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Improvements of the Hermite-Hadamard inequality for the simplex

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Abstract

In this study, the simplex whose vertices are barycenters of the given simplex facets plays an essential role. The article provides an extension of the Hermite-Hadamard inequality from the simplex barycenter to any point of the inscribed simplex except its vertices. A two-sided refinement of the generalized inequality is obtained in completion of this work.

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1 Introduction

A concise approach to the concept of affinity and convexity is as follows. Let $\mathbb{X}$ be a linear space over the field $\mathbb{R}$. Let $P_1, \ldots, P_m \in \mathbb{X}$ be points, and let $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ be coefficients. A linear combination

$$\sum_{j=1}^{m} \lambda_j P_j \tag{1}$$

is affine if $\sum_{j=1}^{m} \lambda_j = 1$. An affine combination is convex if all coefficients λ_j are nonnegative.

Let $\mathcal{S} \subseteq \mathbb{X}$ be a set. The set containing all affine combinations of points of $\mathcal{S}$ is called the affine hull of the set $\mathcal{S}$, and it is denoted with $\operatorname{aff}\mathcal{S}$. A set $\mathcal{S}$ is affine if $\mathcal{S} = \operatorname{aff}\mathcal{S}$. Using the adjective convex instead of affine, and the prefix conv instead of aff, we obtain the characterization of the convex set.

A convex function $f : \operatorname{conv}\mathcal{S} \to \mathbb{R}$ satisfies the Jensen inequality

$$f\left(\sum_{j=1}^{m} \lambda_j P_j\right) \le \sum_{j=1}^{m} \lambda_j f(P_j) \tag{2}$$

for all convex combinations of points $P_j \in \mathcal{S}$. An affine function $f : \operatorname{aff}\mathcal{S} \to \mathbb{R}$ satisfies the equality in equation (2) for all affine combinations of points $P_j \in \mathcal{S}$.

Throughout the paper, we use the n-dimensional space $\mathbb{X} = \mathbb{R}^n$ over the field $\mathbb{R}$.

2 Convex functions on the simplex

The section is a review of the known results on the Hermite-Hadamard inequality for simplices, and it refers to its generic background. The main notification is concentrated in Lemma 2.1, which is also the generalization of the Hermite-Hadamard inequality.

Let $A_1,\dots,A_{n+1} \in \mathbb{R}^n$ be points so that the points $A_1 - A_{n+1},\dots,A_n - A_{n+1}$ are linearly independent. The convex hull of the points A_i written in the form of $A_1 \cdots A_{n+1}$ is called the n-simplex in $\mathbb{R}^n$, and the points A_i are called the vertices. So, we use the denotation

$$A_1 \cdots A_{n+1} = \operatorname{conv}\{A_1,\dots,A_{n+1}\}. \tag{3}$$

The convex hull of n vertices is called the facet or $(n-1)$-face of the given n-simplex.

The analytic presentation of points of an n-simplex $\mathcal{A} = A_1 \cdots A_{n+1}$ in $\mathbb{R}^n$ arises from the n-volume by means of the Lebesgue measure or the Riemann integral. We will use the abbreviation vol instead of vol_n.

Let $A \in \mathcal{A}$ be a point, and let $\mathcal{A}_i$ be the convex hull of the set containing the point A and vertices A_j for $j \neq i$, formally as

$$\mathcal{A}_i = \operatorname{conv}\{A_1,\dots,A_{i-1},A,A_{i+1},\dots,A_{n+1}\}. \tag{4}$$

Each $\mathcal{A}_i$ is a facet or n-subsimplex of $\mathcal{A}$, so $\mathrm{vol}(\mathcal{A}_i) = 0$ or $0 < \mathrm{vol}(\mathcal{A}_i) \leq \mathrm{vol}(\mathcal{A})$, respectively. The sets $\mathcal{A}_i$ satisfy $\mathcal{A} = \bigcup_{i=1}^{n+1} \mathcal{A}_i$ and $\mathrm{vol}(\mathcal{A}_i \cap \mathcal{A}_j) = 0$ for $i \neq j$, and so it follows that $\mathrm{vol}(\mathcal{A}) = \sum_{i=1}^{n+1} \mathrm{vol}(\mathcal{A}_i)$.

The point A can be uniquely represented as the convex combination of the vertices A_i by means of

$$A = \sum_{i=1}^{n+1} \alpha_i A_i, \tag{5}$$

where we have the coefficients

$$\alpha_i = \frac{\mathrm{vol}(\mathcal{A}_i)}{\mathrm{vol}(\mathcal{A})}. \tag{6}$$

If the point A belongs to the interior of the n-simplex $\mathcal{A}$, then all sets $\mathcal{A}_i$ are n-simplices, and consequently all coefficients α_i are positive. Furthermore, the reverse implications are valid.

