Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

CrossMark

Coexistence of an unstirred chemostat model with B-D functional response by fixed point index theory

Xiao-zhou Feng[1,2]*, Jian-hui Tian[3] and Xiao-li Ma[1]

*Correspondence:
flxzfxz8@163.com
[1]Department of Mathematics and
Physics, Xi'an Technological
University, Xi'an, 710032, China
[2]Institute of Mathematics, Shaanxi
Normal University, Xi'an, 710062,
China
Full list of author information is
available at the end of the article

Abstract

This paper deals with an unstirred chemostat model with the Beddington-DeAngelis functional response. First, some prior estimates for positive solutions are proved by the maximum principle and the method of upper and lower solutions. Second, the calculation on the fixed point index of chemostat model is obtained by degree theory and the homotopy invariance theorem. Finally, some sufficient condition on the existence of positive steady-state solutions is established by fixed point index theory and bifurcation theory.

Keywords: chemostat; degree theory; the fixed point index theory; bifurcation theory

1 Introduction

The chemostat is a laboratory apparatus used for the continuous culture of microorganisms. Mathematical models of the chemostat are surprisingly amenable to analysis. Early results can be found in the articles of Levin [1] and Hsu [2]. For a general discussion of competition, Smith and Waltman [3] discussed the well-unstirred model in detail. Recently, Wu [4–6] and Nie [7, 8] studied the coexistence and asymptotic behavior of chemostat models from the viewpoint of partial differential equations theories. The response functions are mainly the Michaelis-Menten functional response $f(S) = S/(1 + \kappa S)$, κ is a constant. Until very recently, both ecologists and mathematicians chose to base their studies on the Beddington-DeAngelis (denoted by B-D) functional response introduced by Beddington [9] and DeAngelis [10]. The Beddington-DeAngelis functional response has some of the same qualitative features as the Michaelis-Menten form but has an extra term βu in the denominator which models mutual interference between predators. It has been the source of controversy and can provide a better description of predator feeding over a range of predator-prey abundances, which are strongly supported by numerous field and laboratory experiments and observations.

In this paper, we are interested in the following unstirred chemostat model with Beddington-DeAngelis functional response:

$$\begin{cases} S_t = d\Delta S - auf(S,u) - bvg(S,v), & x \in \Omega, t > 0, \\ u_t = d\Delta u + (1-q)auf(S,u), & x \in \Omega, t > 0, \\ v_t = d\Delta v + bvg(S,v) + qauf(S,u), & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial v} + \gamma S = -S^0, \quad \frac{\partial u}{\partial v} + \gamma u = 0, \quad \frac{\partial v}{\partial v} + \gamma v = 0, & x \in \partial\Omega, t > 0 \\ S(x,0) = S_0(x) \geq 0, & x \in \Omega, \\ u(x,0) = u_0(x) \geq 0, \neq 0, & x \in \Omega, \\ v(x,0) = v_0(x) \geq 0, \neq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $a, b, m_i, k_i, i = 1, 2$ and γ are positive constants, a and b are the maximal growth rates of the two competitors, respectively. u stands the density of the plasmid-bearing organism; v denotes plasmid-free organism; The parameter q, $0 < q < 1$ denotes the fraction of plasmid-bearing organism converting into plasmid-free organism. Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $f(S,u) = \frac{S}{1+m_1 S+k_1 u}$, $g(S,v) = \frac{S}{1+m_2 S+k_2 v}$ are Beddington-DeAngelis functions; m_i, $i = 1, 2$ are the Michaelis-Menton constants; k_i $(i = 1, 2)$ model mutual interference of between predators.

In the present paper, we shall investigate non-negative steady-state solutions of system (1.1). Hence, we will concentrate on the following elliptic system:

$$\begin{cases} d\Delta S - auf(S,u) - bvg(S,v) = 0, & x \in \Omega, \\ d\Delta u + (1-q)auf(S,u) = 0, & x \in \Omega, \\ d\Delta v + bvg(S,v) + qauf(S,u) = 0, & x \in \Omega, \\ \frac{\partial S}{\partial v} + \gamma S = -S^0, \quad \frac{\partial u}{\partial v} + \gamma u = 0, \quad \frac{\partial v}{\partial v} + \gamma v = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Let $z = S + u + v$, then $S = z - u - v$ (refer to [4–7]) and z satisfies

$$\Delta z = 0, \quad x \in \Omega, \qquad \frac{\partial z}{\partial v} + \gamma z = -S^0, \quad x \in \partial\Omega.$$

Then one can argue in the exactly same way as in [4, 5, 7] to obtain the limiting system of (1.2), which can be written as

$$\begin{cases} d\Delta u + (1-q)auf(z-u-v,u) = 0, & x \in \Omega, \\ d\Delta v + bvg(z-u-v) + qauf(z-u-v,u) = 0, & x \in \Omega, \\ \frac{\partial u}{\partial v} + ru = 0, \quad \frac{\partial v}{\partial v} + rv = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $f(S,u) = \frac{S}{1+m_1 S+k_1 u}$, $g(S,v) = \frac{S}{1+m_2 S+k_2 v}$, and $S = z - u - v$. We only concern on the case that $S(x), u(x), v(x)$ are non-negative, so we redefine the response function as follows:

$$\hat{f}(S,u) = \begin{cases} f(S,u), & S \geq 0, u \geq 0, \\ 0, & \text{others}, \end{cases} \qquad \hat{g}(S,v) = \begin{cases} g(S,v), & S \geq 0, v \geq 0, \\ 0, & \text{others}. \end{cases}$$

Then $\hat{f}(S,u), \hat{g}(S,v) \in C^1(R)$. We will denote $\hat{f}(S,u), \hat{g}(S,v)$ by $f(S,u), g(S,v)$ for the sake of simplicity, respectively. Thus, the solution of equation (1.3) satisfies

$$S(x) + u(x) + v(x) = z(x), \quad x \in \bar{\Omega}.$$

In the following, we set up the fixed point index theory on this paper. Let E be a Banach Space. $W \subset E$ is called a wedge if W is a closed convex set and $\alpha W \subset W$ for all $\alpha \geq 0$.

For $y \in W$, we define $W_y = \{x \in E : \exists r = r(x) > 0, \text{s.t.}, y + rx \in W\}$, $S_y = \{x \in \bar{W}_y : -x \in \bar{W}_y\}$, we always assume that $E = \overline{W - W}$. Let $T : W_y \to W_y$ be a compact linear operator on E. We say that T has property α on $\bar{W}_y$ if there exists $t \in (0,1)$ and $\omega \in \bar{W}_y \backslash S_y$, such that $\omega - tT\omega \in S_y$.

Suppose that $F : W \to W$ is a compact operator, and $y_0 \in W$ is an isolated fixed point of F, such that $Fy_0 = y_0$, let $L = F'(y_0)$ is Fréchet differentiable at y_0, it follows that $L : \overline{W} \to \overline{W}$.

