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Multilinear fractional integral operators on non-homogeneous metric measure spaces

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Abstract

In this paper, the boundedness in Lebesgue spaces for multilinear fractional integral operators and commutators generated by multilinear fractional integrals with an RBMO(μ) function on non-homogeneous metric measure spaces is obtained.

MSC: Primary 42B25; secondary 47B47

Keywords: multilinear fractional integrals; commutators; non-homogeneous metric measure spaces

1 Introduction and preliminaries

A measure μ is called a doubling measure, if there exists a positive constant C such that $\mu(B(x,2l)) \le C\mu(B(x,l))$, for all $x \in \operatorname{supp}\mu$ and all $l > 0$, which is the main condition in homogeneous spaces. Also μ is a non-doubling measure, if there exists an integer $k \in (0,n]$ and a positive constant C_0, such that

$$\mu\big(B(x,l)\big) \le C_0 l^k. \tag{1.1}$$

This innovation caused the tremendous development of harmonic analysis (see [1–8]). It is worthy to mention that this theory solves the Painlevé's problem and Vitushkin's conjectures (see [7, 9]). Hytönen [10] introduced the non-homogeneous metric measure spaces (X,d,μ), which contains the homogeneous spaces and non-doubling measure spaces. Many researchers obtained the boundedness of operators on the non-homogeneous metric measure spaces; see, *e.g.*, [10–25].

For multilinear integral operators, the bilinear theory for Calderón-Zygmund operators was studied by Coifman-Meyers [26], then, the boundedness on Lebesgue spaces or Hardy spaces for multilinear singular integrals was proved by Gorafakos-Torres [27, 28]. In non-doubling measure spaces, Xu [29, 30] and Lian-Wu [31] obtained the boundedness of multilinear singular integrals or multilinear fractional integrals and commutators respectively. In non-homogeneous metric measure spaces, Hu *et al.* [32] established the weighted norm inequalities for multilinear Calderón-Zygmund operators. The authors of [23] proved the boundedness on Lebesgue spaces for commutators of multilinear singular integrals.

In this paper, we introduce multilinear fractional integrals and its commutators on non-homogeneous metric spaces, then we study the boundedness in Lebesgue spaces for these

operators, provided that fractional integral is bounded from $L^r(\mu)$ to $L^s(\mu)$, for some $r \in (1, 1/\beta)$ and $1/s = 1/r - \beta$ with $0 < \beta < 1$. Our results include both the results for the homogeneous spaces and the non-doubling measure spaces.

Throughout this paper, $L_c^\infty(\mu)$ denotes $L^\infty(\mu)$ with compact support. C always denotes a positive constant independent of the main parameters involved, but it may be different in different currents. And p' is the conjugate index of p, namely, $1/p + 1/p' = 1$. Next let us give some definitions and notations.

Definition 1.1 ([10]) A metric space (X, d) is geometrically doubling, if there is a positive integer N_0 such that, for all ball $B(x, r) \subset X$, one can find a finite ball covering $\{B(x_j, r/2)\}_{j=1}^{N_0}$.

Definition 1.2 ([10]) For a metric measure space (X, d, μ), if μ is a Borel measure on X, and there is a function $\lambda : X \times (0, +\infty) \to (0, +\infty)$ and a positive constant C_λ, such that for all $x \in X$, the function $l \longmapsto \lambda(x, l)$ is non-decreasing, and for all $x \in X$, $l > 0$, the following holds:

$$\mu\big(B(x, l)\big) \le \lambda(x, l) \le C_\lambda \lambda(x, l/2), \tag{1.2}$$

then (X, d, μ) is called upper doubling.

Remark 1.3

(i) If $\lambda(x, l)$ equals to $\mu(B(x, l))$, then the homogeneous spaces is upper doubling spaces. Also, if $\lambda(x, l)$ equals Cl^k, then a metric space (X, d, μ) satisfying (1.1) is upper doubling.