If μ is a positive measure on $\mathbb{R}^n$, and if $\mathcal{S} \subseteq \mathbb{R}^n$ is a measurable set such that $\mu(\mathcal{S}) > 0$, then the integral mean point

$$S = \left(\frac{\int_{\mathcal{S}} x_1 \, d\mu(x)}{\mu(\mathcal{S})},\dots,\frac{\int_{\mathcal{S}} x_n \, d\mu(x)}{\mu(\mathcal{S})} \right) \tag{7}$$

is called the μ-barycenter of the set $\mathcal{S}$. In the above integrals, points $x \in \mathcal{S}$ are used as $x = (x_1,\dots,x_n)$. The μ-barycenter S belongs to the convex hull of $\mathcal{S}$. When we use the Lebesgue measure, we say just barycenter. If $\mathcal{S}$ is closed and convex, then a μ-integrable continuous convex function $f : \mathcal{S} \to \mathbb{R}$ satisfies the inequality

$$f\left(\frac{\int_{\mathcal{S}} x_1 \, d\mu(x)}{\mu(\mathcal{S})},\dots,\frac{\int_{\mathcal{S}} x_n \, d\mu(x)}{\mu(\mathcal{S})} \right) \leq \frac{\int_{\mathcal{S}} f(x) \, d\mu(x)}{\mu(\mathcal{S})} \tag{8}$$

as a special case of Jensen's inequality for multivariate convex functions; see the excellent McShane paper in [1]. If f is affine, then the equality is valid in (8).

We consider a convex function f defined on the n-simplex $\mathcal{A} = A_1 \cdots A_{n+1}$. The following lemma presents a basic inequality for a convex function on the simplex, and it refers to the connection of the simplex barycenter with simplex vertices.

Lemma 2.1 *Let μ be a positive measure on $\mathbb{R}^n$. Let $\mathcal{A} = A_1 \cdots A_{n+1}$ be an n-simplex in $\mathbb{R}^n$ such that $\mu(\mathcal{A}) > 0$. Let A be the μ-barycenter of $\mathcal{A}$, and let $\sum_{i=1}^{n+1} \alpha_i A_i$ be its unique convex combination by means of*

$$A = \left(\frac{\int_{\mathcal{A}} x_1 \, d\mu(x)}{\mu(\mathcal{A})}, \dots, \frac{\int_{\mathcal{A}} x_n \, d\mu(x)}{\mu(\mathcal{A})} \right) = \sum_{i=1}^{n+1} \alpha_i A_i. \tag{9}$$

Then each convex function $f : \mathcal{A} \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{i=1}^{n+1} \alpha_i A_i \right) \le \frac{\int_{\mathcal{A}} f(x) \, d\mu(x)}{\mu(\mathcal{A})} \le \sum_{i=1}^{n+1} \alpha_i f(A_i). \tag{10}$$

Proof We have three cases depending on the position of the μ-barycenter A within the simplex $\mathcal{A}$.

If A is an interior point of $\mathcal{A}$, then we take a supporting hyperplane $x_{n+1} = h_1(x)$ at the graph point $(A, f(A))$, and the secant hyperplane $x_{n+1} = h_2(x)$ passing through the graph points $(A_1, f(A_1)), \dots, (A_{n+1}, f(A_{n+1}))$. Using the affinity of the functions h_1 and h_2, we get

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} \alpha_i A_i \right) &= h_1(A) = \frac{\int_{\mathcal{A}} h_1(x) \, d\mu(x)}{\mu(\mathcal{A})} \\ &\le \frac{\int_{\mathcal{A}} f(x) \, d\mu(x)}{\mu(\mathcal{A})} \\ &\le \frac{\int_{\mathcal{A}} h_2(x) \, d\mu(x)}{\mu(\mathcal{A})} = h_2(A) \\ &= \sum_{i=1}^{n+1} \alpha_i h_2(A_i) = \sum_{i=1}^{n+1} \alpha_i f(A_i) \end{aligned} \tag{11}$$

because $h_2(A_i) = f(A_i)$. So, formula (10) works for the interior point A.

If A is a relative interior point of a certain k-face where $1 \le k \le n-1$, then we can apply the previous procedure to the respective k-simplex. For example, if $A_1 \cdots A_{k+1}$ is the observed k-face, then the coefficients $\alpha_1, \dots, \alpha_{k+1}$ are positive, and the coefficients $\alpha_{k+2}, \dots, \alpha_{n+1}$ are equal to zero.

