} \frac{1}{|x|^p}. \tag{3.9}$$

Putting (3.9) into the right-hand side of (1.8),

$$\sum_{i=1}^{n} \left(\frac{p-1}{p} \right)^p \int_A \frac{|u|^p}{|x_i|^p}\, dx \geq \left(\frac{p-1}{p} \right)^p n^{\frac{p+2}{2}} \int_A \frac{|u|^p}{|x|^p}\, dx. \tag{3.10}$$

On the other hand,

$$\int_A \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^p dx \leq \int_A \left(\sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{\frac{p}{2}} dx = \int_A |\nabla u|^p\, dx. \tag{3.11}$$

Hence (3.7) is proved via (3.10) and (3.11). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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