(ii) By [18], we know that there exists another function $\tilde{\lambda} \le \lambda$, $\forall x, y \in X$ with $d(x, y) \le l$, and the following holds:

$$\tilde{\lambda}(x, l) \le \widetilde{C}\tilde{\lambda}(y, l). \tag{1.3}$$

Thus one always assumes that λ satisfies (1.3) throughout this paper. Because the singularity of the commutators is stronger than that of the fractional integral, we need to assume $\lambda(x, al) \ge a^m \lambda(x, l)$, for all $x \in X$ and $a, l > 0$, in the proof of boundedness of commutators.

(iii) The upper doubling condition is equivalent to the weak growth condition introduced by Tan-Li in [33].

A measure μ is (α, β)-doubling, if $\mu(\alpha B) \le \beta \mu(B)$, for $\alpha, \beta \in (1, +\infty)$ and all ball $B \subset X$. Bui-Duong [11] pointed out that there exist many doubling balls. One always means that (α, β)-doubling ball is a $(6, \beta_0)$-doubling ball throughout the paper, for some fixed number $\beta_0 > \max\{C_\lambda^{3\log_2 6}, 6^n\}$, where $n = \log_2 N_0$ is viewed as a geometric dimension of the space, except α and β are designated.

Definition 1.4 ([15]) For $0 \le \gamma < 1$, B and R be two arbitrary balls with $B \subset R$ and $N_{B,R}$ be the smallest integer satisfying $6^{N_{B,R}} l_B \ge l_R$. One defines

$$K_{B,R}^{(\gamma)} = 1 + \sum_{j=1}^{N_{B,R}} \left[\frac{\mu(6^j B)}{\lambda(x_B, 6^j l_B)} \right]^{(1-\gamma)}. \tag{1.4}$$

For $\gamma = 0$, one simply writes $K_{B,R}^{(0)} = K_{B,R}$.

Definition 1.5 Let $\alpha \in (0, m)$. We call K is an m-linear fractional integral kernel, if

$$K(\cdot, \dots, \cdot) \in L^1_{\mathrm{loc}}\big((X)^{m+1} \backslash \big\{(x, y_1 \cdots, y_i, \dots, y_m) : x = y_i, 1 \le i \le m\big\}\big),$$

and the following two items hold:

(i)

$$\big|K(x, y_1, \dots, y_i, \dots, y_m)\big| \le \frac{C}{[\sum_{i=1}^m \lambda(x, d(x, y_i))]^{m-\alpha}}, \tag{1.5}$$

$\forall (x, y_1, \dots, y_i, \dots, y_m) \in (X)^{m+1}$, with $x \ne y_i$ for some i;

(ii) there is a constant $0 < \delta \le 1$,

$$\begin{aligned}&\big|K(x, y_1, \dots, y_i, \dots, y_m) - K\big(x', y_1, \dots, y_i, \dots, y_m\big)\big| \\ &\quad \le \frac{Cd(x, x')^\delta}{[\sum_{i=1}^m d(x, y_i)]^\delta [\sum_{i=1}^m \lambda(x, d(x, y_i))]^{m-\alpha}},\end{aligned} \tag{1.6}$$

if $Cd(x, x') \le \max_{1 \le i \le m} d(x, y_i)$, and for every i,

$$\begin{aligned}&\big|K(x, y_1, \dots, y_i, \dots, y_m) - K\big(x, y_1, \dots, y_i', \dots, y_m\big)\big| \\ &\quad \le \frac{Cd(y_i, y_i')^\delta}{[\sum_{i=1}^m d(x, y_i)]^\delta [\sum_{i=1}^m \lambda(x, d(x, y_i))]^{m-\alpha}},\end{aligned} \tag{1.7}$$

if $Cd(y_i, y_i') \le \max_{1 \le i \le m} d(x, y_i)$.

For any m compactly supported bounded functions $f_1, \dots, f_m$, and any point $x \notin \bigcap_{i=1}^m \operatorname{supp} f_i$, the multilinear fractional integral operators $I_{\alpha,m}$ is defined by

$$\begin{aligned}&I_{\alpha,m}(f_1, \dots, f_m)(x) \\ &\quad = \int_{X^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\mu(y_1) \cdots d\mu(y_m).\end{aligned} \tag{1.8}$$

Remark 1.6 As $\max_{1 \le i \le m} d(x, y_i) \le \sum_{i=1}^m d(x, y_i) \le m \max_{1 \le i \le m} d(x, y_i)$, (ii) in Definition 1.5 is equivalent to the following:

(ii′) There is a constant $0 < \delta \le 1$,

$$\begin{aligned}&\big|K(x, y_1, \dots, y_i, \dots, y_m) - K\big(x', y_1, \dots, y_i, \dots, y_m\big)\big| \\ &\quad \le \frac{Cd(x, x')^\delta}{[\max_{1 \le i \le m} d(x, y_i)]^\delta [\sum_{i=1}^m \lambda(x, d(x, y_i))]^{m-\alpha}},\end{aligned}$$

if $Cd(x, x') \le \max_{1 \le i \le m} d(x, y_i)$, and for every i,

$$\begin{aligned}&\big|K(x, y_1, \dots, y_i, \dots, y_m) - K\big(x, y_1, \dots, y_i', \dots, y_m\big)\big| \\ &\quad \le \frac{Cd(y_j, y_i')^\delta}{[\max_{1 \le i \le m} d(x, y_i)]^\delta [\sum_{i=1}^m \lambda(x, d(x, y_i))]^{m-\alpha}},\end{aligned}$$

if $Cd(y_i, y_i') \le \max_{1 \le i \le m} d(x, y_i)$.

Definition 1.7 ([11]) Given $\rho > 1$, $b \in L^1_{\mathrm{loc}}(\mu)$ is an RBMO(μ) function, if there is a positive constant C, for all B, we have

$$\frac{1}{\mu(\rho B)} \int_B \left| b(x) - m_{\widetilde{B}} b \right| d\mu(x) \le C, \tag{1.9}$$

and for all two doubling balls B, R with $B \subset R$,

$$\left| m_B(b) - m_R(b) \right| \le C K_{B,R}, \tag{1.10}$$

where $\widetilde{B}$ is the smallest (α, β)-doubling ball with the form $6^k B$, $k \in \mathbf{N} \cup \{0\}$, and

$$m_{\widetilde{B}}(b) = \frac{1}{\mu(\widetilde{B})} \int_{\widetilde{B}} b(x)\, d\mu(x).$$

The RBMO(μ) norm of b, denoted by $\|b\|_*$, is the minimal constant C in (1.9) and (1.10).

For $1 \le j \le m$, let C_j^m be the family of subsets $\sigma = \{\sigma(1), \sigma(2), \dots, \sigma(j)\}$ of $\{1, 2, \dots, m\}$ with j different elements. For each $\sigma \in C_j^m$, $\sigma' = \{1, 2, \dots, m\} \backslash \sigma$. For $b_j \in \mathrm{RBMO}(\mu)$, $j = 1, \dots, m$, set $\vec{b} = (b_1, b_2, \dots, b_m)$, $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma(x) = b_{\sigma(1)}(x) \cdots b_{\sigma(j)}(x)$. Denote $\vec{f} = (f_1, \dots, f_m)$, $\vec{f}_\sigma = (f_{\sigma(1)}, \dots, f_{\sigma(j)})$, and $\vec{b}_{\sigma'} \vec{f}_{\sigma'} = (b_{\sigma'(j+1)} f_{\sigma'(j+1)}, \dots, b_{\sigma'(m)} f_{\sigma'(m)})$.

Definition 1.8 For $b_j \in \mathrm{RBMO}(\mu)$, $j = 1, \dots, m$, and multilinear fractional integral operators $I_{\alpha,m}$, we define the commutators $[\vec{b}, I_{\alpha,m}]$ by

$$[\vec{b}, I_{\alpha,m}](\vec{f})(x) = \sum_{j=0}^{m} \sum_{\sigma \in C_j^m} (-1)^{m-j} b_\sigma(x) I_{\alpha,m}(\vec{f}_\sigma, \vec{b}_{\sigma'} \vec{f}_{\sigma'})(x).$$