If A is a simplex vertex, suppose that $A = A_1$, then the trivial inequality $f(A_1) \le f(A_1) \le f(A_1)$ represents formula (10). □

More generally, if the μ-barycenter A lies in the interior of $\mathcal{A}$, the inequality in formula (10) holds for all μ-integrable functions $f : \mathcal{A} \to \mathbb{R}$ that admit a supporting hyperplane at A, and satisfy the supporting-secant hyperplane inequality

$$h_1(x) \le f(x) \le h_2(x) \tag{12}$$

for every point x of the simplex $\mathcal{A}$.

Lemma 2.1 was obtained in [2], Corollary 1, the case $\alpha_i = 1/(n+1)$ was obtained in [3], Theorem 2, and a similar result was obtained in [4], Theorem 2.4.

By applying the Lebesgue measure or the Riemann integral in Lemma 2.1, the condition in (9) gives the barycenter

$$A = \left(\frac{\int_{\mathcal{A}} x_1\, dx}{\mathrm{vol}(\mathcal{A})}, \dots, \frac{\int_{\mathcal{A}} x_n\, dx}{\mathrm{vol}(\mathcal{A})} \right) = \frac{\sum_{i=1}^{n+1} A_i}{n+1}, \tag{13}$$

and its use in formula (10) implies the Hermite-Hadamard inequality

$$f\left(\frac{\sum_{i=1}^{n+1} A_i}{n+1} \right) \le \frac{\int_{\mathcal{A}} f(x)\, dx}{\mathrm{vol}(\mathcal{A})} \le \frac{\sum_{i=1}^{n+1} f(A_i)}{n+1}. \tag{14}$$

The above inequality was introduced by Neuman in [5]. An approach to this inequality can be found in [6].

The discrete version of Lemma 2.1 contributes to the Jensen inequality on the simplex.

Corollary 2.2 *Let $\mathcal{A} = A_1 \cdots A_{n+1}$ be an n-simplex in $\mathbb{R}^n$, and let $P_1, \dots, P_m \in \mathcal{A}$ be points. Let $A = \sum_{j=1}^m \lambda_j P_j$ be a convex combination of the points P_j, and let $\sum_{i=1}^{n+1} \alpha_i A_i$ be the unique convex combination of the vertices A_i such that*

$$A = \sum_{j=1}^{m} \lambda_j P_j = \sum_{i=1}^{n+1} \alpha_i A_i. \tag{15}$$

Then each convex function $f : \mathcal{A} \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{i=1}^{n+1} \alpha_i A_i \right) \le \sum_{j=1}^{m} \lambda_j f(P_j) \le \sum_{i=1}^{n+1} \alpha_i f(A_i). \tag{16}$$

Proof The discrete measure μ concentrated at the points P_j by the rule

$$\mu\left(\{P_j\}\right) = \lambda_j \tag{17}$$

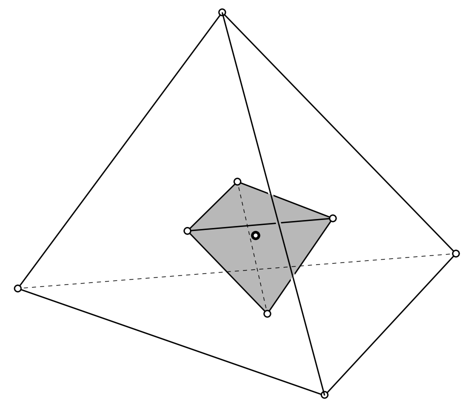
can be utilized in Lemma 2.1 to obtain the discrete inequality in formula (16). □

Putting $\sum_{j=1}^m \lambda_j P_j$ instead of $\sum_{i=1}^{n+1} \alpha_i A_i$ within the first term of formula (16), we obtain the Jensen inequality extended to the right.

Corollary 2.2 in the case $\alpha_i = 1/(n+1)$ was obtained in [3], Corollary 4.

One of the most influential results of the theory of convex functions is the Jensen inequality (see [7] and [8]), and among the most beautiful results is certainly the Hermite-Hadamard inequality (see [9] and [10]). A significant generalization of the Jensen inequality for multivariate convex functions can be found in [1]. Improvements of the Hermite-Hadamard inequality for univariate convex functions were obtained in [11]. As for the Hermite-Hadamard inequality for multivariate convex functions, one may refer to [2, 4, 5, 12–16], and [17].

Figure 1 The inscribed simplex as the barycenter extension.



3 Main results

Throughout the section, we will use an n-simplex $\mathcal{A} = A_1 \cdots A_{n+1}$ in the space $\mathbb{R}^n$, and its two n-subsimplices which will be denoted with $\mathcal{B}$ and $\mathcal{C}$.

