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Some strong convergence properties for arrays of rowwise ANA random variables

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Abstract

In this paper, some complete convergence, complete moment convergence, and mean convergence results for arrays of rowwise asymptotically negatively associated (ANA) random variables are obtained. These theorems not only generalize some well-known ones to ANA cases, but they also improve them.

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Keywords: arrays of rowwise ANA random variables; complete convergence; complete moment convergence; mean convergence

1 Introduction

First of all, we will restate some definitions of dependent structures.

Definition 1.1 A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is called negatively associated (NA) if for any disjoint subsets A and B of $\{1, 2, \dots, n\}$, and any real coordinatewise non-decreasing functions f_1 on \mathbb{R}^A and f_2 on \mathbb{R}^B ,

$$\text{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0, \quad (1.1)$$

whenever this covariance exists. An infinite family of random variables $\{X_n, n \geq 1\}$ is NA if every finite subfamily is NA.

For two nonempty disjoint sets S and T of real numbers, let $\sigma(S)$ and $\sigma(T)$ be the σ -fields separately generated by $\{X_i, i \in S\}$ and $\{X_i, i \in T\}$. Let $\text{dist}(S, T) = \min\{|j - k|, j \in S, k \in T\}$.

Definition 1.2 A sequence of random variables $\{X_n, n \geq 1\}$ is called $\tilde{\rho}$ (or ρ^*)-mixing if

$$\tilde{\rho}(s) = \sup\{\rho(S, T) : S, T \subset \mathbb{N}, \text{dist}(S, T) \geq s\} \rightarrow 0 \quad \text{as } s \rightarrow \infty, \quad (1.2)$$

where

$$\rho(S, T) = \sup\left\{ \frac{|EXY - EXEY|}{\sqrt{\text{Var } X} \cdot \sqrt{\text{Var } Y}}; X \in L_2(\sigma(S)), Y \in L_2(\sigma(T)) \right\}.$$

Definition 1.3 A sequence of random variables $\{X_n, n \geq 1\}$ is said to be ANA if

$$\rho^-(s) = \sup\{\rho^-(S, T) : S, T \subset \mathbb{N}, \text{dist}(S, T) \geq s\} \rightarrow 0 \quad \text{as } s \rightarrow \infty, \tag{1.3}$$

where

$$\rho^-(S, T) = 0 \vee \sup\left\{ \frac{\text{Cov}(f_1(X_i, i \in S), f_2(X_j, j \in T))}{(\text{Var}f_1(X_i, i \in S))^{1/2}(\text{Var}f_2(X_j, j \in T))^{1/2}} \right\},$$

where the supremum is taken over all coordinatewise non-decreasing functions f_1 on \mathbb{R}^S and f_2 on \mathbb{R}^T .

An array of random variables $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is called rowwise ANA random variables if for every $n \geq 1$, $\{X_{ni}, 1 \leq i \leq n\}$ is a sequence of ANA random variables.

The concept of NA was introduced by Joag-Dev and Proschan [1], the concept of $\tilde{\rho}$ -mixing was introduced by Bradley [2], and the concept of ANA was introduced by Zhang and Wang [3]. It is easily seen that $\rho^-(s) \leq \tilde{\rho}(s)$, and a sequence of ANA random variables is NA if and only if $\rho^-(1) = 0$. Hence, sequences of ANA random variables are a family of very wide scope, which contain NA random variable sequences and $\tilde{\rho}$ -mixing random variable sequences.

Since the notion of ANA random variables was introduced, many applications have been found. We can refer the reader to [3–13], and so forth.

The concept of complete convergence was first given by Hsu and Robbins [14]. A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to a constant λ if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - \lambda| > \varepsilon) < \infty.$$

In view of the Borel-Cantelli lemma, the above result implies that $X_n \rightarrow \lambda$ almost surely. Therefore, the notion of complete convergence is a very important tool in establishing almost sure convergence of summation of random variables.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0, b_n > 0, q > 0$. If for all $\varepsilon \geq 0$,

$$\sum_{n=1}^{\infty} a_n E(b_n^{-1} |X_n| - \varepsilon)_+^q < \infty,$$

then the above result was called the complete moment convergence by Chow [15].

Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise NA random variables, and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\psi_n(t), n \geq 1\}$ be a sequence of positive, even functions such that

$$\frac{\psi_n(|t|)}{|t|} \uparrow \quad \text{and} \quad \frac{\psi_n(|t|)}{|t|^p} \downarrow \quad \text{as } |t| \uparrow \tag{1.4}$$

for some nonnegative integer p . Introduce the following conditions:

$$EX_{ni} = 0, \quad 1 \leq i \leq n, n \geq 1, \tag{1.5}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)} < \infty, \tag{1.6}$$

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^n E \left(\frac{|X_{ni}|}{a_n} \right)^r \right)^s < \infty, \tag{1.7}$$

where $0 < r \leq 2$ and $s > 0$.

Gan and Chen [16] showed the following complete convergence theorems for NA cases.