Proposition 1.1 ([11] Dancer index theorem) *Assume that $I - L$ is invertible on E, then we have*

(1) *L has property α on $\bar{W}$, then* $\text{index}_W(F, y_0) = 0$;

(2) *L does not have property α on $\bar{W}$, then* $\text{index}_W(F, y_0) = \text{index}_E(L, \theta) = \pm 1$.

Proposition 1.2 ([11]) *Assume that $F(\theta) = \theta$, $A_0 = F'(\theta)$ is Fréchet differentiable of F at θ in W. If the eigenvalue problem*

$$A_0 h = \lambda h, \quad h \in W \tag{1.4}$$

has no eigenvalue equal to 1, *then θ is isolate fixed point of F, and*

(1) *if (1.4) has no eigenvalue larger than* 1, *then* $\text{index}_W(F, \theta) = 1$;

(2) *if (1.4) has an eigenvalue $\lambda > 1$, then* $\text{index}_W F(F, \theta) = 0$.

Proposition 1.3 ([12]) *Assuming that T is a positive compact linear operator in ordered Banach spaces, u is a positive element in Banach space, $r(T)$ is the spectral radius of the operator T, then*

(1) *if $Tu > u$, then $r(T) > 1$;*

(2) *if $Tu < u$, then $r(T) < 1$;*

(3) *if $Tu = u$, then $r(T) = 1$.*

Proposition 1.4 ([13]) *Assume that $q(x) \in C(\Omega)$, $q(x) + p > 0$ on $\overline{\Omega}$, p is a positive real constant, λ_1 is the principal eigenvalue of the following problem*:

$$-\Delta\phi - q(x)\phi = \lambda\phi, \quad x \in \Omega, \qquad \frac{\partial\phi}{\partial\nu} + r\phi = 0, \quad x \in \partial\Omega.$$

If $\lambda_1 > 0$ (or < 0), then all eigenvalues of the following problem:

$$-\Delta\phi - p\phi = t\big(q(x) + p\big)\phi, \quad x \in \Omega, \qquad \frac{\partial\phi}{\partial\nu} + r\phi = 0, \quad x \in \partial\Omega$$

are larger than 1 (*or less than* 1).

The organization of our paper is as follows. In Section 2, some prior estimates for positive solutions are proved by the maximum principle and the upper and lower solution method. In Section 3, the calculations on the fixed point index of chemostat model by degree theory and the homotopy invariance theorem. In Section 4, some sufficient conditions on the existence of positive steady-state solutions is established by the fixed point index theory in cone and bifurcation theory.

2 Some prior estimates for positive solutions

The main purpose of this section is to give prior upper and lower positive bounds for positive solutions of (1.3) by using the maximum principle and the upper and lower solution method.

Let λ_1, μ_1 be, respectively, the principal eigenvalue of the following problem:

$$\Delta\varphi + \lambda\varphi f(z,0) = 0, \quad x \in \Omega, \qquad \frac{\partial\varphi}{\partial\nu} + r\varphi = 0, \quad x \in \partial\Omega, \tag{2.1}$$

$$\Delta\psi + \mu\psi g(z,0) = 0, \quad x \in \Omega, \qquad \frac{\partial\psi}{\partial\nu} + r\psi = 0, \quad x \in \partial\Omega, \tag{2.2}$$

the corresponding principal eigenvalue function denoted by $\varphi_1(x)$, $\psi_1(x)$, and

$$\|\varphi_1\| = 1, \qquad \|\psi_1\| = 1.$$

First, we consider the single species equation as follows:

$$d\Delta u + a(1-q)uf_1(z-u,u) = 0, \quad x \in \Omega, \qquad \frac{\partial u}{\partial\nu} + ru = 0, \quad x \in \partial\Omega. \tag{2.3}$$

By [9, 14, 15], we can directly get the following conclusions.

Lemma 2.1 *If $a \leq \frac{\lambda_1 d}{1-q}$, then 0 is the unique non-negative solution of (2.3); If $a > \frac{\lambda_1 d}{1-q}$, (2.3) has a unique positive solution, denoted by Θ, satisfying the following properties*:

(i) $0 < \Theta < z$;

(ii) Θ *is continuously differentiable for $a \in (\frac{\lambda_1 d}{1-q}, +\infty)$ and is point wisely increasing when a is increasing.*

(iii) $\lim_{a \to \frac{\lambda_1 d}{1-q}} \Theta = 0$ *uniformly for $x \in \bar{\Omega}$, $\lim_{a \to \infty} \Theta = z(x)$ for almost every $x \in \Omega$;*

(iv) *Let $L_{(a,d)} = d\Delta + a(1-q)(f(z-\Theta,\Theta) - \Theta f_1'(z-\Theta,\Theta) + \Theta f_2'(z-\Theta,\Theta))$ be the linearized operator of (2.3) at Θ, then $L_{(a,d)}$ is differentiable in $C_B^2(\bar{\Omega}) = \{u \in C^2(\bar{\Omega}) : \frac{\partial u}{\partial\nu} + ru = 0\}$, and all eigenvalues of $L_{(a,d)}$ are strictly negative.*

Remark 2.1 For (2.4), we have the same conclusion as Lemma 2.1. Suppose $b > d\mu_1$, we denote the unique positive solution by θ for the following problem:

$$d\Delta\upsilon + b\upsilon g(z-\upsilon,\upsilon) = 0, \quad x \in \Omega, \qquad \frac{\partial\upsilon}{\partial\nu} + r\upsilon = 0, \quad x \in \partial\Omega, \tag{2.4}$$

let $L_{(b,d)} = d\Delta + b(g(z-\theta,\theta) - \theta g_1'(z-\theta,\theta) + \theta g_2'(z-\theta,\theta))$ be the linearized operator of (2.4) at θ.

Next, let $\hat{\lambda}_1$ be the principal eigenvalue of the following eigenvalue problem:

$$\Delta\varphi + \hat{\lambda}\varphi f(z-\theta,0) = 0, \quad x \in \Omega, \qquad \frac{\partial\varphi}{\partial\nu} + r\varphi = 0, \quad x \in \partial\Omega, \tag{2.5}$$

the corresponding eigenfunction denoted by $\hat{\varphi}_1(x)$ and uniquely determined by normalization $\|\hat{\varphi}_1\| = 1$.

In order to accurately estimate the positive solution of (1.3), we consider the following boundary value problem:

$$d\Delta \upsilon + b\upsilon g(z-\upsilon,\upsilon) + aq\Theta f(z-\upsilon,0) = 0, \quad x \in \Omega, \qquad \frac{\partial \upsilon}{\partial \nu} + r\upsilon = 0, \quad x \in \partial\Omega. \quad (2.6)$$

Lemma 2.2 *Suppose* $a > \frac{\lambda_1 d}{1-q}$, *there exists a unique positive solution of* (2.6). *Then* $0 < \upsilon < z$, *and* $\theta < \upsilon < z$, *when* $b > d\mu_1$.

Proof Let $\omega = z - \upsilon$, then

$$d\Delta\omega + b\upsilon g(\omega,\upsilon) + aq\Theta f(\omega,0) = 0, \quad x \in \Omega, \qquad \frac{\partial \omega}{\partial \nu} + r\omega = S^0, \quad x \in \partial\Omega.$$

Suppose $\inf_{x \in \bar{\Omega}} \omega(x) = \omega(x_0) < 0$, then $x_0 \notin \Omega$. Otherwise, $\Delta\omega(x_0) \geq 0$. However,

$$d\Delta\omega(x_0) = b\upsilon(x_0)g\big(\omega(x_0),\upsilon(x_0)\big) + aq\Theta f\big(\omega(x_0),0\big) < 0,$$

a contradiction. If $x_0 \in \partial\Omega$, it follows from the definition of $r(x)$ and $S^0(x)$ that

$$\frac{\partial \omega}{\partial \nu}\bigg|_{x_0} = S^0(x_0) - r\omega(x_0) > 0.$$

This contradicts $\frac{\partial \omega}{\partial \nu}|_{x_0} < 0$, hence, $\upsilon \leq z$ on $\bar{\Omega}$, and $\upsilon \leq z, \upsilon \neq z$, then $\omega = z - \upsilon \geq 0, \neq 0$. If $\omega(x_0) = 0$, for some point $x_0 \in \bar{\Omega}$, then it follows from the maximum principle [16] that $x_0 \in \partial\Omega$. Furthermore, from the Holp lemma [16], we can get $\frac{\partial \omega}{\partial \nu}|_{x_0} < 0$, this contradicts the boundary condition.

Thus, $\upsilon < z$ on $\bar{\Omega}$, and for

$$d\Delta\upsilon + b\upsilon g(z-\upsilon,\upsilon) + aq\Theta f(z-\upsilon,0) > d\Delta\upsilon + b\upsilon g(z-\upsilon,\upsilon).$$

Hence $\upsilon > \theta$ when $b > d\mu_1$.

Next, we will prove the existence and uniqueness of solutions. For sufficiently small $\delta > 0$, $\delta\phi_1, z$ are the upper and lower solutions of (2.6). It follows from the comparison principle [16] that (2.6) exists the minimum solution υ_1 and maximum solution υ_2, satisfying $\delta\phi_1 \leq \upsilon_1 \leq \upsilon_2 \leq z$.

In the following, we prove the uniqueness. Thanking to υ_1, υ_2 are the solution of (2.6), we have

$$d\Delta\upsilon_1 + b\upsilon_1 g(z-\upsilon_1,\upsilon_1) + aq\Theta f(z-\upsilon_1,0) = 0, \quad x \in \Omega,$$

$$d\Delta\upsilon_2 + b\upsilon_2 g(z-\upsilon_2,\upsilon_2) + aq\Theta f(z-\upsilon_2,0) = 0, \quad x \in \Omega.$$

Multiplying the second equation and first equation by υ_1, υ_2, respectively, and applying the Green's formula, we obtain $I = \int_\Omega d(\Delta\upsilon_1 \cdot \upsilon_2 - \Delta\upsilon_2 \cdot \upsilon_1) = 0$. Then

$$\int_\Omega b\upsilon_1\upsilon_2\big(g(z-\upsilon_1,\upsilon_1) - g(z-\upsilon_2,\upsilon_2)\big)\,dx + aq\int_\Omega \Theta\big[\upsilon_2 f(z-\upsilon_1,0) - \upsilon_1 f(z-\upsilon_2,0)\big]\,dx = 0.$$

According to the monotonicity of f, g, and $\upsilon_1 \leq \upsilon_2$, we get $\upsilon_1 \equiv \upsilon_2$. $\qquad\square$

In conclusion, we can get prior estimates on the system (1.3).

Theorem 2.3 *Suppose* (u, v) *is non-negative solution of* (1.3), *and* $u \neq 0, v \neq 0$. *Then*
  (i) $0 < u < \Theta < z, 0 < v \leq \hat{v} < z, x \in \bar{\Omega}$;
 (ii) $u + v < z, x \in \bar{\Omega}$;
(iii) $a > \frac{\lambda_1 d}{1-q}$.

Proof The proof is in [8, 14], we omit it. □

3 Calculations of fixed point index

In this section, we will calculate the fixed point index of (1.3) by using the standard fixed point index theory in cone.

Let $C_0(\bar{\Omega}) = \{y \in C(\bar{\Omega}) | \frac{\partial y}{\partial n} + ry = 0\}$, $E = [C_0(\bar{\Omega}) \times C_0(\bar{\Omega})]$. For a sufficiently large $P > 0$, and $\tau \in [0,1]$, we consider the following equations:

$$\begin{cases} (-d\Delta + \tau P)u = \tau(P + a(1-q)f(z - u - v, u))u, & x \in \Omega, \\ (-d\Delta + \tau P)v = \tau(Pv + bg(z - u - v, v)v + qauf(z - u - v, u)), & x \in \Omega, \\ \frac{\partial u}{\partial v} + ru = 0, \qquad \frac{\partial v}{\partial v} + rv = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

For $(u, v)^T \in E$, $\tau \in [0,1]$, $(U, V)^T \in [C^{1+\alpha}(\bar{\Omega})] \times [C^{1+\alpha}(\bar{\Omega})]$ is the unique solution of the following linear problem:

$$\begin{cases} (-d\Delta + \tau P)U = \tau(P + a(1-q)f(z - u - v, u))u, & x \in \Omega, \\ (-d\Delta + \tau P)V = \tau(Pv + bg(z - u - v, v)v + qauf(z - u - v, u)), & x \in \Omega, \\ \frac{\partial U}{\partial v} + rU = 0, \qquad \frac{\partial V}{\partial v} + rV = 0, & x \in \partial\Omega, \end{cases}$$

define $F_\tau : [0,1] \times E \to E$, $F_\tau(u, v)^T = (U, V)^T$, it follows from [17] that F_τ is compact.

It is clear to see that $(u, v)^T \in E$ is fixed point of F_τ if only and if $(u, v)^T \in E$ is a positive solution of (1.3). Let

$$K_0 = \left\{ u \in C_0(\bar{\Omega}) | u(x) \geq 0, x \in \Omega \right\},$$

$$W = \left\{ (u, v) \in E | u(x) \geq 0, v(x) \geq 0, x \in \Omega \right\},$$

$$D = \left\{ (u, v) \in W | 0 \leq u(x) \leq \Theta, 0 \leq v(x) \leq \max_{\bar{\Omega}} \hat{v} + 1, x \in \Omega \right\},$$

$$D' = (\text{int}\, D) \cap W.$$

Then W is a cone in E, D is bounded set in W, let $F = F_1$. It follows that there exists $K > 0$ such that $f(z - u - v, u) \geq f(z - u, u) - Kv$.

Suppose that P is sufficiently large, for all $(u, v) \in D$, we can get

$$P + a(1-q)f(z - u - v, u) > 0, \qquad P + bg(z - u - v, v) - aquK > 0.$$

Then $F : D' \to W$ is continuously differentiable, hence, (1.3) had non-negative solution if only and if F has fixed point on D. According to the homotopy invariance of degree, we have

$$\deg_W(I - F_\tau, D', (0,0)) = \deg_W(I - F, D', (0,0)), \qquad \tau \in [0,1].$$

In the following, we calculate the index number of $(0,0)$ and $(0,\theta)$ by using the fixed point theory.

Lemma 3.1 *For the index number of the operator F at $(0,0)$ the following results hold*:
  (i) *if $a \neq \frac{\lambda_1 d}{1-q}$, $b > \mu_1 d$, then* $\mathrm{index}_W(F,(0,0)) = 0$;
  (ii) *Suppose that $b < \mu_1 d$. If $a > \frac{\lambda_1 d}{1-q}$, then* $\mathrm{index}_W(F,(0,0)) = 0$; *If $a < \frac{\lambda_1 d}{1-q}$, then*
      $\mathrm{index}_W(F,(0,0)) = 1$.

Proof Define the operator at $(0,0)$ of F as follows:

$$L_0(\varphi,\psi) = F'(0,0)\begin{pmatrix}\varphi\\\psi\end{pmatrix} = (-d\Delta + P)^{-1}\begin{pmatrix} P + a(1-q)f(z,0) & 0 \\ aqf(z,0) & P + bg(z,0)\end{pmatrix}\begin{pmatrix}\varphi\\\psi\end{pmatrix}.$$

Hence, $F'(0,0)\cdot(\varphi,\psi)^T = \lambda(\varphi,\psi)^T$ is equivalent to

$$\begin{cases} -d\Delta\varphi + P\varphi = \frac{1}{\lambda}(P\varphi + a(1-q)f(z,0)\varphi), & x \in \Omega, \\ -d\Delta\psi + P\psi = \frac{1}{\lambda}(P\psi + b\psi g(z,0) + qa\varphi f(z,0)), & x \in \Omega, \\ (\varphi,\psi) \in W. \end{cases} \tag{3.2}$$

Take $y_0 = (0,0)$, then $\bar{W}_{y_0} = K_0 \times K_0$, $S_{y_0} = \{(0,0)\}$, $\bar{W}_{y_0}\setminus S_{y_0} = K_0 \times K_0\setminus\{(0,0)\}$.
  If 1 is an eigenvalue of (3.2), then

$$\begin{cases} d\Delta\varphi + a(1-q)f(z,0)\varphi = 0, & x \in \Omega, \\ d\Delta\psi + b\psi g(z,0) + aq\varphi f(z,0) = 0, & x \in \Omega, \end{cases} \tag{3.3}$$

if $a \neq \frac{\lambda_1 d}{1-q}$, $b \neq \mu_1 d$, form the first equation of (3.3) and definition of λ_1, we have $\varphi \equiv 0$. Similarly, it follows from the second equation that $\psi \equiv 0$. Thus 1 is not an eigenvalue of L_0, it is to see that $(0,0)$ is an isolated fixed point.
  (i) Suppose that $a \neq \frac{\lambda_1 d}{1-q}, b > \mu_1 d$. If $\hat{\lambda}_1$ is the principal eigenvalue of $-d\Delta\psi - bg(z,0)\psi = \hat{\lambda}\psi$, then $\hat{\lambda}_1 < 0$. From Proposition 1.4, there exists $\frac{1}{\lambda} < 1$ is an eigenvalue of $(-d\Delta + P)\psi = \frac{1}{\lambda}(P + bg(z,0))\psi$, ψ is the corresponding eigenfunction, then the L_0 has no eigenvalue greater than 1, so $(0,\psi)$ is the corresponding eigenfunction. It follows from Proposition 1.2 that we have $\mathrm{index}_W(F,(0,0)) = 0$.
  (ii) Suppose that $b < \mu_1 d$. If $a > \frac{\lambda_1 d}{1-q}$, 1 is not an eigenvalue of (3.2), and $\lambda_1(-d\Delta - (1-q)af(z,0)) < 0$. From Proposition 1.4, then there exists $\frac{1}{\lambda_0} < 1$ which is an eigenvalue of the first equation of (3.2), the corresponding eigenfunction denoted by φ_1, and putting it into the second equation, we have

$$\left(-d\Delta + \left(1 - \frac{1}{\hat{\lambda}_0}\right)P - \frac{1}{\hat{\lambda}_0}bg(z,0)\right)\psi = \frac{1}{\hat{\lambda}_0}aqf(z,0)\varphi_1.$$

Thanks to $b < \mu_1 d$ and $\frac{1}{\hat{\lambda}_0} < 1$, all eigenvalues of the operator $(-d\Delta + (1 - \frac{1}{\hat{\lambda}_0})P - \frac{1}{\hat{\lambda}_0}bg(z,0))I$ are larger than 0, that is the operator is inverse, hence, let

$$\psi_1 = \left(-d\Delta + \left(1 - \frac{1}{\hat{\lambda}_0}\right)P - \frac{1}{\hat{\lambda}_0}bg(z,0)\right)^{-1}\frac{1}{\hat{\lambda}_0}aqf(z,0)\varphi_1.$$

Then there exists the eigenvalue of L_0 is larger than 1, and the corresponding eigenfunction denoted by (φ_1, ψ_1), then from Proposition 1.1, we can get $\mathrm{index}_W(F(0,0)) = 0$.

When $a < \frac{\lambda_1 d}{1-q}$, 1 is not the eigenvalue of (3.2). Next, we prove that L_0 does not have the α property on $\bar{W}_{y_0}$. Suppose that L_0 on $\bar{W}_{y_0}$ has α property, then there exist $t_1 \in (0,1)$ and the function $(\phi, \psi) \in \bar{W}_{y_0} \backslash S_{y_0} = K_0 \times K_0 \backslash \{(0,0)\}$ such that $(I - t_1 L_0)(\varphi, \psi)^T \in S_{y_0}$, that is,

$$\begin{cases} (-d\Delta + P)\varphi = t_1(P + a(1-q)f(z,0))\varphi, & x \in \Omega, \\ (-d\Delta + P)\psi = t_1((P + bg(z,0))\psi + qaf(z,0)\varphi), & x \in \Omega. \end{cases} \tag{3.4}$$

For (3.4), if $\psi \equiv 0$, then $\varphi \equiv 0$, since (φ, ψ) is an eigenfunction, hence $\psi \neq 0$, we discuss the following two cases:

(a) If $\varphi \equiv 0$, then from the second equation of (3.4), we see that $\lambda_1(-d\Delta - bg(z,0)) > 0$ when $b < \mu_1 d$, it follows from the Proposition 1.4 that (3.4) has no eigenvalues which are equal to or less than 1. This is a contradiction hypothesis.

(b) If $\varphi \not\equiv 0$, then we can get the following results from equation (3.4), when

$$a < \frac{\lambda_1 d}{1-q}, \quad \lambda_1\big(-d\Delta - a(1-q)f(z,0)\big) > 0.$$

It follows from Proposition 1.4 that (3.4) has no eigenvalues which are equal to or less than 1, this is a contradiction with the hypothesis. Hence, L_0 as no property α on $\bar{W}_{y_0}$. By Proposition 1.4, we can get that

$$\mathrm{index}_W(F, y_0) = \mathrm{index}_W\big(L_0, (0,0)\big) = \pm 1.$$

It is easy to see that $\mathrm{index}_W(L_0, (0,0)) = (-1)^\sigma$, where σ is the sum of the algebraic multiplicities of the eigenvalues of L_0 which are greater than 1. From above results and (3.2), we see that L_0 has no eigenvalues which are greater than 1, then $\sigma = 0$, that is, $\mathrm{index}_W(F, (0,0)) = 1$. □

Lemma 3.2 $\mathrm{index}_W(F, D') = 1$.

Proof Taking $\tau \in (0,1)$ sufficiently small, such that $\tau a(1-q) < \lambda_1 d, \tau b < \lambda_1 d$, it follows from Lemma 3.1 that $\mathrm{index}_W(F, (0,0)) = 1$, so as $0 < \tau << 1$, $\deg_W(I - F_\tau, D', (0,0)) = 1$, by the homotopy invariance, we get $\deg_W(I - F_\tau, D', (0,0)) = 1$, thus $\mathrm{index}_W(F, D') = 1$. □

Lemma 3.3 *Suppose $b > \mu_1 d$, the index on the point of $(0,\theta)$ satisfies the following results*:
  (i) *If $a < \frac{\hat{\lambda}_1 d}{1-q}$, then $\mathrm{index}_W(F, (0,\theta)) = 1$; if $a > \frac{\hat{\lambda}_1 d}{1-q}$, then $\mathrm{index}_W(F, (0,\theta)) = 0$.*
  (ii) *If $a = \frac{\hat{\lambda}_1 d}{1-q}$, then either* (1.3) *has a positive solution or $\mathrm{index}_W(F, (0,\theta)) = 1$.*

Proof Defining the linearized operator F on $(0,\theta)$ as follows:

$$L_1 = F'(0,\theta) = (-d\Delta + P)^{-1} \begin{pmatrix} P + a(1-q)f(z-\theta,0) & 0 \\ aqf(z-\theta,0) - b\theta g_1'(z-\theta,0) & P + L_b \end{pmatrix},$$

where $L_b = b(g(z-\theta,\theta) - \theta g_1'(z-\theta,\theta) + \theta g_2'(z-\theta,\theta))$. Taking $y_1 = (0,\theta)$, then $\bar{W}_{y_1} = K_0 \times C_0(\Omega), S_{y_1} = \{0\} \times C_0(\bar{\Omega}), \bar{W}_{y_1} \backslash S_{y_0} = K_0 \backslash \{0\} \times C_0(\bar{\Omega})$.

First, we will prove that $I - L_1$ is invertible on $\bar{W}_{y_1}$ as $a \neq \frac{\hat{\lambda}_1 d}{1-q}$. Assume that there exists $\phi, \psi \in C_0(\bar{\Omega})$ such that $L_1(\phi, \psi)^T = (\phi, \psi)^T$, then

$$d\Delta\phi + a(1-q)f(z-\theta,0)\phi = 0,$$

$$d\Delta\psi + L_b\psi + qaf(z-\theta,0)\phi - b\theta g'(z-\theta,\theta)\phi = 0.$$

From the definition of $a \neq \frac{\hat{\lambda}_1 d}{1-q}$ and λ_1, we can get $\phi \equiv 0$, and taking $\phi \equiv 0$ into the second equation, then $d\Delta\psi + L_b\psi = 0$.

It follows from Lemma 2.1 that $\psi \equiv 0$, hence $I - L_1$ is inverse on $\bar{W}_{y_1}$.

(i) When $a < \frac{\hat{\lambda}_1 d}{1-q}$, we can prove that L_1 has no property α on $\bar{W}_{y_1}$.

We assume that L_1 has property α on $\bar{W}_{y_1}$, then there exist $t_1 \in (0,1)$ and the function $(\phi, \psi) \in \bar{W}_{y_1} \setminus S_{y_1} = K_0 \setminus \{0\} \times C_0\{\bar{\Omega}\}$, such that $I - t_1 L_1(\phi, \psi)^T \in S_{y_1} = \{0\} \times C_0(\bar{\Omega})$, that is

$$\begin{cases} (-d\Delta + P)\phi - t_1(P + (1-q)af(z-\theta,0))\phi = 0, \\ (-d\Delta + P)\psi - t_1(aqf(z-\theta,0)\phi - b\theta g_1'(z-\theta,\theta)\phi + P\psi + L_b\psi) \in C_0(\bar{\Omega}). \end{cases} \tag{3.5}$$

It follows from the definition of ϕ, ψ which satisfies (3.5), let $T = (-d\Delta + P)^{-1}(P + (1-q)af(z-\theta,0))I$, for (3.5), $T\phi = \frac{1}{t_1}\phi > \phi$, from Proposition 1.3, we can obtain $r(T) > 1$. However, from $a < \frac{\hat{\lambda}_1 d}{1-q}$, we know that there exists $\phi > 0$ such that

$$d\Delta\phi + a(1-q)f(z-\theta,0)\phi = \lambda_1(d\Delta + a(1-q)f(z-\theta,0))\phi < 0.$$

Adding both sides with $P\phi > 0$, we can get $(d\Delta + a(1-q)f(z-\theta,0) + P)\phi < P\phi$, that is

$$T\phi = (-d\Delta + P)^{-1}(a(1-q)f(z-\theta,0) + P)\phi < \phi.$$

From Proposition 1.3, we can obtain $r(T) < 1$, and then get a contradiction. Hence L_1 has no property α on $\bar{W}_{y_0}$. From Proposition 1.1, we know

$$\mathrm{index}_W(F, y_1) = \mathrm{index}_W(L_1, (0, \theta)) = \pm 1.$$

It is easy to see that $\mathrm{index}_W(L_1, (0, \theta)) = (-1)^\sigma$, where σ is the sum of the algebraic multiplicities of the eigenvalues of L_1 which are greater than 1. Set λ be the eigenvalue of L_1, the corresponding eigenfunction named by (ϕ, ψ), then $L_1(\phi, \psi)^T = \lambda(\phi, \psi)^T$, that is

$$\begin{cases} (-d\Delta + P)^{-1}(P + (1-q)af(z-\theta,0))\phi = \lambda\phi, \\ (-d\Delta + P)^{-1}(aqf(z-\theta,0)\phi - b\theta g_1'(z-\theta,\theta)\phi + P\psi + L_b\psi) = \lambda\psi. \end{cases}$$

By the definition of the above operator T, we see that $T\phi = \lambda\phi$ and $r(T) < 1$, then all eigenvalues of L_1 are less than 1, so

$$\mathrm{index}_W(F, (0, \theta)) = \mathrm{index}_E(L_1, (0, 0)) = (-1)^\sigma = (-1)^0 = 1.$$

Next, we prove that L_1 has property α on $\bar{W}_{y_1}$ when $a > \frac{\hat{\lambda}_1 d}{1-q}$. Suppose that η is the principal eigenvalue of $d\Delta\phi + a(1-q)f(z-\theta,0)\phi = \eta\phi$, thanks to $a > \frac{\hat{\lambda}_1 d}{1-q}$, $\eta > 0$, from Proposition 1.4, we know that there exists $t_1 \in (0,1)$ is eigenvalue of $(-d\Delta + P)\phi = t_1(P + a(1-$

$q)f(z - \theta, 0))\phi$, take $\phi \in K_0 \backslash \{0\}$ as the corresponding eigenfunction, then $(\phi, 0) \in \bar{W}_{y_1} \backslash S_{y_1}$ such that

$$(-d\Delta + P)\phi - t_1 (P + a(1 - q)f(z - \theta, 0))\phi = 0,$$

$$-t_1(-d\Delta + P)^{-1}(aqf(z - \theta, 0)\phi - b\theta g_1'(z - \theta, \theta)\phi) \in C_0(\bar{\Omega}),$$

that is, $I - t_1 L_1(\phi, \psi)^T \in S_{y_1}$, thus, L_1 has property α on $\bar{W}_{y_1}$. By the Proposition 1.1, we can obtain $\text{index}_W(F, (0, \theta)) = 0$.

(ii) First, we show that $I - L_1$ is not invertible on $\bar{W}_{y_1}$. That is, there exists $(\omega, \chi) \in \bar{W}_{y_1}$ such that $L_1(\omega, \chi)^T = (\omega, \chi)^T$, it follows that

$$\begin{cases} d\Delta\omega + \alpha(1 - q)f(z - \theta, 0)\omega = 0, \\ d\Delta\chi + L_b\chi + qaf(z - \theta, 0)\omega - b\theta g_1'(z - \theta, \theta)\omega = 0. \end{cases} \tag{3.6}$$

If $a = \frac{\hat{\lambda}_1 d}{1-q}$, then we take $\hat{\phi}_1 > 0$ into the second equation of (3.6) as follows:

$$L_{(a,b)}(d\Delta + L_b)\chi = -qaf(z - \theta, 0)\omega_1 + b\theta g_1'(z - \theta, \theta)\hat{\phi}_1.$$

From Remark 2.1, we see that all the eigenvalues of $L_{(b,d)}$ less than 0, that is, the operator $L_{(b,d)}$ is invertible, then there exists the unique

$$\chi_1 = L_{(a,b)}^{-1}(-qaf(z - \theta, 0)\omega_1 + b\theta g_1'(z - \theta, \theta)\hat{\phi}_1),$$

hence there exists $(\hat{\phi}_1, \chi_1) \in \bar{W}_{y_1} = K_0 \times C_0(\bar{\Omega})$ such that $L_1(\omega, \chi)^T = (\omega, \chi)^T$. Thus, we have proved that $I - L_1$ is not invertible on $\bar{W}_{y_1}$. Hence, the results cannot be proved by Proposition 1.1 in this case.

In the following, we prove that (1.3) can bifurcate from $(a, 0, \theta)$ by the bifurcation theory [18, 19], then we get the result. Defining the function as $F(a, u, v) = (d\Delta u + a(1 - q)uf(z - u - v, u), d\Delta v + bvg(z - v - u, v) + aqf(z - u - v, u)u)$. Clearly, $F(a, 0, \theta) = 0$. Define the operator as follows:

$$L_1(a, 0, \theta) = D_{(u,v)}F(a, 0, \theta) = \begin{pmatrix} d\Delta + a(1 - q)f(z - \theta, 0) & 0 \\ -b\theta g_1'(z - \theta, \theta) + aqf(z - \theta, 0) & d\Delta + L_b \end{pmatrix},$$

where $D_{(u,v)}F(a, 0, \theta)$ stands for the Fréchet derivatives of F at the point (u, v).

Let $L_1(a, 0, \theta)(\omega, \chi)^T = (0, 0)^T$. From the analysis of (3.6), we see that the nuclear space of $L_1(a, 0, \theta)$ satisfies $N(L_1(a, 0, \theta)) = \text{span}\{(\hat{\phi}_1, \chi_1)\}$. Hence $\dim N(L_1(a, 0, \theta)) = 1$.

The adjoint operator of $L_1(a, 0, \theta)$ can be written as

$$L_1^*(a, 0, \theta) = \begin{pmatrix} d\Delta + a(1 - q)f(z - \theta, 0) & -b\theta g_1'(z - \theta, \theta) + aqf(z - \theta, 0) \\ 0 & d\Delta + L_b \end{pmatrix}.$$

Let $L_1^*(a, 0, \theta)(\omega, \chi)^T = (0, 0)^T$, it is easy to get $(\omega, \chi) = (\hat{\phi}_1, 0)$, so $N(L_1^*(a, 0, \theta)) = \text{span}\{(\hat{\phi}_1, 0)\}$. According to the Fredholm theorem [18], we can obtain $R(L_1(a, 0, \theta)) = \{(\omega, \chi) \in E | \int_\Omega \omega\hat{\phi}_1 \, dx = 0\}$, so $\text{codim} R(L(a, 0, \theta)) = 1$.

Define $L_2(a, 0, \theta) = D_a D_{(u,v)} F(a, 0, \theta)$, then

$$L_2(a, 0, \theta) \cdot (\hat{\phi}_1, \chi_1) = \begin{pmatrix} (1-q)f(z-\theta, 0) & 0 \\ qf(z-\theta, 0) & 0 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} (1-q)f(z-\theta, 0)\hat{\phi}_1 \\ qf(z-\theta, 0)\hat{\phi}_1 \end{pmatrix}.$$

Owing to $N(L_1^*)$ and $R(L_1)$ are orthogonal, however $\int_\Omega (1-q)f(z-\theta, 0)\hat{\phi}_1^2 \neq 0$.

Hence $L_2(a, 0, \theta)(\hat{\phi}_1, \chi_1)^T \notin R(L_1)$. To sum up, when $a = \frac{\lambda_1 d}{1-q}, b > \mu_1 d$, it follows from the bifurcation theorem [18, 19] that $F(a, u, v) = 0$ $(a \in R)$ can produce the bifurcation at the point $(a, 0, \theta)$. That is, there exist $|s| < \delta$ $(\delta > 0)$ and the C^1 function as $(a(s), \phi(s), \varphi(s))$: $(-\delta, \delta) \to R \times E$, such that $u(s) = s(\hat{\phi}_1 + \phi(s)), v(s) = \theta - s(\chi_1 + \varphi(s))$,

$$a(0) = a, \qquad \phi(0) = \varphi(0) = 0, \qquad \phi(s), \varphi(s) \in Z, \quad Z \oplus N(L_1) = E,$$

and satisfying $F(a(s), u(s), v(s)) = 0$ $(|s| < \delta)$.

We shall discuss two possible cases as follows:

Case 1: if $a'(s) \equiv 0$, then $a(s) = a$, for s submitting to $|s| < \sigma$, and $F(a(s), u(s), v(s)) = 0$, that is, $(u(s), v(s))$ is the solution (1.3), since $\hat{\phi}_1 > 0$, and $|s|$ is very small, then there exists $\varepsilon > 0$ such that $u(s) > 0, v(s) > 0$, when $0 < s < \varepsilon \leq \delta$. Hence, (1.3) have positive solutions.

Case 2: if $a'(s) \not\equiv 0$, then there exists $\delta' > 0, a'(s)$ is not equal to 0 when $|s| < \delta \leq \delta'$. According to the uniqueness of the bifurcation solution of $F(a(s), u(s), v(s)) = 0$, $(0, \theta)$ is isolated fixed point of the operator F in the neighborhood of $(a, 0, \theta)$. Owing to 1 is the eigenvalue of $F'(0, \theta)$, we cannot calculate $\text{index}_W(F, (0, \theta)) = (-1)^\sigma$. Hence, we can define the operator as

$$F_t(u, v) = (-d\Delta + P)^{-1} \begin{pmatrix} Pu + (1-q)af(z-u-v, u)u - tu \\ M + bg(z-u-v, v)v + qaf(z-u-v, u)u \end{pmatrix}.$$

Obviously, $(0, \theta)$ is the fixed point of F_t. Defining the derived operator of F_t at $(0, \theta)$ as follows:

$$L_t = F_t'(0, \theta) = (-d\Delta + P)^{-1} \begin{pmatrix} p + (1-q)af(z-\theta, 0) - t & 0 \\ -b\theta g_1'(z-\theta, \theta) + aqf(z-\theta, 0) & P + L_b \end{pmatrix}.$$

The compact operator F_t deduce that F_t' is compact operator. Suppose that 1 is the eigenvalue of L_t when $t \in (0, 1]$, then $L_t(\phi, \psi)^T = (\phi, \psi)^T$, that is,

$$\begin{cases} d\Delta\phi + a(1-q)f(z-\theta, 0)\phi - t\phi = 0, \\ d\Delta\psi + L_b\psi + aqf(z-\theta, 0)\phi - b\theta g_1'(z-\theta, \theta)\phi = 0. \end{cases}$$

Thanks to $a(1-q) = \lambda_1 d$, then $\lambda_1(d\Delta + (1-q)af(z-\theta, 0) - t) < 0$, that is, $d\Delta + (1-q)af(z-\theta, 0) - t$ is invertible, then $\phi \equiv 0$, and from Lemma 2.1, we know that the operator L is invertible. Then, similarly, we see that $\psi \equiv 0$. Thus, 1 is not the eigenvalue of L_t.

Due to $(0, \theta)$ is the isolated fixed point of F_t, $\text{index}_W(L_t(0, 0)) = (-1)^\sigma$. To show that L_t has no eigenvalue greater than 1, suppose that $\sigma > 1$ is an eigenvalue of L_t. Let

$$(\omega, \chi) \neq (0, 0), \quad L_t(\omega, \chi)^T = \sigma(\omega, \chi)^T, \quad \text{then}$$

$$d\Delta\omega + P\omega = \frac{1}{\sigma}\big(P\omega + a(1-q)f(z-\theta,0)\omega - t\omega\big). \tag{3.7}$$

Equation (3.7) can be rewritten as follows:

$$d\Delta\omega + \frac{1}{\sigma}(1-q)af(z-\theta,0)\omega = \left(P + \frac{t}{\sigma} - \frac{P}{\sigma}\right)\omega.$$

Thanks to $\sigma > 1$, then $\nu = P + \frac{t}{\sigma} - \frac{P}{\sigma} > 0$ is some eigenvalue of $d\Delta + \frac{1}{\sigma}(1-q)af(z-\theta,0)$. However, from

$$\lambda_1\left(d\Delta + \frac{1}{\sigma}(1-q)af(z-\theta,0)\right) \le \lambda_1\big(d\Delta + (1-q)af(z-\theta,0)\big),$$

we obtain $\lambda_1(d\Delta + \frac{1}{\sigma}(1-q)af(z-\theta,0)) \le 0$, this contradicts with $\nu > 0$. Hence, $\mathrm{index}_W(F_t, (0,\theta)) = 1$ as $t \in [0,1]$. Because $(0,\theta)$ is isolated fixed point of (1.3), we can take some neighborhood U of $(0,\theta)$ in D' such that F_t has no fixed point at ∂U. Using the homotopy invariance property, we get $\mathrm{index}_W(F,(0,\theta)) = \mathrm{index}_W(F_t(0,\theta)) = 1$. □

4 Coexistence of the chemostat model

In this section, by using the fixed point index calculation method, combined with Lemmas 3.1-3.3, we can show that there exists the sufficient condition of existence of nonnegative solutions to equation (1.3).

Theorem 4.1
  (i) *If $a < \frac{\lambda_1 d}{1-q}, b < \mu_1 d$, then the unique non-negative solution of (1.3) is zero*;
  (ii) *if $a > \frac{\lambda_1 d}{1-q}, b < \mu_1 d$, then (1.3) have at least one positive solution besides $(0,0)$*;
  (iii) *if $a > \frac{\lambda_1 d}{1-q}, b > \mu_1 d$, then (1.3) have at least one positive solution besides $(0,0),(0,\theta)$*.

Proof (i) From Lemma 2.1, (1.3) can have no semi-trivial solution as $(0,\theta)$ when $a < \frac{\lambda_1 d}{1-q}$, $b < \mu_1 d$ then there exists only a zero solution as $(0,0)$. It follows from Lemma 3.1 that

$$\mathrm{index}_W\big(F,(0,0)\big) = 1, \quad \text{then } \mathrm{index}_W\big(F,D'\big) = \mathrm{index}_W\big(F,(0,0)\big).$$

Hence, we cannot prove the existence of positive solution by the fixed point index theory.
  In the following, we show it by the upper and lower solution method.
  When $a < \frac{\lambda_1 d}{1-q}$, we suppose (1.3) have non-negative solution $(\tilde{u},\tilde{v})$. It follows that either $\tilde{u} > 0$ or $\tilde{u} \equiv 0$. If $\tilde{u} > 0$, then

$$0 = d\Delta\tilde{u} + a(1-q)f(z-\tilde{u}-\tilde{v},\tilde{u})\tilde{u} \le d\Delta\tilde{u} + a(1-q)f(z-\tilde{u},\tilde{u})\tilde{u}, \quad x \in \Omega.$$

Hence $\tilde{u}$ is a positive lower solution of (1.3), and there exists an upper solution as $C_1 > 0$, $\tilde{u} \le C_1, x \in \bar{\Omega}$. Therefore, (1.3) has a positive solution u_+ such that $\tilde{u} \le u_+ \le C_1$, $x \in \bar{\Omega}$. This contradicts Lemma 2.1. Thus $\tilde{u} \equiv 0$, then when $b < \mu_1 d$, assume that (1.3) have non-negative solution $(0,\tilde{v})$, and $\tilde{v} > 0$, this contradicts with Remark 2.1. Hence, the non-negative solution of (1.3) has the only zero.
  (ii) If $a > \frac{\lambda_1 d}{1-q}, b < \mu_1 d$, from Lemma 2.1, we know that (1.3) has a semi-trivial solution $(0,\theta)$, then there exists only $(0,0)$. It follows from Lemma 3.1 that

$$\mathrm{index}_W\big(F,(0,0)\big) = 0,$$

and according to Lemma 3.2, $\text{index}_W(F, D') = 1$, then

$$\text{index}_W(F, D') \neq \text{index}_W(F, (0, 0)).$$

Hence (1.3) has at least a positive solution on D'.

(iii) Similar to the proof of (ii), we can prove that $(0, 0), (0, \theta)$ are the non-negative solutions, if $a > \frac{\hat{\lambda}_1 d}{1-q}, b > \mu_1 d$. Thanks to Lemmas 3.1-3.3,

$$\text{index}_W(F, D') \neq \text{index}_W(F, (0, 0)) + \text{index}_W(F, (0, \theta)).$$

Hence (1.3) has at least a positive solution on D'. □

Remark 4.1 When $a = \frac{\lambda_1 d}{1-q}, b > \mu_1 d$, it follows from Lemma 3.3 that we can obtain either $\text{index}_W(F, (0, \theta)) = 1$, or for (1.3) there exists a positive solution.

When $\text{index}_W(F, (0, \theta)) = 1$,

$$\text{index}_W(F, D') = \text{index}_W(F, (0, 0)) + \text{index}_W(F, (0, \theta)),$$

the method of the index calculation cannot judge the existence of positive solutions. If another result of Lemma 3.3 holds, then (1.3) has a positive solution. To sum up, we cannot determine whether we have the existence of the positive solution.

5 Conclusion

The coexistence of an unstirred chemostat model with B-D functional response is studied by fixed point index theory in our paper. First of all, some prior estimates for positive solutions are proved by the maximum principle and the upper and lower solution method. Second, the calculations are performed on the fixed point index of chemostat model by degree theory and the homotopy invariance theorem. Finally, some sufficient condition on the existence of positive steady-state solutions is established by fixed point index theory in cone and bifurcation theory.

Author details
^{1}Department of Mathematics and Physics, Xi'an Technological University, Xi'an, 710032, China. ^{2}Institute of Mathematics, Shaanxi Normal University, Xi'an, 710062, China. ^{3}School of Mechanical Engineering, Xi'an Technological University, Xi'an, 710032, China.

References
1. Levin, BR, Stewart, FM: Partitioning of resources and the outcome of interspecic competition: a model and some general considerations. Am. Nat. **107**, 171-198 (1973)
2. Hsu, SB, Hubbell, SP, Waltman, P: A mathematical model for single nutrient competition in continuous cultures of micro-organisms. SIAM J. Appl. Math. **32**, 366-383 (1977)

3. Smith, HL, Waltman, P: The Theory of the Chemostat. Cambridge University Press, Cambridge (1995)
4. Wu, JH: Global bifurcation of coexistence state for the competition model in the chemostat. Nonlinear Anal. **39**, 817-835 (2000)
5. Wu, JH, Wolkowicz, GSK: A system of resource-based growth models with two resources in the unstirred chemostat. J. Differ. Equ. **172**, 300-332 (2001)
6. Wu, JH, Nie, H, Wolkowicz, GSK: A mathematical model of competition for two essential resources in the unstirred chemostat. SIAM J. Appl. Math. **65**, 209-229 (2004)
7. Nie, H, Wu, JH: A system of reaction-diffusion equations in the unstirred chemostat with an inhibitor. Int. J. Bifurc. Chaos Appl. Sci. Eng. **16**(4), 989-1009 (2006)
8. Nie, H, Wu, JH: Asymptotic behavior of an unstirred chemostat model with internal in-hibitor. J. Math. Anal. Appl. **334**, 889-908 (2007)
9. DeAngelis, DL, Goldstein, RA, O'Neill, RV: A model for trophic interaction. Ecology **56**, 881-892 (1975)
10. Hsu, SB, Waltman, P: On a system of reaction-diffusion equations arising from competition in an unstirred chemostat. SIAM J. Appl. Math. **53**, 1026-1044 (1993)
11. Dancer, EN: On the indices of fixed points of mappings in cones and applications. J. Math. Anal. Appl. **91**(1), 131-151 (1983)
12. Li, L, Logan, R: Positive solution to general elliptic competition models. Differ. Integral Equ. **4**(4), 817-834 (1991)
13. Wang, MX: Nonlinear Parabolic Equations. Science Press, Beijing (1997)
14. Wu, JH, Nie, H, Wolkowicz, GSK: The e ect of inhibitor on the plasmid-bearing and plasmid-free model in the unstirred chemostat. SIAM J. Math. Anal. **38**(6), 1860-1885 (2007)
15. Wu, JH: Global bifurcation of coexistence state for the competition model in the chemostat. Nonlinear Anal. **39**(7), 817-835 (2000)
16. Ye, QX, Li, ZY: The Introduction of Reaction Diffusion Equations, pp. 107-139. Science Press, Beijing (1994)
17. Hess, P, Lazer, AC: On an abstract competition in an unstirred chemostat. Can. Appl. Math. Q. **2**, 461-484 (1994)
18. Blat, J, Brown, KJ: Global bifurcation of positive solutions in some systems of elliptic equations. SIAM J. Math. Anal. **17**(6), 1339-1353 (1986)
19. Smoller, J: Shock Waves and Reaction-Diffusion Equations, pp. 93-122, 126-156, 167-181. Spring, New York (1983)