For $m = 2$,

$$\begin{aligned}[b_1, b_2, I_{\alpha,2}](f_1, f_2)(x) &= b_1(x) b_2(x) I_{\alpha,2}(f_1, f_2)(x) - b_1(x) I_{\alpha,2}(f_1, b_2 f_2)(x) \\ &\quad - b_2(x) I_{\alpha,2}(b_1 f_1, f_2)(x) + I_{\alpha,2}(b_1 f_1, b_2 f_2)(x).\end{aligned} \tag{1.11}$$

$[b_1, I_{\alpha,2}]$ and $[b_2, I_{\alpha,2}]$ are defined thus:

$$\begin{aligned}[b_1, I_{\alpha,2}](f_1, f_2)(x) &= b_1(x) I_{\alpha,2}(f_1, f_2)(x) - I_{\alpha,2}(b_1 f_1, f_2)(x), \\ [b_2, I_{\alpha,2}](f_1, f_2)(x) &= b_2(x) I_{\alpha,2}(f_1, f_2)(x) - I_{\alpha,2}(f_1, b_2 f_2)(x).\end{aligned}$$

In this paper, one only considers the case of $m = 2$ for simplicity.

Theorem 1.9 *Let* $0 < \alpha < 2$, $1 < p_1, p_2 < +\infty$, $0 < \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \alpha < 1$, $g_1 \in L^{p_1}(\mu)$ *and* $g_2 \in L^{p_2}(\mu)$. *If* I_β *is bounded from* $L^r(\mu)$ *into* $L^s(\mu)$, *for some* $r \in (1, 1/\beta)$ *and* $1/s = 1/r - \beta$, *with* $0 < \beta < 1$, *then there is a positive constant* C,

$$\left\| I_{\alpha,2}(g_1, g_2) \right\|_{L^q(\mu)} \le C \|g_1\|_{L^{p_1}(\mu)} \|g_2\|_{L^{p_2}(\mu)},$$

where I_β is defined by

$$I_\beta f(x) := \int_X \frac{f(y)}{[\lambda(y, d(x,y))]^{1-\beta}}\, d\mu(y).$$

Theorem 1.10 *Set $\|\mu\| = \infty$, $0 < \alpha < 2$, $1 < p_1, p_2 < +\infty$, $0 < \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \alpha < 1$, $g_1 \in L^{p_1}(\mu)$, $g_2 \in L^{p_2}(\mu)$, $b_1, b_2 \in \mathrm{RBMO}(\mu)$ and if I_β is bounded from $L^r(\mu)$ into $L^s(\mu)$ for some $r \in (1, 1/\beta)$, $1/s = 1/r - \beta$ with $0 < \beta < 1$, then there is a positive constant C,*

$$\big\|[b_1, b_2, I_{\alpha,2}](g_1, g_2)\big\|_{L^q(\mu)} \le C\|g_1\|_{L^{p_1}(\mu)}\|g_2\|_{L^{p_2}(\mu)}.$$

Remark 1.11 For the case that $\|\mu\| < \infty$, by Lemma 2.1 in Section 2 below, Theorem 1.10 also holds, if we assume that

$$\int_X I_{\alpha,2}(g_1, g_2)(x)\, d\mu(x) = 0, \qquad \int_X [b_1, I_{\alpha,2}](g_1, g_2)(x)\, d\mu(x) = 0,$$
$$\int_X [b_2, I_{\alpha,2}](g_1, g_2)(x)\, d\mu(x) = 0 \quad \text{and} \quad \int_X [b_1, b_2, I_{\alpha,2}](g_1, g_2)(x)\, d\mu(x) = 0.$$

This paper is organized as follows. Theorem 1.9 and Theorem 1.10 are proved in Section 2. In Section 3, some applications are stated.

2 Proof of main results

Proof of Theorem 1.9 Let $\alpha = \alpha_1 + \alpha_2$, $0 < \alpha_i < 1/p_i < 1$, $i = 1, 2$. It is easy to check that

$$\prod_{i=1}^{2}\big[\lambda\big(x, d(x, y_i)\big)\big]^{1-\alpha_i} \le \left[\sum_{i=1}^{2}\lambda\big(x, d(x, y_i)\big)\right]^{2-\alpha}.$$

Thus

$$\begin{aligned}\big|I_{\alpha,2}(g_1, g_2)(x)\big| &\le C\int_{X^2} \frac{|g_1(y_1)g_2(y_2)|}{[\sum_{i=1}^{2}\lambda(x, d(x, y_i))]^{2-\alpha}}\, d\mu(y_1)\, d\mu(y_2)\\ &\le \prod_{i=1}^{2}\int_X \frac{|g_i(y_i)|}{[\lambda(x, d(x, y_i))]^{1-\alpha_i}}\, d\mu(y_i)\\ &= \prod_{i=1}^{2} I_{\alpha_i}\big(|g_i|\big)(x).\end{aligned}$$

Let $1/q_i = 1/p_i - \alpha_i$ and $1/q_1 + 1/q_2 = 1/q$, $1 < q_i < \infty$. It follows from the Hölder's inequality and the $L^{p_i}(\mu) - L^{q_i}(\mu)$ boundedness of I_{α_i}, $i = 1, 2$, that

$$\begin{aligned}&\big\|I_{\alpha,2}(g_1, g_2)\big\|_{L^q(\mu)}\\ &\le \left\|\prod_{i=1}^{2} I_{\alpha_i}\big(|g_i|\big)\right\|_{L^q(\mu)}\\ &\le \big\|I_{\alpha_1}\big(|g_1|\big)\big\|_{L^{q_1}(\mu)}\big\|I_{\alpha_2}\big(|g_2|\big)(x)\big\|_{L^{q_2}(\mu)}\\ &\le \|g_1\|_{L^{p_1}(\mu)}\|g_2\|_{L^{p_2}(\mu)}.\end{aligned}$$

Thus the proof of Theorem 1.9 is completed. □

In order to prove Theorem 1.10, we need some lemmas.

For $f \in L^1_{\mathrm{loc}}(\mu)$ and $0 < \beta < 1$, one defines the sharp maximal operator

$$M^{\sharp,(\beta)}f(x) = \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B \left|f(y) - m_{\widetilde{B}}(f)\right| d\mu(y) + \sup_{(B,R)\in\Delta_x} \frac{|m_B(f) - m_R(f)|}{K_{B,R}^{(\beta)}},$$

here $\Delta_x := \{(B,R) : x \in B \subset R \text{ and } B, R \text{ are doubling balls}\}$.

One defines the non-centered doubling maximal operator

$$Nf(x) = \sup_{B \ni x, B \text{ doubling}} \frac{1}{\mu(B)} \int_B \left|f(y)\right| d\mu(y).$$

It is easy to see

$$\left|f(x)\right| \le Nf(x),$$

for every $f \in L^1_{\mathrm{loc}}(\mu)$ and μ-a.e. $x \in X$.

For $\rho > 1$, $\alpha \in (0,1)$ and $t \in (1,\infty)$, one defines the non-centered maximal operator $M^{(\alpha)}_{t,(\rho)}f$ as follows:

$$M^{(\alpha)}_{t,(\rho)}f(x) = \sup_{B \ni x} \left\{ \frac{1}{[\mu(\rho B)]^{1-\alpha t}} \int_B \left|f(y)\right|^t d\mu(y) \right\}^{1/t}.$$

For simplicity, write $M^{(0)}_{1,(\rho)}f(x)$ as $M_{(\rho)}f$. If $\rho \ge 5$ and for every $p > 1$, then $\|M_{(\rho)}f\|_{L^p(\mu)} \le C\|f\|_{L^p(\mu)}$ and for $p \in (t, 1/\alpha)$ and $1/q = 1/p - \alpha$, $\|M^{(\alpha)}_{t,(\rho)}\|_{L^q(\mu)} \le C\|f\|_{L^p(\mu)}$ (see [15]).

Lemma 2.1 ([15]) *For $f \in L^1_{\mathrm{loc}}(\mu)$, $\int_X f(x)\,d\mu(x) = 0$ if $\|\mu\| < \infty$. