Kilicman and Borgohain *Journal of Inequalities and Applications* (2017) 2017:15
DOI 10.1186/s13660-016-1284-9

Journal of Inequalities and Applications
a SpringerOpen Journal

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Some new lacunary statistical convergence with ideals

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Abstract

In this paper, the idea of lacunary I_λ-statistical convergent sequence spaces is discussed which is defined by a Musielak-Orlicz function. We study relations between lacunary I_λ-statistical convergence with lacunary I_λ-summable sequences. Moreover, we study the I_λ-lacunary statistical convergence in probabilistic normed space and discuss some topological properties.

Keywords: Musielak-Orlicz function; ideal convergence; lacunary sequences; probabilistic normed space

1 Introduction

The concept of statistical convergence [1] which is the extended idea of convergence of real sequences has become an important tool in many branches of mathematics. For references one may see [2–8] and many more.

Similarly, I-convergence is also an extended notion of statistical convergence ([9]) of real sequences. A family of sets $I \subseteq 2^A$ (power sets of A) is an ideal if I is additive, *i.e.* $S, T \in I \Rightarrow S \cup T \in I$, and hereditary *i.e.* $S \in I$, $T \subseteq S \Rightarrow T \in I$, where A is any non-empty set.

A lacunary sequence is an increasing integer sequence $\theta = (i_j)$ such that $i_0 = 0$ and $h_j = i_j - i_{j-1} \to \infty$ as $j \to \infty$. As regards ideal convergence and lacunary ideal convergence, one may refer to [10–19] etc.

Note: Throughout this paper, θ will be determined by the interval $K_j = (k_{j-1}, k_j]$ and the ratio $\frac{k_j}{k_{j-1}}$ will be defined by ϕ_j.

2 Preliminary concepts

A sequence (x_i) of real numbers is statistically convergent to M if, for arbitrary $\xi > 0$, the set $K(\xi) = \{i \in \mathbb{N} : |x_i - M| \geq \xi\}$ has natural density zero, *i.e.*,

$$\lim_i \frac{1}{i} \sum_{j=1}^{i} \chi_{K(\xi)}(j) = 0,$$

where $\chi_{K(\xi)}$ denotes the characteristic function of $K(\xi)$.

A sequence (x_i) of elements of $\mathbb{R}$ is I-convergent to $M \in \mathbb{R}$ if, for each $\xi > 0$,

$$\{i \in \mathbb{N} : |x_i - M| \geq \xi\} \in I.$$

For any lacunary sequence $\theta = (i_j)$, the space N_θ is defined as (Freedman *et al.* [5])

$$N_\theta = \left\{ (x_i) : \lim_{j\to\infty} i_j^{-1} \sum_{i\in K_j} |x_i - M| = 0, \text{ for some } M \right\}.$$

The concept of a Musielak-Orlicz function is defined as $\mathcal{M} = (M_j)$. The sequence $\mathcal{N} = (N_i)$ is defined by

$$N_i(a) = \sup\{ |a|b - M_j(b) : b \geq 0 \}, \quad i = 1, 2, \ldots,$$

which is named the complementary function of a Musielak-Orlicz function $\mathcal{M}$ (see [20]) (throughout the paper $\mathcal{M}$ is a Musielak-Orlicz function).

If $\lambda = (\lambda_i)$ is a non-decreasing sequence of positive integers such that Λ denotes the set of all non-decreasing sequences of positive integers. We call a sequence $\{x_i\}_{i\in\mathbb{N}}$ lacunary I_λ-statistically convergent of order α to M, if, for each $\gamma > 0$ and $\xi > 0$,

$$\left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^\alpha} \left| \left\{ j \leq i : \frac{1}{h_i} \sum_{j\in I_i} M_j\left(\frac{|x_j - M|}{\rho^{(j)}} \right) \geq \gamma \right\} \right| \geq \xi \right\} \in I.$$

We denote the class of all lacunary I_λ-statistically convergent sequences of order α defined by a Musielak-Orlicz function by $S^\alpha_{I_\lambda}(\mathcal{M}, \theta)$.

Some particular cases:

1. If $M_j(x) = M(x)$, for all $j \in \mathbb{N}$, then $S^\alpha_{I_\lambda}(\mathcal{M}, \theta)$ is reduced to $S^\alpha_{I_\lambda}(M, \theta)$.
   Also, if $M_j(x) = x$, for all $j \in \mathbb{N}$, then $S^\alpha_{I_\lambda}(\mathcal{M}, \theta)$ will be changed as $S^\alpha_{I_\lambda}(\theta)$.
2. If $\lambda_i = i$, for all $i \in \mathbb{N}$, then $S^\alpha_{I_\lambda}(\mathcal{M}, \theta)$ will be reduced to $S^\alpha_I(\mathcal{M}, \theta)$.
3. If $\alpha = 1$, then α-density of any set is reduced to the natural density of the set. So, the set $S^\alpha_{I_\lambda}(\mathcal{M}, \theta)$ reduces to $S_{I_\lambda}(\mathcal{M}, \theta)$ for $\alpha = 1$.
4. If $\theta = (2^r)$ and $\alpha = 1$, then (x_j) is said to be I_λ-statistically convergent defined by a Musielak-Orlicz function, *i.e.* $(x_j) \in S_{I_\lambda}(\mathcal{M})$.
5. if $M_j(x) = x$, $\theta = (2^r)$, $\lambda_j = j$, $\alpha = 1$, then I_λ-lacunary statistically convergence of order α defined by Musielak-Orlicz function reduces to I-statistical convergence.

In this article, we define the concept of lacunary I_λ-statistically convergence of order α defined by $\mathcal{M}$ and investigate some results on these sequences. Later on, we investigate some results of lacunary I_λ-statistically convergence of real sequences in probabilistic normed space too.

3 Main results

Theorem 3.1 *Let $\lambda = (\lambda_i)$ and $\mu = (\mu_i)$ be two sequences in Λ such that $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$ for fixed reals α and β. If $\liminf_{i\to\infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} > 0$, then $S^\beta_{I_\mu}(\mathcal{M}, \theta) \subseteq S^\alpha_{I_\lambda}(\mathcal{M}, \theta)$.*

Proof Suppose that $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$ and $\liminf_{i\to\infty} \frac{\lambda_i^\alpha}{\mu_i^\beta} > 0$. Since $I_i \subset J_i$, where $J_i = [i - \mu_i + 1, i]$, so for $\gamma > 0$, we can write

$$\{ j \in J_i : |x_j - M| \geq \gamma \} \supset \{ j \in I_i : |x_j - M| \geq \gamma \},$$

which implies

$$\frac{1}{\mu_i^{\beta}}\left|\left\{j \in J_i : |x_j - M| \geq \gamma\right\}\right| \geq \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} \cdot \frac{1}{\lambda_i^{\alpha}}\left|\left\{j \in I_i : |x_j - M| \geq \gamma\right\}\right|,$$

for all $i \in \mathbb{N}$.

Assume that $\liminf_{i\to\infty} \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} = a$, so from the definition we see that $\{i \in \mathbb{B} : \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} < \frac{a}{2}\}$ is finite. Now for $\xi > 0$,

$$\left\{i \in \mathbb{N} : \frac{1}{\lambda_i^{\beta}}\left|\left\{j \in J_i : |x_j - M| \geq \gamma\right\}\right| \geq \xi\right\} \subset \left\{i \in \mathbb{N} : \frac{1}{\mu_i^{\alpha}}\left|\left\{j \in I_i : |x_j - M| \geq \gamma\right\}\right| \geq \frac{a}{2}\xi\right\} \cup \left\{i \in \mathbb{N} : \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} < \frac{a}{2}\right\}.$$

Since I is admissible and (x_j) is a lacunary I_μ-statistically convergent sequence of order β defined by $\mathcal{M}$, by using the continuity of $\mathcal{M}$, we see with the lacunary sequence $\theta = (h_i)$, the right hand side belongs to I, which completes the proof. □

Theorem 3.2 *If* $\lim_{i\to\infty} \frac{\mu_i}{\lambda_i^{\beta}} = 1$, *for* $\lambda = (\lambda_i)$ *and* $\mu = (\mu_i)$ *two sequences of* Λ *such that* $\lambda_i \leq \mu_i$, $\forall i \in \mathbb{N}$ *and* $0 < \alpha \leq \beta \leq 1$ *for fixed* α, β *reals, then* $S_{I_\lambda}^{\alpha}(\mathcal{M},\theta) \subseteq S_{I_\mu}^{\beta}(\mathcal{M},\theta)$.

Proof Let (x_j) be lacunary I_λ-statistically convergent to M of order α defined by $\mathcal{M}$. Also assume that $\lim_{i\to\infty} \frac{\mu_i}{\lambda_i^{\beta}} = 1$. Choose $m \in \mathbb{N}$ such that $|\frac{\mu_i}{\lambda_i^{\beta}} - 1| < \frac{\xi}{2}$, $\forall i \geq m$.

Since $I_i \subset J_i$, for $\gamma > 0$, we may write

$$\begin{aligned}
\frac{1}{\mu_i^{\beta}}\left|\left\{j \in J_i : |x_j - M| \geq \gamma\right\}\right| &= \frac{1}{\mu_i^{\beta}}\left|\left\{i - \mu_i + 1 \leq j \leq i - \lambda_i : |x_j - M| \geq \gamma\right\}\right| \\
&\quad + \frac{1}{\mu_i^{\beta}}\left|\left\{j \in I_i : |x_j - M| \geq \gamma\right\}\right| \\
&\leq \frac{\mu_i - \lambda_i}{\mu_i^{\beta}} + \frac{1}{\mu_i^{\beta}}\left|\left\{j \in I_i : |x_j - M| \geq \gamma\right\}\right| \\
&\leq \frac{\mu_i - \lambda_i^{\beta}}{\lambda_i^{\beta}} + \frac{1}{\mu_i^{\beta}}\left|\left\{j \in I_i : |x_j - M| \geq \gamma\right\}\right| \\
&\leq \left(\frac{\mu_i}{\lambda_i^{\beta}} - 1\right) + \frac{1}{\lambda_i^{\alpha}}\left|\left\{j \in I_i : |x_j - M| \geq \gamma\right\}\right| \\
&= \frac{\xi}{2} + \frac{1}{\lambda_i^{\alpha}}\left|\left\{j \in I_i : |x_j - M| \geq \gamma\right\}\right|.
\end{aligned}$$

Hence,

$$\left\{i \in \mathbb{N} : \frac{1}{\mu_i^{\beta}}\left|\left\{j \leq i : |x_j - M| \geq \gamma\right\}\right| \geq \xi\right\} \subset \left\{i \in \mathbb{N} : \frac{1}{\lambda_i^{\alpha}}\left|\left\{j \in I_i : |x_j - M| \geq \gamma\right\}\right| \geq \frac{\xi}{2}\right\} \cup \{1, 2, 3, \ldots, m\}.$$

Since (x_j) is lacunary I_λ-statistically convergent sequence of order α defined by $\mathcal{M}$ and since I is admissible, by using the continuity of $\mathcal{M}$, it follows that the set on the right hand side with the lacunary sequence $\theta = (h_i)$ belongs to I and

$$S^{\alpha}_{I_\lambda}(\mathcal{M},\theta) \subseteq S^{\beta}_{I_\mu}(\mathcal{M},\theta). \qquad \square$$

We define the lacunary I_λ-summable sequence of order α defined by $\mathcal{M}$ as

$$w^{\alpha}_{I_\lambda}(\mathcal{M},\theta) = \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^{\alpha}} \left(j \le i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{|x_j - M|}{\rho^{(j)}} \right) \ge \gamma \right) \right\} \in I.$$

Theorem 3.3 *Given* $\lambda = (\lambda_i), \mu = (\mu_i) \in \Lambda$. *Suppose that* $\lambda_i \le \mu_i$ *for all* $i \in \mathbb{N}$, $0 < \alpha \le \beta \le 1$. *Then*:

1. *If* $\liminf_{i\to\infty} \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} > 0$, *then* $w^{\beta}_{\mu}(\mathcal{M},\theta) \subset w^{\alpha}_{\lambda}(\mathcal{M},\theta)$.
2. *If* $\lim_{i\to\infty} \frac{\mu_i}{\lambda_i^{\beta}} = 1$, *then* $\ell_\infty \cap w^{\alpha}_{\lambda}(\mathcal{M},\theta) \subset w^{\beta}_{\mu}(\mathcal{M},\theta)$.

Theorem 3.4 *Let* $\lambda_i \le \mu_i$ *for all* $i \in \mathbb{N}$, *where* $\lambda, \mu \in \Lambda$. *Then, if* $\liminf_{i\to\infty} \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} > 0$, *and if* (x_j) *is lacunary* I_μ*-summable of order* β *defined by* $\mathcal{M}$, *then it is lacunary* I_λ*-statistically convergent of order* α *defined by* $\mathcal{M}$. *Here* $0 < \alpha \le \beta \le 1$, *for fixed reals* α *and* β.

Proof For any $\gamma > 0$, we have

$$\begin{aligned}
\sum_{j \in J_i} |x_j - M| &= \sum_{j \in J_i, |x_j - M| \ge \varepsilon} |x_j - M| + \sum_{j \in J_i, |x_j - M| < \varepsilon} |x_j - M| \\
&\ge \sum_{j \in I_i, |x_j - M| \ge \varepsilon} |x_j - M| + \sum_{j \in I_i, |x_j - M| \ge \varepsilon} |x_j - M| \\
&\ge \sum_{j \in I_i, |x_j - M| \ge \varepsilon} |x_j - M| \\
&\ge \left| \left\{ j \in I_i : |x_j - M| \ge \gamma \right\} \right| . \gamma .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{\mu_i^{\beta}} \sum_{j \in J_i} |x_j - M| &\ge \frac{1}{\mu_i^{\beta}} \left| \left\{ j \in I_i : |x_j - M| \ge \gamma \right\} \right| . \gamma \\
&\ge \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} . \frac{1}{\lambda_i^{\alpha}} \left| \left\{ j \in I_i : |x_j - M| \ge \gamma \right\} \right| . \gamma .
\end{aligned}$$

If $\liminf_{i\to\infty} \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} = a$, then $\{ i \in \mathbb{N} : \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} < \frac{a}{2} \}$ is finite. So, for $\delta > 0$, we get

$$\begin{aligned}
&\left\{ i \in \mathbb{N} : \frac{1}{\lambda_i^{\alpha}} \left| \left\{ j \le i : \sum_{j \in J_i} |x_j - M| \ge \gamma \right\} \right| \ge \xi \right\} \\
&\subset \left\{ i \in \mathbb{N} : \frac{1}{\mu_i^{\beta}} \left\{ j \in I_i : |x_j - M| \ge \gamma \right\} \ge \frac{a}{2} \xi \right\} \\
&\quad \cup \left\{ i \in \mathbb{N} : \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} < \frac{a}{2} \right\}.
\end{aligned}$$

Since I is admissible and (x_j) is lacunary I_μ-summable sequence of order β defined by $\mathcal{M}$, using its continuity and using the lacunary sequence $\theta = (h_i)$, we can conclude that $w^\beta_{I_\mu}(\mathcal{M},\theta) \subseteq S^\alpha_{I_\lambda}(\mathcal{M},\theta)$. □

Theorem 3.5 *Let* $\lim_{i\to\infty} \frac{\mu_i}{\lambda_i^\beta} = 1$, *where* $0 < \alpha \le \beta \le 1$ *for fixed reals* α *and* β *and* $\lambda_i \le \mu_i$, *for all* $i \in \mathbb{N}$, *where* $\lambda, \mu \in \Lambda$. *Also let* $\theta!$ *be a refinement of* θ. *Let* (x_j) *to be a bounded sequence. If* (x_j) *is lacunary* I_λ*-statistically convergent sequence of order* α *defined by* $\mathcal{M}$, *then it is also a lacunary* I_μ*-summable sequence of order* β *defined by* $\mathcal{M}$. *i.e.* $S^\alpha_{I_\lambda}(\mathcal{M},\theta) \subseteq w^\beta_{I_\mu}(\mathcal{M},\theta!)$.

Proof Suppose that (x_j) is lacunary I_λ-statistically convergent sequence of order α defined by $\mathcal{M}$.

Given that $\lim_{i\to\infty} \frac{\mu_i}{\lambda_i^\beta} = 1$, we can choose $s \in \mathbb{N}$ such that $|\frac{\mu_i}{\lambda_i^\beta} - 1| < \frac{\delta}{2}$, $\forall i \ge s$.

Assume that there are a finite number of points $\theta! = (j_i^!)$ in the interval $I_i = (j_{i-1}, j_i]$. Let there exists exactly one point $j_i^!$ of $\theta!$ in the interval I_i, that is, $j_{i-1} = j^!_{p-1} < j^!_p < j^!_{p+1} = j_i$, for $p \in \mathbb{N}$.

Let $I_i^1 = (j_{i-1}, j_p]$, $I_i^2 = (j_p, j_i]$, $h_i^1 = j_p - j_{i-1}$, $h_i^2 = j_i - j_p$. Since $I_i^1 \subset I_i$ and $I_i^2 \subset I_i$, both h_i^1 and h_i^2 tend to ∞ as $i \to \infty$. We have

$$
\begin{aligned}
&\frac{1}{\mu_i^\beta}\Big(h_i^{-1}\sum_{j\in J_i}|x_j - M|\Big) \\
&\quad\le \frac{1}{\mu_i^\beta}\Big(\big(h_i^{-1}h_i^1\big)\big(h_i^1\big)^{-1}\sum_{j\in I_i^1}|x_j - M| + \big(h_i^{-1}h_i^2\big)\big(h_i^2\big)^{-1}\sum_{j\in I_i^2}|x_j - M|\Big) \\
&\quad\le \Big(\frac{\mu_i - \lambda_i}{\mu_i^\beta}\Big)\big(h_i^{-1}h_i^1\big)\big(h_i^1\big)^{-1}L + \frac{1}{\mu_i^\beta}\Big(\big(h_i^{-1}h_i^2\big)\big(h_i^2\big)^{-1}\sum_{j\in I_i^2}|x_j - M|\Big) \\
&\quad\le \Big(\frac{\mu_i - \lambda_i^\beta}{\lambda_i^\beta}\Big)\big(h_i^{-1}h_i^1\big)\big(h_i^1\big)^{-1}L + \frac{1}{\mu_i^\beta}\Big(\big(h_i^{-1}h_i^2\big)\big(h_i^2\big)^{-1}\sum_{j\in I_i^2}|x_j - M|\Big) \\
&\quad\le \Big(\frac{\mu_i}{\lambda_i^\beta} - 1\Big)\big(h_i^{-1}h_i^1\big)\big(h_i^1\big)^{-1}L + \frac{1}{\mu_i^\beta}\Big(\big(h_i^{-1}h_i^2\big)\big(h_i^2\big)^{-1}\sum_{j\in I_i^2, |x_j - M|\ge\varepsilon}|x_j - M|\Big) \\
&\qquad + \frac{1}{\mu_i^\beta}\Big(\big(h_i^{-1}h_i^2\big)\big(h_i^2\big)^{-1}\sum_{j\in I_i^2, |x_j - M|<\varepsilon}|x_j - M|\Big) \\
&\quad\le \Big(\frac{\mu_i}{\lambda_i^\beta} - 1\Big)\big(h_i^{-1}h_i^1\big)\big(h_i^1\big)^{-1}L + \frac{L}{\lambda_i^\alpha}\big|\big\{j \in I_i : \big(h_i^{-1}h_i^2\big)\big(h_i^2\big)^{-1}|x_j - M| \ge \varepsilon\big\}\big| \\
&\qquad + \varepsilon\big(h_i^{-1}h_i^2\big)\big(h_i^2\big)^{-1}, \quad \forall i \in \mathbb{N} \\
&\quad= \frac{\delta}{2}\big(h_i^{-1}h_i^1\big)\big(h_i^1\big)^{-1}L + \frac{L}{\lambda_i^\alpha}\big|\big\{j \in I_i : \big(h_i^{-1}h_i^2\big)\big(h_i^2\big)^{-1}|x_j - M| \ge \varepsilon\big\}\big| + \varepsilon\big(h_i^{-1}h_i^2\big)\big(h_i^2\big)^{-1}.
\end{aligned}
$$

Since $x \in w^\beta_{I_\mu}(\mathcal{M},\theta!)$, we have $0 < h_i^{-1}h_i^1 \le 1$ and $0 < h_i^{-1}h_i^2 \le 1$.

Hence, for $\xi > 0$,

$$\left\{i \in \mathbb{N} : \frac{1}{\mu_i^{\beta}}\left(\frac{1}{h_i}\sum_{j\in J_i}|x_j - M| \geq \gamma\right) \geq \xi\right\} \subset \left\{i \in \mathbb{N} : \frac{L}{\lambda_i^{\alpha}}\left|\left\{j \in I_i : \frac{1}{h_i^2}|x_j - M| \geq \gamma\right\}\right| \geq \xi\right\} \cup \{1, 2, 3, \ldots, s\}.$$

Since (x_j) is lacunary I_λ-statistically convergent sequence of order α defined by $\mathcal{M}$ and since I is admissible, by using the continuity of $\mathcal{M}$, we can say that

$$S_{I_\lambda}^{\alpha}(\mathcal{M}, \theta) \subseteq w_{I_\mu}^{\beta}(\mathcal{M}, \theta!). \qquad \square$$

Corollary 3.1 *Let* $\lambda \leq \mu_i$ *for all* $i \in \mathbb{N}$ *and* $0 < \alpha \leq \beta \leq 1$. *Let* $\liminf_{i\to\infty} \frac{\lambda_i^{\alpha}}{\mu_i^{\beta}} > 0$, $\theta!$ *be the refinement of* θ. *Also let* $\mathcal{M} = (M_i)$ *be a Musielak-Orlicz function where* (M_i) *is pointwise convergent. Then* $w_{I_\mu}^{\beta}(\mathcal{M}, \theta!) \subset S_{I_\lambda}^{\alpha}(\mathcal{M}, \theta)$ *iff* $\lim_i M_i(\frac{\gamma}{\rho^{(i)}}) > 0$, *for some* $\gamma > 0$, $\rho^{(i)} > 0$.

Corollary 3.2 *Let* $\mathcal{M} = (M_i)$ *be a Musielak-Orlicz function and* $\lim_{i\to\infty} \frac{\mu_i}{\lambda_i^{\beta}} = 1$, *for fixed numbers* α *and* β *such that* $0 < \alpha \leq \beta \leq 1$. *Then* $S_{I_\lambda}^{\alpha}(\mathcal{M}, \theta) \subset w_{I_\mu}^{\beta}(\mathcal{M}, \theta)$ *iff* $\sup_\nu \sup_i(\frac{\nu}{\rho^{(i)}})$.

4 Lacunary I_λ-statistical convergence in probabilistic normed spaces

Let X be a real linear space and $\nu : X \to D$, where D is the set of all distribution functions $g : \mathbb{R} \to \mathbb{R}_0^+$ such that it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} g(t) = 0$ and $\sup_{t\in\mathbb{R}} g(t) = 1$. The probabilistic norm or ν-norm is a t-norm [21] satisfying the following conditions:

1. $\nu_p(0) = 0$,
2. $\nu_p(t) = 1$ for all $t > 0$ iff $p = 0$,
3. $\nu_{\alpha p}(t) = \nu_p(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}\backslash\{0\}$ and for all $t > 0$,
4. $\nu_{p+q}(s + t) \geq \tau(\nu_p(s), \nu_q(t))$ for all $p, q \in X$ and $s, t \in \mathbb{R}_0^+$;

(X, ν, τ) is named a probabilistic normed space, in short PNS.

A sequence $x = (x_i)$ is I-convergent to $M \in X$ in (X, ν, τ) for each $\xi > 0$ and $t > 0$, $\{i \in \mathbb{N} : \nu_{x_i - M}(t) \leq 1 - \xi\} \in I$ (here I is a non-trivial ideal of $\mathbb{N}$) [19].

We define a sequence $x = (x_i)$ to be lacunary I_λ-statistical convergent to M in (X, ν, τ) defined by $\mathcal{M}$, if, for each $\nu > 0$, $M > 0$, $\mu > 0$, $\xi > 0$ and $t > 0$,

$$\left\{i \in \mathbb{N} : \frac{1}{\lambda_i}\left|\left\{j \leq i : \frac{1}{h_i}\sum_{j\in I_i} M_j\left(\frac{\nu_{x_j - M}(t)}{\rho^{(j)}}\right) \leq 1 - \mu\right\}\right| \leq 1 - \xi\right\} \in I.$$

We write it as $I_\lambda^{\nu}(\theta)\lim x = \psi$.

Example: Let $(\mathbb{R}, \nu, \tau)$ be a PNS with the probabilistic norm $\nu_p(t) = \frac{t}{t+|p|}$ (for all $p \in \mathbb{R}$ and every $t > 0$) and $\tau(a, b) = ab$. Also, let I be a non-trivial admissible ideal such that $I = \{B \subset \mathbb{N} : \delta(B) = 0\}$. Define a sequence x as follows:

$$x_i = \begin{cases} \frac{1}{i} & \text{if } i = k^2, i \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have, for each $\nu > 0$, $M > 0$, $\mu > 0$, $\xi > 0$ and $t > 0$, $\delta(K) = 0$, where

$$K = \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \left| \left\{ j \leq i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{\nu_{x_j - M}(t)}{\rho^{(j)}} \right) \leq 1 - \mu \right\} \right| \leq 1 - \xi \right\},$$

which implies $K \in I$ and $I_\lambda^\nu(\theta) - \lim = 0$.

Theorem 4.1 *Let (X, ν, τ) be a PNS. If $x = (x_i)$ is lacunary I_λ^ν-statistical convergent, then it has a unique limit.*

Proof Suppose $x = (x_i)$ to be lacunary I_λ^ν-statistical convergent in X which has two limits, M_1 and M_2.

For $\beta > 0$ and $t > 0$, let us choose $\xi > 0$ such that $\tau((1 - \xi), (1 - \xi)) \geq 1 - \beta$.

Take the following sets:

$$K_1(\xi, t) = \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \left| \left\{ j \leq i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{\nu_{x_j - M_1}(t)}{\rho^{(j)}} \right) \leq 1 - \mu \right\} \right| \leq 1 - \xi \right\},$$

$$K_2(\xi, t) = \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \left| \left\{ j \leq i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{\nu_{x_j - M_2}(t)}{\rho^{(j)}} \right) \leq 1 - \mu \right\} \right| \leq 1 - \xi \right\}.$$

Since $x = (x_i)$ is lacunary I_λ^ν-statistical convergent to M_1, we have $K_1(\xi, t) \in I$. Similarly, $K_2(\xi, t) \in I$.

Now, let $K(\xi, t) = K_1(\xi, t) \cup K_2(\xi, t) \in I$. We see that $K(\xi, t)$ belongs to I, from which it is clear that $K^C(\xi, t)$ is non-empty set in $F(I)$, where $F(I)$ is the filter associated with the ideal I [9].

If $i \in K^C(\xi, t)$, then we have $i \in K_1^C(\xi, t) \cap K_2^C(\xi, t)$ and so

$$\nu_{M_1 - M_2}(t) \geq \tau \left(\nu_{x_i - M_1} \left(\frac{t}{2} \right), \nu_{x_i - M_2} \left(\frac{t}{2} \right) \right) > \tau((1 - \xi), (1 - \xi)).$$

Since $\tau((1 - \xi), (1 - \xi)) \geq 1 - \beta$, it follows that $\nu_{M_1 - M_2}(t) > 1 - \beta$.

For arbitrary $\beta > 0$, we get $\nu_{M_1 - M_2}(t) = 1$ for all $t > 0$, which proves $M_1 = M_2$. □

Theorem 4.2 *Let (X, ν, τ) be a PNS. If x is lacunary I^ν-statistical convergent, then it is lacunary I_λ^ν-statistical convergent if* $\lim_i \frac{\lambda_i}{i} > 0$.

Proof For given $\mu > 0$, $\xi > 0$, and $t > 0$,

$$\left\{ j \leq i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{\nu_{x_j - M}(t)}{\rho^{(j)}} \right) \leq 1 - \mu \right\} \supset \left\{ j \in I_i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{\nu_{x_j - M}(t)}{\rho^{(j)}} \right) \leq 1 - \mu \right\}.$$

Therefore,

$$\begin{aligned} &\frac{1}{i} \left\{ j \leq i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{\nu_{x_j - M}(t)}{\rho^{(j)}} \right) \leq 1 - \mu \right\} \\ &\quad \geq \frac{1}{i} \left\{ j \in I_i : \frac{1}{h_i} \sum_{j \in I_i} M_j \left(\frac{\nu_{x_j - M}(t)}{\rho^{(j)}} \right) \leq 1 - \mu \right\} \end{aligned}$$

$$\geq \frac{1}{\lambda_i}.\frac{\lambda_i}{i}\left\{j \in I_i : \frac{1}{h_i}\sum_{j\in I_i} M_j\left(\frac{\nu_{x_j-M}(t)}{\rho^{(j)}}\right) \leq 1-\mu\right\},$$

$$\left\{i \in \mathbb{N} : \frac{1}{i}\left\{j \leq i : \frac{1}{h_i}\sum_{j\in I_i} M_j\left(\frac{\nu_{x_j-M}(t)}{\rho^{(j)}}\right) \leq 1-\mu\right\} \leq 1-\xi\right\}$$

$$\geq \frac{\lambda_i}{i}\left\{i \in \mathbb{N} : \frac{1}{\lambda_i}\left\{j \in I_i : \frac{1}{h_i}\sum_{j\in I_i} M_j\left(\frac{\nu_{x_j-M}(t)}{\rho^{(j)}}\right) \leq 1-\mu\right\} \leq 1-\xi\right\}.$$

Since $\lim_i \frac{\lambda_i}{i} > 0$ and taking the limit $i \to \infty$, we get $I_\lambda^\nu(\theta) - \lim x = M$. □

We define $x = (x_i)$ to be lacunary λ-statistically convergent to M with respect to ν as

$$\delta\left(\left\{i \in \mathbb{N} : \frac{1}{\lambda_i}\left|\left\{j \leq i : \frac{1}{h_i}\sum_{j\in I_i} M_j\left(\frac{\nu_{x_j-M}(t)}{\rho^{(j)}}\right) \leq 1-\mu\right\}\right| \leq 1-r\right\}\right) = 0.$$

Theorem 4.3 *Let (X,ν,τ) be a PNS.*

1. *If x is lacunary λ-statistically convergent to M, then it is also lacunary I_λ^ν-statistically convergent to M.*
2. *If $I_\lambda^\nu(\theta) - \lim x = M_1$, $I_\lambda^\nu(\theta) - \lim y = M_2$, then $I_\lambda^\nu(\theta) - \lim(x_k + y_k) = (M_1 + M_2)$.*
3. *If $I_\lambda^\nu(\theta) - \lim x = M$,then $I_\lambda^\nu(\theta) - \lim \alpha x = \alpha M$.*

Theorem 4.4 *Let (X,ν,τ) be a PNS. If x is lacunary λ-statistical convergent to M, then $I_\lambda^\nu(\theta) - \lim x = M$.*

Proof Let $x = (x_i)$ be lacunary λ-statistically convergent to M, then, for every $t > 0$, $\xi > 0$ and $\mu > 0$, there exists $i_0 \in \mathbb{N}$ such that

$$\delta\left(\left\{i \in \mathbb{N} : \frac{1}{\lambda_i}\left\{j \leq i : \frac{1}{h_i}\sum_{j\in I_i} M_j\left(\frac{\nu_{x_j-\psi}(t)}{\rho^{(j)}}\right) \leq 1-\mu\right\} \leq 1-\xi\right\}\right) = 0,$$

for all $i \geq i_0$. Therefore the set

$$B = \left\{i \in \mathbb{N} : \left\{j \leq i : \frac{1}{h_i}\sum_{j\in I_i} M_j\left(\frac{\nu_{x_j-\psi}(t)}{\rho^{(j)}}\right) \leq 1-\mu\right\} \leq 1-\xi\right\} \subseteq \{1,2,3,\dots i_0-1\}.$$

Since I is admissible, we have $B \in I$. Hence $I_\lambda^\nu(\theta) - \lim x = \psi$. □

Theorem 4.5 *Let (X,ν,τ) be a PNS. If x is lacunary λ-statistical convergent, then it has a unique limit.*

Theorem 4.6 *Let (X,ν,τ) be a PNS. If x is lacunary λ-statistically convergent, then there exists a subsequence (x_{m_k}) of x such that it is also lacunary λ-statistically convergent to M.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both of the authors jointly worked on deriving the results and approved the final manuscript.

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Acknowledgements
The authors would like to extend their sincere appreciation to the referees for very useful comments and remarks for the earlier version of the manuscript.

Received: 29 August 2016 Accepted: 23 December 2016 Published online: 10 January 2017

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