Let B_i stand for the barycenter of the facet of $\mathcal{A}$ not containing the vertex A_i by

$$B_i = \frac{\sum_{i \neq j=1}^{n+1} A_j}{n}, \tag{18}$$

and let $\mathcal{B} = B_1 \cdots B_{n+1}$ be the n-simplex of the vertices B_i.

The simplices $\mathcal{A}$ and $\mathcal{B}$ in our three-dimensional space are tetrahedrons presented in Figure 1. Our aim is to extend the Hermite-Hadamard inequality to all points of the inscribed simplex $\mathcal{B}$ excepting its vertices. So, we focus on the non-peaked simplex $\mathcal{B}' = \mathcal{B} \setminus \{B_1, \dots, B_{n+1}\}$.

Lemma 3.1 *Let $\mathcal{A} = A_1 \cdots A_{n+1}$ be an n-simplex in $\mathbb{R}^n$, and let $A = \sum_{i=1}^{n+1} \alpha_i A_i$ be a convex combination of the vertices A_i.*

The point A belongs to the n-simplex $\mathcal{B} = B_1 \cdots B_{n+1}$ if and only if the coefficients α_i satisfy $\alpha_i \leq 1/n$.

The point A belongs to the non-peaked simplex $\mathcal{B}' = \mathcal{B} \setminus \{B_1, \dots, B_{n+1}\}$ if and only if the coefficients α_i satisfy $0 < \alpha_i \leq 1/n$.

Proof The first statement, relating to the simplex $\mathcal{B}$, will be covered as usual by proving two directions.

Let us assume that the coefficients α_i satisfy the limitations $\alpha_i \leq 1/n$. Then the coefficients

$$\beta_i = 1 - n\alpha_i \tag{19}$$

are nonnegative, and their sum is equal to 1. Since $\beta_i = 1 - \sum_{i \neq j=1}^{n+1} \beta_j$, the reverse connection

$$\alpha_i = \frac{\sum_{i \neq j=1}^{n+1} \beta_j}{n} \tag{20}$$

follows. The last of the convex combinations

$$
\begin{aligned}
A &= \sum_{i=1}^{n+1} \alpha_i A_i \\
&= \sum_{i=1}^{n+1} \frac{\sum_{i\neq j=1}^{n+1} \beta_j}{n} A_i = \sum_{i=1}^{n+1} \beta_i \frac{\sum_{i\neq j=1}^{n+1} A_j}{n} \\
&= \sum_{i=1}^{n+1} \beta_i B_i
\end{aligned}
\tag{21}
$$

confirms that the point A belongs to the simplex $\mathcal{B}$.

Let us assume that the point A belongs to the simplex $\mathcal{B}$. Then we have the convex combination $A = \sum_{i=1}^{n+1} \lambda_i B_i$. Using equation (21) in the reverse direction, we get the convex combinations equality

$$
\sum_{i=1}^{n+1} \lambda_i B_i = \sum_{i=1}^{n+1} \alpha_i A_i
\tag{22}
$$

with the coefficient connections $\alpha_i = \sum_{i\neq j=1}^{n+1} \lambda_j / n$ from which we may conclude that $\alpha_i \leq 1/n$.

The second statement, relating to the non-peaked simplex $\mathcal{B}'$, follows from the first statement and the convex combinations in formula (18) which uniquely represent the facet barycenters B_i. □

We need another subsimplex of $\mathcal{A}$. Let A be a point belonging to the interior of $\mathcal{A}$. In this case, the sets $\mathcal{A}_i$ defined by formula (4) are n-simplices. Let C_i stand for the barycenter of the simplex $\mathcal{A}_i$ by means of

$$
C_i = \frac{A + \sum_{i\neq j=1}^{n+1} A_j}{n+1},
\tag{23}
$$

and let $\mathcal{C} = C_1 \cdots C_{n+1}$ be the n-simplex of the vertices C_i.

Lemma 3.2 *Let $\mathcal{A} = A_1 \cdots A_{n+1}$ be an n-simplex in $\mathbb{R}^n$, and let $A = \sum_{i=1}^{n+1} \alpha_i A_i$ be a convex combination of the vertices A_i with coefficients α_i satisfying $\alpha_i > 0$.*

The point A belongs to the non-peaked simplex $\mathcal{C}' = \mathcal{C} \setminus \{C_1, \dots, C_{n+1}\}$ if and only if the coefficients α_i satisfy the additional limitations $\alpha_i \leq 1/n$.