Theorem A *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise NA random variables, and let $\{\psi_n(t), n \geq 1\}$ satisfy (1.4) for some integer $1 < p \leq 2$. Then (1.5) and (1.6) imply*

$$\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| \rightarrow 0 \text{ completely.} \tag{1.8}$$

Theorem B *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise NA random variables, and let $\{\psi_n(t), n \geq 1\}$ satisfy (1.4) for some integer $p > 2$. Then (1.5), (1.6), and (1.7) imply (1.8).*

Zhu [17] obtained the corresponding result for ρ^* -mixing cases.

Theorem C *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise ρ^* -mixing random variables, and let $\psi(t)$ be a positive, even function satisfying (1.4) for some integer $p \geq 2$. Then (1.5), (1.6), and*

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^n E \left(\frac{X_{ni}}{a_n} \right)^2 \right)^{v/2} < \infty \text{ for } v \geq p, \tag{1.9}$$

imply (1.8).

Inspired by the above obtained theorems, in this work, we will not only extend Theorems A, B, and C to ANA random variables, but also one obtains some much stronger conclusions under some more general conditions. The goal of this paper is to study complete convergence, complete moment convergence, and mean convergence for arrays of rowwise ANA random variables.

Throughout this paper, let $I(A)$ be the indicator function of the set A . The symbol C always stands for a generic positive constant, which may vary from one place to another, and $a_n = O(b_n)$ stands for $a_n \leq Cb_n$.

2 Main results

Now, the main results are presented in this section. The proofs will be given in the next section.

Theorem 2.1 *Let N be a positive integer, $M \geq 2$ and $0 \leq s < (1/6M)^{M/2}$. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise ANA random variables with $\rho^-(N) \leq s$ in each row, and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\psi_n(t), n \geq 1\}$ be a sequence of positive, even functions such that*

$$\frac{\psi_n(|t|)}{|t|^q} \uparrow \text{ and } \frac{\psi_n(|t|)}{|t|^p} \downarrow \text{ as } |t| \uparrow \tag{2.1}$$

for some $1 \leq q < p$.

(1) If $1 < p \leq 2$, then conditions (1.5) and (1.6) imply

$$\sum_{n=1}^{\infty} P\left(\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

(2) If $p > 2$, then conditions (1.5), (1.6), and (1.9) imply (2.2).

Theorem 2.2 Let N be a positive integer, $M \geq 2$ and $0 \leq s < (1/6M)^{M/2}$. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise ANA random variables with $\rho^-(N) \leq s$ in each row, and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\psi_n(t), n \geq 1\}$ be a sequence of positive, even functions satisfying (2.1) for some $1 \leq q < p$.

(1) If $1 < p \leq 2$, then conditions (1.5) and (1.6) imply

$$\sum_{n=1}^{\infty} a_n^{-q} E\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| - \varepsilon a_n\right)_+^q < \infty \quad \text{for all } \varepsilon > 0. \tag{2.3}$$

(2) If $p > 2$, then conditions (1.5), (1.6), and (1.9) imply (2.3).

Theorem 2.3 Let N be a positive integer, $M \geq 2$ and $0 \leq s < (1/6M)^{M/2}$. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise ANA random variables with $\rho^-(N) \leq s$ in each row, and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\psi_n(t), n \geq 1\}$ be a sequence of positive, even functions satisfying (2.1) for some $1 \leq q < p$.

(1) If $1 < p \leq 2$, then condition (1.5) and

$$\sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{2.4}$$

imply

$$\lim_{n \rightarrow \infty} E\left(\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right|\right)^q = 0. \tag{2.5}$$

(2) If $p > 2$, then conditions (1.5), (2.4), and

$$\sum_{i=1}^n \frac{E|X_{ni}|^2 I(|X_{ni}| \leq a_n)}{a_n^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{2.6}$$

imply (2.5).

Remark 2.1 Since NA random variables and $\tilde{\rho}$ -mixing random variables are two special cases of ANA random variables, Theorem 2.1 is an extension and improvement of Theorems A and B for NA random variables, Theorem C for $\tilde{\rho}$ -mixing random variables. In addition, in this work, we consider the case $1 \leq q < p$, which has a wider scope than the case $q = 1$ in Gan and Chen [16] and Zhu [17].

Remark 2.2 Under the conditions of Theorem 2.2, one has

$$\begin{aligned}
 \infty &> \sum_{n=1}^{\infty} a_n^{-q} E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| - \varepsilon a_n \right)_+^q \\
 &= \sum_{n=1}^{\infty} a_n^{-q} \int_0^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| - \varepsilon a_n > t^{1/q} \right) dt \\
 &\geq C \sum_{n=1}^{\infty} a_n^{-q} \int_0^{\varepsilon^q a_n^q} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > a_n \varepsilon + t^{1/q} \right) dt \\
 &\geq C \sum_{n=1}^{\infty} a_n^{-q} \int_0^{\varepsilon^q a_n^q} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > a_n \varepsilon + a_n \varepsilon \right) dt \\
 &\geq C \varepsilon^q \sum_{n=1}^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > 2 \varepsilon a_n \right) \quad \text{for all } \varepsilon > 0.
 \end{aligned} \tag{2.7}$$

Hence, from (2.7), one can clearly know that the complete moment convergence implies the complete convergence. Compared with the corresponding results of Gan and Chen [16] and Zhu [17], it is worth pointing out that Theorem 2.2 is much stronger and conditions are more general and much weaker.

3 Proofs

To prove the main results, the following lemmas are needed.

Lemma 3.1 (Wang and Lu [7]) *Let $\{X_n, n \geq 1\}$ be a sequence of ANA random variables, and let $\{f_n, n \geq 1\}$ be a sequence of real functions all of which are monotone non-decreasing (or all monotone non-increasing), then $\{f_n(X_n), n \geq 1\}$ is still a sequence of ANA random variables.*

Lemma 3.2 (Wang and Lu [7]) *For a positive integer $N \geq 1$, real numbers $M \geq 2$ and $0 \leq s < (\frac{1}{6M})^{M/2}$, let $\{X_n, n \geq 1\}$ be a sequence of ANA random variables with $\rho^-(N) \leq s$, $EX_n = 0$ and $E|X_n|^M < \infty$ for every $n \geq 1$. Then there exists a positive constant $C = C(M, N, s)$ such that*

$$E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right|^M \right) \leq C \left(\sum_{i=1}^n E|X_i|^M + \left(\sum_{i=1}^n EX_i^2 \right)^{M/2} \right). \tag{3.1}$$

In particular, for $M = 2$,

$$E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right|^2 \right) \leq C \sum_{i=1}^n EX_i^2.$$

Proof of Theorem 2.1 For any $1 \leq i \leq n, n \geq 1$, define

$$\begin{aligned}
 Y_{ni} &= -a_n I(X_{ni} < -a_n) + X_{ni} I(|X_{ni}| \leq a_n) + a_n I(X_{ni} > a_n), \\
 Z_{ni} &= X_{ni} - Y_{ni} = (X_{ni} + a_n) I(X_{ni} < -a_n) + (X_{ni} - a_n) I(X_{ni} > a_n).
 \end{aligned}$$

It is easy to check that, for all $\varepsilon > 0$,

$$\begin{aligned}
 P\left(\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > \varepsilon\right) &\leq P\left(\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_{ni} \right| > \varepsilon\right) + P\left(\max_{1 \leq i \leq n} |X_{ni}| > a_n\right) \\
 &= P\left(\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right| > \varepsilon - \frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right|\right) \\
 &\quad + P\left(\max_{1 \leq i \leq n} |X_{ni}| > a_n\right). \tag{3.2}
 \end{aligned}$$

First of all, we will show that

$$\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.3}$$

For $1 \leq i \leq n, n \geq 1, EX_{ni} = 0$, then $EY_{ni} = -EZ_{ni}$. If $X_{ni} > a_n, 0 < Z_{ni} = X_{ni} - a_n < X_{ni}$. If $X_{ni} < -a_n, X_{ni} < Z_{ni} = X_{ni} + a_n \leq 0$. So, $|Z_{ni}| \leq |X_{ni}|I(|X_{ni}| > a_n)$. Then from conditions (2.1) and (1.6), one has

$$\begin{aligned}
 \frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right| &= \frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EZ_{ni} \right| \\
 &\leq \frac{1}{a_n} \sum_{i=1}^n E|Z_{ni}| \\
 &\leq C \sum_{i=1}^n \frac{E|X_{ni}|I(|X_{ni}| > a_n)}{a_n} \\
 &\leq C \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.4}
 \end{aligned}$$

Hence, for n large enough,

$$\begin{aligned}
 P\left(\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > \varepsilon\right) &\leq P\left(\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right| > \frac{\varepsilon}{2}\right) \\
 &\quad + P\left(\max_{1 \leq i \leq n} |X_{ni}| > a_n\right).
 \end{aligned}$$

To prove (2.2), it suffices to show that

$$I_1 \triangleq \sum_{n=1}^{\infty} P\left(\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right| > \frac{\varepsilon}{2}\right) < \infty, \tag{3.5}$$

$$I_2 \triangleq \sum_{n=1}^{\infty} P\left(\max_{1 \leq i \leq n} |X_{ni}| > a_n\right) < \infty. \tag{3.6}$$

By Lemma 3.1, it obviously follows that $\{Y_{ni} - EY_{ni}, 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise ANA random variables with zero mean. For I_1 , note that $|Y_{ni}| \leq a_n$.

(1) If $1 \leq q < p \leq 2$, by the Markov inequality, Lemma 3.2 (for $M = 2$), (2.1), and (1.6), one has

$$\begin{aligned}
 I_1 &\leq C \sum_{n=1}^{\infty} \frac{1}{a_n^2} E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right|^2 \right) \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{a_n^2} \sum_{i=1}^n E |Y_{ni} - EY_{ni}|^2 \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{a_n^2} \sum_{i=1}^n E Y_{ni}^2 \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E |Y_{ni}|^p}{a_n^p} \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E \psi_i(|Y_{ni}|)}{\psi_i(a_n)} \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E \psi_i(|X_{ni}|)}{\psi_i(a_n)} < \infty.
 \end{aligned}
 \tag{3.7}$$

(2) If $1 \leq q < p$ and $p > 2$, by the Markov inequality, Lemma 3.2 (for $M > p > 2$), (2.1), (1.6), and (1.9), one also has

$$\begin{aligned}
 I_1 &\leq C \sum_{n=1}^{\infty} \frac{1}{a_n^M} E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right|^M \right) \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{a_n^M} \left(\sum_{i=1}^n E |Y_{ni} - EY_{ni}|^M + \left(\sum_{i=1}^n E |Y_{ni} - EY_{ni}|^2 \right)^{M/2} \right) \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E |Y_{ni}|^M}{a_n^M} + C \sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E |Y_{ni}|^2}{a_n^2} \right)^{M/2} \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E \psi_i(|Y_{ni}|)}{\psi_i(a_n)} + C \sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E |Y_{ni}|^2}{a_n^2} \right)^{M/2} \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E \psi_i(|X_{ni}|)}{\psi_i(a_n)} + C \sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E |X_{ni}|^2}{a_n^2} \right)^{M/2} < \infty.
 \end{aligned}
 \tag{3.8}$$

Note that $|Z_{ni}| \leq |X_{ni}| I(|X_{ni}| > a_n)$. By a standard argument, one has

$$\begin{aligned}
 I_2 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n P(|X_{ni}| > a_n) \\
 &= C \sum_{n=1}^{\infty} \sum_{i=1}^n E I(|X_{ni}| > a_n) \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E |X_{ni}|^q I(|X_{ni}| > a_n)}{a_n^q} \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E \psi_i(|X_{ni}|)}{\psi_i(a_n)} < \infty.
 \end{aligned}
 \tag{3.9}$$

The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2 For all $\varepsilon > 0$ and any $t \geq 0$, since

$$\begin{aligned} \sum_{n=1}^{\infty} a_n^{-q} E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| - \varepsilon a_n \right)_+^q &= \sum_{n=1}^{\infty} a_n^{-q} \int_0^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| - \varepsilon a_n > t^{1/q} \right) dt \\ &= \sum_{n=1}^{\infty} a_n^{-q} \int_0^{a_n^q} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > \varepsilon a_n + t^{1/q} \right) dt \\ &\quad + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > \varepsilon a_n + t^{1/q} \right) dt \\ &\leq \sum_{n=1}^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > \varepsilon a_n \right) \\ &\quad + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > t^{1/q} \right) dt \\ &\triangleq J_1 + J_2. \end{aligned} \tag{3.10}$$

By Theorem 2.1, one has $J_1 < \infty$. To prove (2.3), one needs only to show that $J_2 < \infty$. For any $1 \leq i \leq n, n \geq 1$, define

$$\begin{aligned} Y_{ni} &= -t^{1/q} I(X_{ni} < -t^{1/q}) + X_{ni} I(|X_{ni}| \leq t^{1/q}) + t^{1/q} I(X_{ni} > t^{1/q}), \\ Z_{ni} &= X_{ni} - Y_{ni} = (X_{ni} + t^{1/q}) I(X_{ni} < -t^{1/q}) + (X_{ni} - t^{1/q}) I(X_{ni} > t^{1/q}). \end{aligned}$$

It is easy to check that, for all $\varepsilon > 0$,

$$\begin{aligned} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > t^{1/q} \right) &\leq P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_{ni} \right| > t^{1/q} \right) + P \left(\bigcup_{i=1}^n (|X_{ni}| > t^{1/q}) \right) \\ &\leq P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_{ni} \right| > t^{1/q} \right) + \sum_{i=1}^n P(|X_{ni}| > t^{1/q}). \end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned} J_2 &= \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > t^{1/q} \right) dt \\ &\leq \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^n \int_{a_n^q}^{\infty} P(|X_{ni}| > t^{1/q}) dt \\ &\quad + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_{ni} \right| > t^{1/q} \right) dt \\ &\triangleq J_{21} + J_{22}. \end{aligned} \tag{3.12}$$

For J_{21} , by conditions (2.1) and (1.6), one has

$$\begin{aligned}
 J_{21} &= \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^n \int_{a_n^q}^{\infty} P(|X_{ni}| > t^{1/q}) dt \\
 &\leq \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^n \int_0^{\infty} P(|X_{ni}| I(|X_{ni}| > a_n) > t^{1/q}) dt \\
 &\leq \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^n E|X_{ni}|^q I(|X_{ni}| > a_n) \\
 &\leq \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} < \infty.
 \end{aligned} \tag{3.13}$$

For J_{22} , we will first show that

$$\max_{t \geq a_n^q} \frac{1}{t^{1/q}} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.14}$$

Similar to the proof of (3.4), by conditions (1.5), (1.6), and (2.1), one has

$$\begin{aligned}
 \max_{t \geq a_n^q} \frac{1}{t^{1/q}} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right| &= \max_{t \geq a_n^q} \frac{1}{t^{1/q}} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EZ_{ni} \right| \\
 &\leq C \max_{t \geq a_n^q} \frac{1}{t^{1/q}} \sum_{i=1}^n E|Z_{ni}| \\
 &\leq C \max_{t \geq a_n^q} \frac{1}{t^{1/q}} \sum_{i=1}^n E|X_{ni}| I(|X_{ni}| > t^{1/q}) \\
 &\leq C \sum_{i=1}^n \frac{E|X_{ni}| I(|X_{ni}| > a_n)}{a_n} \\
 &\leq C \sum_{i=1}^n \frac{E|X_{ni}|^q I(|X_{ni}| > a_n)}{a_n^q} \\
 &\leq C \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{3.15}$$

Hence, while n is sufficiently large, for $t \geq a_n^q$,

$$\max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right| \leq \frac{t^{1/q}}{2},$$

which implies

$$P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_{ni} \right| > t^{1/q} \right) \leq P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right| > \frac{t^{1/q}}{2} \right). \tag{3.16}$$

For $J_{22} < \infty$, we will consider the following two cases. Let $d_n = [a_n] + 1$.

(1) If $1 \leq q < p \leq 2$, by (3.16), the c_r inequality and Lemma 3.2, one has

$$\begin{aligned}
 J_{22} &\leq \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right| > \frac{t^{1/q}}{2}\right) dt \\
 &\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-2/q} \sum_{i=1}^n E(Y_{ni} - EY_{ni})^2 dt \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \int_{a_n^q}^{\infty} EY_{ni}^2 t^{-2/q} dt \\
 &= C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \int_{a_n^q}^{\infty} EX_{ni}^2 I(|X_{ni}| \leq d_n) t^{-2/q} dt \\
 &\quad + C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \int_{a_n^q}^{\infty} EX_{ni}^2 I(d_n < |X_{ni}| \leq t^{1/q}) t^{-2/q} dt \\
 &\quad + C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \int_{a_n^q}^{\infty} P(|X_{ni}| > t^{1/q}) dt \\
 &\triangleq J_{221} + J_{222} + J_{223}.
 \end{aligned} \tag{3.17}$$

For J_{221} , by $1 \leq q < p \leq 2$ and (1.6), one has

$$\begin{aligned}
 J_{221} &= C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \int_{a_n^q}^{\infty} EX_{ni}^2 I(|X_{ni}| \leq d_n) t^{-2/q} dt \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{EX_{ni}^2 I(|X_{ni}| \leq d_n)}{a_n^2} \\
 &= C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{EX_{ni}^2 I(|X_{ni}| \leq a_n)}{a_n^2} + C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{EX_{ni}^2 I(a_n < |X_{ni}| \leq d_n)}{a_n^2} \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E|X_{ni}|^p I(|X_{ni}| \leq a_n)}{a_n^p} \\
 &\quad + C \sum_{n=1}^{\infty} \sum_{i=1}^n \left(\frac{a_n + 1}{a_n}\right)^2 \frac{E|X_{ni}|^2 I(|X_{ni}| \leq d_n)}{d_n^2} \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} + C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E|X_{ni}|^p I(|X_{ni}| \leq d_n)}{d_n^p} \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} + C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(d_n)} \\
 &\leq 2C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} < \infty.
 \end{aligned} \tag{3.18}$$

For J_{222} , since

$$\sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \int_{a_n^q}^{d_n^q} EX_{ni}^2 I(d_n < |X_{ni}| \leq t^{1/q}) t^{-2/q} dt = 0,$$

which implies

$$J_{222} = C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \int_{a_n^q}^{\infty} EX_{ni}^2 I(d_n < |X_{ni}| \leq t^{1/q}) t^{-2/q} dt.$$

Let $t = x^q$, by (2.1), (1.6), and $1 \leq q < 2$, one has

$$\begin{aligned} J_{222} &= C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} q \int_{d_n}^{\infty} EX_{ni}^2 I(d_n < |X_{ni}| \leq x) x^{q-3} dx \\ &= C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \sum_{m=d_n}^{\infty} \int_m^{m+1} EX_{ni}^2 I(d_n < |X_{ni}| \leq x) x^{q-3} dx \\ &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \sum_{m=d_n}^{\infty} EX_{ni}^2 I(d_n < |X_{ni}| \leq m+1) m^{q-3} \\ &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \sum_{m=d_n}^{\infty} m^{q-3} \sum_{j=d_n}^m EX_{ni}^2 I(j < |X_{ni}| \leq j+1) \\ &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \sum_{j=d_n}^{\infty} EX_{ni}^2 I(j < |X_{ni}| \leq j+1) \sum_{m=j}^{\infty} m^{q-3} \\ &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} \sum_{j=d_n}^{\infty} j^{q-2} EX_{ni}^2 I(j < |X_{ni}| \leq j+1) \\ &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n a_n^{-q} E|X_{ni}|^q I(|X_{ni}| > d_n) \\ &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E|X_{ni}|^q I(|X_{ni}| > a_n)}{a_n^q} \\ &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} < \infty. \end{aligned} \tag{3.19}$$

For J_{223} , by an argument similar to that in the proof of $J_{21} < \infty$, one can prove $J_{223} < \infty$. Therefore, one can obtain $J_{22} < \infty$ for $1 \leq q < p \leq 2$.

(2) If $1 \leq q < p$ and $p > 2$, by (3.16), the Markov inequality, Lemma 3.2, and the c_r -inequality, one has

$$\begin{aligned} J_{22} &\leq \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right| > \frac{t^{1/q}}{2}\right) dt \\ &\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-p/q} E\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right|\right)^p dt \\ &\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-p/q} \left(\sum_{i=1}^n E|Y_{ni}|^p + \left(\sum_{i=1}^n EY_{ni}^2\right)^{p/2}\right) dt \\ &\leq C \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^n \int_{a_n^q}^{\infty} E|Y_{ni}|^p t^{-p/q} dt + C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-p/q} \left(\sum_{i=1}^n EY_{ni}^2\right)^{p/2} dt \\ &\triangleq K_1 + K_2. \end{aligned} \tag{3.20}$$

For K_1 , one has

$$\begin{aligned}
 K_1 &= C \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^n \int_{a_n^q}^{\infty} E|X_{ni}|^p t^{-p/q} I(|X_{ni}| \leq a_n) dt \\
 &\quad + C \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^n \int_{a_n^q}^{\infty} E|X_{ni}|^p t^{-p/q} I(a_n < |X_{ni}| \leq t^{1/q}) dt \\
 &\quad + C \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^n \int_{a_n^q}^{\infty} P(|X_{ni}| > t^{1/q}) dt \\
 &\triangleq K_{11} + K_{12} + K_{13}.
 \end{aligned}
 \tag{3.21}$$

By an argument similar to that in the proofs of J_{221} and J_{222} (replacing the exponent 2 by p), one easily has $K_{11} < \infty$ and $K_{12} < \infty$. Similarly, from the proof of $J_{21} < \infty$, one can obtain $K_{13} < \infty$.

For K_2 , since $p > 2$, one has

$$\begin{aligned}
 K_2 &= C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-p/q} \left(\sum_{i=1}^n EY_{ni}^2 \right)^{p/2} dt \\
 &\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-p/q} \left(\sum_{i=1}^n EX_{ni}^2 I(|X_{ni}| \leq a_n) \right)^{p/2} dt \\
 &\quad + C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-p/q} \left(\sum_{i=1}^n EX_{ni}^2 I(a_n < |X_{ni}| \leq t^{1/q}) \right)^{p/2} dt \\
 &\quad + C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{i=1}^n P(|X_{ni}| > t^{1/q}) \right)^{p/2} dt \\
 &\triangleq K_{21} + K_{22} + K_{23}.
 \end{aligned}
 \tag{3.22}$$

For K_{21} , by $p > q, p > 2$, and (1.9), one has

$$\begin{aligned}
 K_{21} &= C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-p/q} \left(\sum_{i=1}^n EX_{ni}^2 I(|X_{ni}| \leq a_n) \right)^{p/2} dt \\
 &\leq C \sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{EX_{ni}^2 I(|X_{ni}| \leq a_n)}{a_n^2} \right)^{p/2} < \infty.
 \end{aligned}
 \tag{3.23}$$

For K_{22} , we will consider the following two cases:

(1) When $1 \leq q \leq 2$ and $p > 2$. By (2.1) and (1.6), one has

$$\begin{aligned}
 K_{22} &= C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-p/q} \left(\sum_{i=1}^n EX_{ni}^2 I(a_n < |X_{ni}| \leq t^{1/q}) \right)^{p/2} dt \\
 &= C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(t^{-2/q} \sum_{i=1}^n EX_{ni}^2 I(a_n < |X_{ni}| \leq t^{1/q}) \right)^{p/2} dt
 \end{aligned}$$

$$\begin{aligned}
 &\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(t^{-1} \sum_{i=1}^n E|X_{ni}|^q I(a_n < |X_{ni}| \leq t^{1/q}) \right)^{p/2} dt \\
 &\leq C \sum_{n=1}^{\infty} a_n^{-q} \left(\sum_{i=1}^n E|X_{ni}|^q I(|X_{ni}| > a_n) \right)^{p/2} \int_{a_n^q}^{\infty} t^{-p/2} dt \\
 &\leq C \sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E|X_{ni}|^q I(|X_{ni}| > a_n)}{a_n^q} \right)^{p/2} \\
 &\leq C \left(\sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} \right)^{p/2} < \infty.
 \end{aligned} \tag{3.24}$$

(2) When $2 < q < p$. By (2.1) and (1.6) again, one can have

$$\begin{aligned}
 K_{22} &= C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(t^{-2/q} \sum_{i=1}^n EX_{ni}^2 I(a_n < |X_{ni}| \leq t^{1/q}) \right)^{p/2} dt \\
 &\leq C \sum_{n=1}^{\infty} a_n^{-q} \left(\sum_{i=1}^n E|X_{ni}|^2 I(|X_{ni}| > a_n) \right)^{p/2} \int_{a_n^q}^{\infty} t^{-p/q} dt \\
 &\leq C \sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E|X_{ni}|^2 I(|X_{ni}| > a_n)}{a_n^2} \right)^{p/2} \\
 &\leq C \sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E|X_{ni}|^q I(|X_{ni}| > a_n)}{a_n^q} \right)^{p/2} \\
 &\leq C \left(\sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} \right)^{p/2} < \infty.
 \end{aligned} \tag{3.25}$$

For K_{23} , by (2.1), it follows that $\psi_i(|t|) \uparrow$ as $|t| \uparrow$. By (1.6), one has

$$\sup_{t \geq a_n^q} \sum_{i=1}^n P(|X_{ni}| > t^{1/q}) \leq \sum_{i=1}^n P(|X_{ni}| > a_n) \leq \sum_{i=1}^n \frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, while n is sufficiently large, for $t \geq a_n^q$, one can have

$$\sum_{i=1}^n P(|X_{ni}| > t^{1/q}) < 1.$$

By (3.13), it follows that

$$\begin{aligned}
 K_{23} &= C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{i=1}^n P(|X_{ni}| > t^{1/q}) \right)^{p/2} dt \\
 &\leq C \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^n \int_{a_n^q}^{\infty} P(|X_{ni}| > t^{1/q}) dt \\
 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)} < \infty.
 \end{aligned} \tag{3.26}$$

The proof of Theorem 2.2 is completed. □

Proof of Theorem 2.3 Following the notations in the proof of Theorem 2.2, we will first prove (2.5) for the case of $1 < p \leq 2$. By (3.11), for all $\varepsilon > 0$, one has

$$\begin{aligned}
 E\left(\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right|\right)^q &= \frac{1}{a_n^q} \int_0^\infty P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > t^{1/q}\right) dt \\
 &= \frac{1}{a_n^q} \int_0^{\varepsilon a_n^q} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > t^{1/q}\right) dt \\
 &\quad + \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^\infty P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_{ni} \right| > t^{1/q}\right) dt \\
 &\leq \varepsilon + \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^\infty P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_{ni} \right| > t^{1/q}\right) dt \\
 &\quad + \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^\infty \sum_{i=1}^n P(|X_{ni}| > t^{1/q}) dt \\
 &\triangleq \varepsilon + L_1 + L_2. \tag{3.27}
 \end{aligned}$$

Without loss of generality, one may assume $0 < \varepsilon < 1$. For L_2 , by the Markov inequality, (2.1), and (2.4), one has

$$\begin{aligned}
 L_2 &\leq \sum_{i=1}^n \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^\infty P(|X_{ni}| I(\varepsilon a_n^q < |X_{ni}| \leq a_n) > t^{1/q}) dt \\
 &\quad + \sum_{i=1}^n \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^\infty P(|X_{ni}| I(|X_{ni}| > a_n) > t^{1/q}) dt \\
 &\leq \sum_{i=1}^n \frac{1}{a_n^q} E|X_{ni}|^p I(\varepsilon a_n^q < |X_{ni}| \leq a_n) \int_{\varepsilon a_n^q}^\infty t^{-p/q} dt \\
 &\quad + \sum_{i=1}^n \frac{1}{a_n^q} \int_0^\infty P(|X_{ni}| I(|X_{ni}| > a_n) > t^{1/q}) dt \\
 &\leq C \varepsilon^{1-\frac{p}{q}} \sum_{i=1}^n \frac{E|X_{ni}|^p I(|X_{ni}| \leq a_n)}{a_n^p} + \sum_{i=1}^n \frac{1}{a_n^q} E|X_{ni}|^q I(|X_{ni}| > a_n) \\
 &\leq C \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.28}
 \end{aligned}$$

Similar to the proof of (3.15), by conditions (2.4), (2.1), and (1.5), one has

$$\begin{aligned}
 \max_{t \geq \varepsilon a_n^q} \frac{1}{t^{1/q}} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right| &= \max_{t \geq \varepsilon a_n^q} \frac{1}{t^{1/q}} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EZ_{ni} \right| \\
 &\leq C \max_{t \geq \varepsilon a_n^q} \frac{1}{t^{1/q}} \sum_{i=1}^n E|Z_{ni}|
 \end{aligned}$$

$$\begin{aligned}
 &\leq C \max_{t \geq a_n^q} \frac{1}{t^{1/q}} \sum_{i=1}^n E|X_{ni}| I(|X_{ni}| > t^{1/q}) \\
 &\leq C \varepsilon^{-1/q} \sum_{i=1}^n \frac{E|X_{ni}|^q I(|X_{ni}| > a_n)}{a_n^q} \\
 &\quad + C \varepsilon^{-p/q} \sum_{i=1}^n \frac{E|X_{ni}|^p I(\varepsilon^{1/q} a_n < |X_{ni}| \leq a_n)}{a_n^p} \\
 &\leq C(\varepsilon^{-1/q} + \varepsilon^{-p/q}) \sum_{i=1}^n \frac{E\psi_i(X_{ni})}{\psi_i(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.29}
 \end{aligned}$$

Hence, while n is sufficiently large, (3.16) holds uniformly for $t \geq \varepsilon a_n^q$.

For L_1 , let $d_n = [a_n] + 1$, by (3.16), the Markov inequality, Lemma 3.2, and the c_r inequality, one has

$$\begin{aligned}
 L_1 &\leq C \sum_{i=1}^n \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^{\infty} t^{-2/q} E(Y_{ni} - EY_{ni})^2 dt \\
 &\leq C \sum_{i=1}^n \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^{\infty} t^{-2/q} EY_{ni}^2 dt \\
 &= C \sum_{i=1}^n \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^{\infty} t^{-2/q} EX_{ni}^2 I(|X_{ni}| \leq d_n) dt \\
 &\quad + C \sum_{i=1}^n \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^{\infty} t^{-2/q} EX_{ni}^2 I(d_n < |X_{ni}| \leq t^{1/q}) dt \\
 &\quad + C \sum_{i=1}^n \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^{\infty} P(|X_{ni}| > t^{1/q}) dt \\
 &\triangleq L_{11} + L_{12} + L_{13}. \tag{3.30}
 \end{aligned}$$

By (3.28), one has $L_{13} \rightarrow 0$. For L_{11} , by an argument similar to that in the proof of $J_{221} < \infty$ and (2.4), one can obtain

$$L_{11} = C \sum_{i=1}^n \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^{\infty} t^{-2/q} EX_{ni}^2 I(|X_{ni}| \leq d_n) dt \leq C \sum_{i=1}^n \frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For L_{12} ,

$$\sum_{i=1}^n \frac{1}{a_n^q} \int_{\varepsilon a_n^q}^{d_n^q} t^{-2/q} EX_{ni}^2 I(d_n < |X_{ni}| \leq t^{1/q}) dt = 0,$$

which implies

$$L_{12} = C \sum_{i=1}^n \frac{1}{a_n^q} \int_{d_n^q}^{\infty} t^{-2/q} EX_{ni}^2 I(d_n < |X_{ni}| \leq t^{1/q}) dt.$$

Similarly, by an argument similar to the proof of $J_{222} < \infty$ and (2.4), one also has $L_{12} \rightarrow 0$ as $n \rightarrow \infty$.

The proof of (2.5) for the case of $p > 2$ is similar to that of $1 \leq q < p$ and $p > 2$ in Theorem 2.2, so we omit the details. The proof of Theorem 2.3 is completed. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

HH and MO carried out the design of the study and performed the analysis. YJ participated in its design and coordination. All authors read and approved the final manuscript.

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References

1. Joag-Dev, K, Proschan, F: Negative association of random variables with applications. *Ann. Stat.* **11**(1), 286-295 (1983)
2. Bradley, RC: On the spectral density and asymptotic normality of weakly dependent random fields. *J. Theor. Probab.* **5**(2), 355-373 (1992)
3. Zhang, LX, Wang, XY: Convergence rates in the strong laws of asymptotically negatively associated random fields. *Appl. Math. J. Chin. Univ. Ser. B* **14**(4), 406-416 (1999)
4. Zhang, LX: Central limit theorems for asymptotically negatively associated random fields. *Acta Math. Sin. Engl. Ser.* **16**(4), 691-710 (2000)
5. Zhang, LX: A functional central limit theorem for asymptotically negatively dependent random fields. *Acta Math. Hung.* **86**(3), 237-259 (2000)
6. Budsaba, K, Chen, PY, Volodin, A: Limiting behavior of moving average processes based on a sequence of ρ^- -mixing and NA random variables. *Lobachevskii J. Math.* **26**, 17-25 (2007)
7. Wang, JF, Lu, FB: Inequalities of maximum partial sums and weak convergence for a class of weak dependent random variables. *Acta Math. Sin. Engl. Ser.* **22**(3), 693-700 (2006)
8. Wang, JF, Zhang, LX: A Berry-Esseen theorem and a law of the iterated logarithm for asymptotically negatively associated sequences. *Acta Math. Sin. Engl. Ser.* **23**(1), 127-136 (2007)
9. Yuan, DM, Wu, XS: Limiting behavior of the maximum of the partial sum for asymptotically negatively associated random variables under residual Cesàro alpha-integrability assumption. *J. Stat. Plan. Inference* **140**, 2395-2402 (2010)
10. Tan, XL, Zhang, Y, Zhang, Y: An almost sure central limit theorem of products of partial sums for ρ^- -mixing sequences. *J. Inequal. Appl.* **2012**, 51 (2012). doi:10.1186/1029-242X-2012-51
11. Ko, MH: The Hájek-Rényi inequality and strong law of large numbers for ANA random variables. *J. Inequal. Appl.* **2014**, 521 (2014). doi:10.1186/1029-242X-2014-521
12. Zhang, Y: Complete moment convergence for moving average process generated by ρ^- -mixing random variables. *J. Inequal. Appl.* **2015**, 245 (2015). doi:10.1186/s13660-015-0766-5
13. Huang, HW, Peng, JY, Wu, XT, Wang, B: Complete convergence and complete moment convergence for arrays of rowwise ANA random variables. *J. Inequal. Appl.* **2016**, 72 (2016). doi:10.1186/s13660-016-1016-1
14. Hsu, PL, Robbins, H: Complete convergence and the law of large numbers. *Proc. Natl. Acad. Sci. USA* **33**(2), 25-31 (1947)
15. Chow, YS: On the rate of moment complete convergence of sample sums and extremes. *Bull. Inst. Math. Acad. Sin.* **16**, 177-201 (1988)
16. Gan, SX, Chen, PY: On the limiting behavior of the maximum partial sums for arrays of rowwise NA random variables. *Acta Math. Sci. Ser. B* **27**(2), 283-290 (2007)
17. Zhu, MH: Strong laws of large numbers for arrays of rowwise ρ^* -mixing random variables. *Discrete Dyn. Nat. Soc.* **2007**, Article ID 74296 (2007). doi:10.1155/2007/74296