Proof Suppose that the coefficients α_i satisfy $0 < \alpha_i \leq 1/n$. Let β_i be the coefficients as in equation (19). Using the trivial equality $A = A/(n+1) + nA/(n+1)$, and the coefficient connections of equation (20), we get

$$
\begin{aligned}
A &= \sum_{i=1}^{n+1} \alpha_i A_i = \frac{1}{n+1} A + \frac{n}{n+1} \sum_{i=1}^{n+1} \alpha_i A_i \\
&= \sum_{i=1}^{n+1} \beta_i \frac{A}{n+1} + \sum_{i=1}^{n+1} \sum_{i\neq j=1}^{n+1} \beta_j \frac{A_i}{n+1}
\end{aligned}
$$

$$= \sum_{i=1}^{n+1} \beta_i \frac{A}{n+1} + \sum_{i=1}^{n+1} \beta_i \sum_{i\neq j=1}^{n+1} \frac{A_j}{n+1}$$

$$= \sum_{i=1}^{n+1} \beta_i \frac{A + \sum_{i\neq j=1}^{n+1} A_j}{n+1} = \sum_{i=1}^{n+1} \beta_i C_i \tag{24}$$

indicating that the point A lies in the simplex $\mathcal{C}$. To show that the convex combination $\sum_{i=1}^{n+1} \beta_i C_i$ does not represent any vertex, we will assume that some $\beta_{i_0} = 1$. Then $\alpha_{i_0} = 0$ as opposed to the assumption that all α_i are positive.

The proof of the reverse implication goes exactly in the same way as in the proof of Lemma 3.1. □

Each simplex $\mathcal{C}$ is homothetic to the simplex $\mathcal{B}$. Namely, combining equations (23) and (18), we can represent each vertex C_i by the convex combination

$$C_i = \frac{1}{n+1} A + \frac{n}{n+1} B_i. \tag{25}$$

Then it follows that

$$C_i - A = \frac{n}{n+1} (B_i - A),$$

and using free vectors, we have the equality $\overrightarrow{AC_i} = (n/(n+1))\overrightarrow{AB_i}$. So, the simplices $\mathcal{C}$ and $\mathcal{B}$ are similar respecting the homothety with the center at A and the coefficient $n/(n+1)$.

If $A \in \mathcal{B}'$, then $\mathcal{C} \subset \mathcal{B}'$ by the convex combinations in formula (25). Combining Lemma 3.1 and Lemma 3.2, and applying Corollary 2.2 to the simplex inclusions $\mathcal{C} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$, we get the Jensen type inequality as follows.

Corollary 3.3 *Let $\mathcal{A} = A_1 \cdots A_{n+1}$ be an n-simplex in $\mathbb{R}^n$, let $A = \sum_{i=1}^{n+1} \alpha_i A_i$ be a convex combination of the vertices A_i with coefficients α_i satisfying $0 < \alpha_i \leq 1/n$, and let $\beta_i = 1 - n\alpha_i$. Then it follows that*

$$\sum_{i=1}^{n+1} \beta_i C_i = \sum_{i=1}^{n+1} \beta_i B_i = \sum_{i=1}^{n+1} \alpha_i A_i, \tag{26}$$

and each convex function $f : \mathcal{A} \to \mathbb{R}$ satisfies the double inequality

$$\sum_{i=1}^{n+1} \beta_i f(C_i) \leq \sum_{i=1}^{n+1} \beta_i f(B_i) \leq \sum_{i=1}^{n+1} \alpha_i f(A_i). \tag{27}$$

The point A used in the previous corollary lies in the interior of the simplex $\mathcal{A}$ because the coefficients α_i are positive. In that case, the sets $\mathcal{A}_i$ are n-simplices, and they will be used in the main theorem that follows.

Theorem 3.4 *Let $\mathcal{A} = A_1 \cdots A_{n+1}$ be an n-simplex in $\mathbb{R}^n$, let $A = \sum_{i=1}^{n+1} \alpha_i A_i$ be a convex combination of the vertices A_i with coefficients α_i satisfying $0 < \alpha_i \leq 1/n$, and let $\beta_i = 1 - n\alpha_i$. Let $\mathcal{A}_i$ be the simplices defined by formula* (4).

Then each convex function $f : \mathcal{A} \to \mathbb{R}$ *satisfies the double inequality*

$$f\left(\sum_{i=1}^{n+1} \alpha_i A_i\right) \le \sum_{i=1}^{n+1} \beta_i \frac{\int_{\mathcal{A}_i} f(x)\,dx}{\operatorname{vol}(\mathcal{A}_i)} \le \sum_{i=1}^{n+1} \alpha_i f(A_i). \tag{28}$$

Proof Using the convex combinations equality $\sum_{i=1}^{n+1} \alpha_i A_i = \sum_{i=1}^{n+1} \beta_i C_i$, and applying the Jensen inequality to $f(\sum_{i=1}^{n+1} \beta_i C_i)$, we get

$$f\left(\sum_{i=1}^{n+1} \alpha_i A_i\right) \le \sum_{i=1}^{n+1} \beta_i f(C_i) = \sum_{i=1}^{n+1} \beta_i f\left(\frac{A + \sum_{i\neq j=1}^{n+1} A_j}{n+1}\right).$$

Summing the products of the Hermite-Hadamard inequalities for the function f on the simplices $\mathcal{A}_i$ and the coefficients β_i, it follows that

$$\sum_{i=1}^{n+1} \beta_i f\left(\frac{A + \sum_{i\neq j=1}^{n+1} A_j}{n+1}\right) \le \sum_{i=1}^{n+1} \beta_i \frac{\int_{\mathcal{A}_i} f(x)\,dx}{\operatorname{vol}(\mathcal{A}_i)} \le \sum_{i=1}^{n+1} \beta_i \frac{f(A) + \sum_{i\neq j=1}^{n+1} f(A_j)}{n+1}.$$

Repeating the procedure which was used for the derivation of formula (24), we obtain the series of equalities

$$\begin{aligned}
\sum_{i=1}^{n+1} \beta_i \frac{f(A) + \sum_{i\neq j=1}^{n+1} f(A_j)}{n+1} &= \frac{1}{n+1} f(A) + \frac{n}{n+1} \sum_{i=1}^{n+1} \beta_i \frac{\sum_{i\neq j=1}^{n+1} f(A_j)}{n} \\
&= \frac{1}{n+1} f(A) + \frac{n}{n+1} \sum_{i=1}^{n+1} \frac{\sum_{i\neq j=1}^{n+1} \beta_j}{n} f(A_i) \\
&= \frac{1}{n+1} f\left(\sum_{i=1}^{n+1} \alpha_i A_i\right) + \frac{n}{n+1} \sum_{i=1}^{n+1} \alpha_i f(A_i).
\end{aligned}$$

Finally, applying the Jensen inequality to $f(\sum_{i=1}^{n+1} \alpha_i A_i)$, we get the last inequality

$$\frac{1}{n+1} f\left(\sum_{i=1}^{n+1} \alpha_i A_i\right) + \frac{n}{n+1} \sum_{i=1}^{n+1} \alpha_i f(A_i) \le \sum_{i=1}^{n+1} \alpha_i f(A_i).$$

Bringing together all of the above, we obtain the multiple inequality

$$\begin{aligned}
f\left(\sum_{i=1}^{n+1} \alpha_i A_i\right) &\le \sum_{i=1}^{n+1} \beta_i f\left(\frac{A + \sum_{i\neq j=1}^{n+1} A_j}{n+1}\right) \le \sum_{i=1}^{n+1} \beta_i \frac{\int_{\mathcal{A}_i} f(x)\,dx}{\operatorname{vol}(\mathcal{A}_i)} \\
&\le \sum_{i=1}^{n+1} \beta_i \frac{f(A) + \sum_{i\neq j=1}^{n+1} f(A_j)}{n+1} \le \sum_{i=1}^{n+1} \alpha_i f(A_i),
\end{aligned} \tag{29}$$

of which the most important part is the double inequality in formula (28). □

The inequality in formula (29) is a generalization and refinement of the Hermite-Hadamard inequality. Taking the coefficients $\alpha_i = 1/(n+1)$, in which case $\beta_i = 1/(n+1)$, we realize the five terms inequality

$$f\left(\frac{\sum_{i=1}^{n+1} A_i}{n+1}\right) \le \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{A_i + (n+2)\sum_{i\ne j=1}^{n+1} A_j}{(n+1)(n+1)}\right) \le \frac{\int_{\mathcal{A}} f(x)\,dx}{\operatorname{vol}(\mathcal{A})}$$
$$\le \frac{1}{n+1} f\left(\frac{\sum_{i=1}^{n+1} A_i}{n+1}\right) + \frac{n}{n+1}\frac{\sum_{i=1}^{n+1} f(A_i)}{n+1} \le \frac{\sum_{i=1}^{n+1} f(A_i)}{n+1}, \tag{30}$$

where the second and fourth terms refine the Hermite-Hadamard inequality. The third term is generated from all of $n+1$ simplices $\mathcal{A}_i$. In the present case, these simplices have the same volume equal to $\operatorname{vol}(\mathcal{A})/(n+1)$.

The inequality in formula (30) excepting the second term was obtained in [2], Theorem 2. Similar inequalities concerning the standard n-simplex were obtained in [5, 6] and [18]. Special refinements of the left and right-hand side of the Hermite-Hadamard inequality were recently obtained in [19] and [20].

4 Generalization to the function barycenter

If μ is a positive measure on $\mathbb{R}^n$, if $\mathcal{S} \subseteq \mathbb{R}^n$ is a measurable set, and if $g : \mathcal{S} \to \mathbb{R}$ is a nonnegative integrable function such that $\int_{\mathcal{S}} g(x)\,d\mu(x) > 0$, then the integral mean point

$$S = \left(\frac{\int_{\mathcal{S}} x_1 g(x)\,d\mu(x)}{\int_{\mathcal{S}} g(x)\,d\mu(x)},\dots,\frac{\int_{\mathcal{S}} x_n g(x)\,d\mu(x)}{\int_{\mathcal{S}} g(x)\,d\mu(x)}\right) \tag{31}$$

can be called the μ-barycenter of the function g. It is about the following measure. Introducing the measure ν as

$$\nu(S) = \int_{\mathcal{S}} g(x)\,d\mu(x), \tag{32}$$

we get

$$S = \left(\frac{\int_{\mathcal{S}} x_1\,d\nu(x)}{\nu(\mathcal{S})},\dots,\frac{\int_{\mathcal{S}} x_n\,d\nu(x)}{\nu(\mathcal{S})}\right). \tag{33}$$

Thus the μ-barycenter of the function g coincides with the ν-barycenter of its domain $\mathcal{S}$. So, the barycenter S belongs to the convex hull of the set $\mathcal{S}$. By using the unit function $g(x) = 1$ in formula (31), it is reduced to formula (7).

Utilizing the function barycenter instead of the set barycenter, we have the following reformulation of Lemma 2.1.

Lemma 4.1 *Let μ be a positive measure on $\mathbb{R}^n$. Let $\mathcal{A} = A_1 \cdots A_{n+1}$ be an n-simplex in $\mathbb{R}^n$, and let $g : \mathcal{A} \to \mathbb{R}$ be a nonnegative integrable function such that $\int_{\mathcal{A}} g(x)\,d\mu(x) > 0$. Let A be the μ-barycenter of g, and let $\sum_{i=1}^{n+1} \alpha_i A_i$ be its unique convex combination by means of*

$$A = \left(\frac{\int_{\mathcal{A}} x_1 g(x)\,d\mu(x)}{\int_{\mathcal{A}} g(x)\,d\mu(x)},\dots,\frac{\int_{\mathcal{A}} x_n g(x)\,d\mu(x)}{\int_{\mathcal{A}} g(x)\,d\mu(x)}\right) = \sum_{i=1}^{n+1} \alpha_i A_i. \tag{34}$$

Then each convex function $f : \mathcal{A} \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{i=1}^{n+1} \alpha_i A_i\right) \le \frac{\int_{\mathcal{A}} f(x)g(x)\,d\mu(x)}{\int_{\mathcal{A}} g(x)\,d\mu(x)} \le \sum_{i=1}^{n+1} \alpha_i f(A_i). \tag{35}$$

The proof of Lemma 2.1 can be employed as the proof of Lemma 4.1 by using the measure ν in formula (32) or by utilizing the affinity of the hyperplanes h_1 and h_2 in the form of the equalities

$$h_{1,2}\left(\frac{\int_{\mathcal{A}} x_1 g(x)\,d\mu(x)}{\int_{\mathcal{A}} g(x)\,d\mu(x)}, \dots, \frac{\int_{\mathcal{A}} x_n g(x)\,d\mu(x)}{\int_{\mathcal{A}} g(x)\,d\mu(x)}\right) = \frac{\int_{\mathcal{A}} h_{1,2}(x)g(x)\,d\mu(x)}{\int_{\mathcal{A}} g(x)\,d\mu(x)}. \tag{36}$$

Lemma 4.1 is an extension of the Fejér inequality (see [21]) to multivariable convex functions. As regards univariable convex functions, using the Lebesgue measure on $\mathbb{R}$ and a closed interval as 1-simplex in Lemma 4.1, we get the following generalization of the Fejér inequality.

Corollary 4.2 *Let $[a,b]$ be a closed interval in $\mathbb{R}$, and let $g : [a,b] \to \mathbb{R}$ be a nonnegative integrable function such that $\int_a^b g(x)\,dx > 0$. Let c be the barycenter of g, and let $\alpha a + \beta b$ be its unique convex combination by means of*

$$c = \frac{\int_a^b x g(x)\,dx}{\int_a^b g(x)\,dx} = \alpha a + \beta b. \tag{37}$$

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le \alpha f(a) + \beta f(b). \tag{38}$$

Fejér used a nonnegative integrable function g that is symmetric with respect to the midpoint $c = (a+b)/2$. Such a function satisfies $g(x) = g(2c - x)$, and therefore

$$\int_a^b (x - c)g(x)\,dx = 0.$$

As a consequence it follows that

$$\frac{\int_a^b x g(x)\,dx}{\int_a^b g(x)\,dx} = \frac{\int_a^b (x-c)g(x)\,dx}{\int_a^b g(x)\,dx} + \frac{\int_a^b c g(x)\,dx}{\int_a^b g(x)\,dx} = \frac{a+b}{2},$$

and formula (38) with $\alpha = \beta = 1/2$ turns into the Fejér inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le \frac{f(a) + f(b)}{2}. \tag{39}$$

Using the barycenters of the restrictions of g onto simplices $\mathcal{A}_i$ in formula (4), we have the following generalization of Theorem 3.4.

Theorem 4.3 *Let μ be a positive measure on $\mathbb{R}^n$. Let $\mathcal{A} = A_1 \cdots A_{n+1}$ be an n-simplex in $\mathbb{R}^n$, let $A = \sum_{i=1}^{n+1} \alpha_i A_i$ be a convex combination of the vertices A_i with coefficients α_i satisfying $0 < \alpha_i \leq 1/n$, and let $\beta_i = 1 - n\alpha_i$. Let $\mathcal{A}_i$ be the simplices defined by formula (4), and let $g_i : \mathcal{A}_i \to \mathbb{R}$ be nonnegative integrable functions such that $\int_{\mathcal{A}_i} g_i(x)\,d\mu(x) > 0$ and*

$$C_i = \left(\frac{\int_{\mathcal{A}_i} x_1 g_i(x)\,d\mu(x)}{\int_{\mathcal{A}_i} g_i(x)\,d\mu(x)}, \ldots, \frac{\int_{\mathcal{A}_i} x_n g_i(x)\,d\mu(x)}{\int_{\mathcal{A}_i} g_i(x)\,d\mu(x)} \right) = \frac{A + \sum_{i\neq j=1}^{n+1} A_j}{n+1}. \tag{40}$$

Then each convex function $f : \mathcal{A} \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{i=1}^{n+1} \alpha_i A_i\right) \leq \sum_{i=1}^{n+1} \beta_i \frac{\int_{\mathcal{A}_i} f(x) g_i(x)\,d\mu(x)}{\int_{\mathcal{A}_i} g_i(x)\,d\mu(x)} \leq \sum_{i=1}^{n+1} \alpha_i f(A_i). \tag{41}$$

Proof The first step of the proof is to apply Lemma 4.1 to the functions f and g_i on the simplex $\mathcal{A}_i$ in the way of

$$f\left(\frac{A + \sum_{i\neq j=1}^{n+1} A_j}{n+1}\right) \leq \frac{\int_{\mathcal{A}_i} f(x) g_i(x)\,d\mu(x)}{\int_{\mathcal{A}_i} g_i(x)\,d\mu(x)} \leq \frac{f(A) + \sum_{i\neq j=1}^{n+1} f(A_j)}{n+1}.$$

Summing the products of the above inequalities with the coefficients β_i, we obtain the double inequality that may be combined with formula (29), and so we obtain the multiple inequality

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} \alpha_i A_i\right) &\leq \sum_{i=1}^{n+1} \beta_i f\left(\frac{A + \sum_{i\neq j=1}^{n+1} A_j}{n+1}\right) \leq \sum_{i=1}^{n+1} \beta_i \frac{\int_{\mathcal{A}_i} f(x) g_i(x)\,d\mu(x)}{\int_{\mathcal{A}_i} g_i(x)\,d\mu(x)} \\ &\leq \sum_{i=1}^{n+1} \beta_i \frac{f(A) + \sum_{i\neq j=1}^{n+1} f(A_j)}{n+1} \leq \sum_{i=1}^{n+1} \alpha_i f(A_i) \end{aligned} \tag{42}$$

containing the double inequality in formula (41). □

The conditions in formula (40) require that the μ-barycenter of the function g_i coincides with the barycenter $C_i = (A + \sum_{i\neq j=1}^{n+1} A_j)/(n+1)$ of the simplex $\mathcal{A}_i$.

Using the Lebesgue measure and functions $g_i(x) = 1$, the inequality in formula (42) reduces to the inequality in formula (29).

Competing interests
The author declares that he has no competing interests.

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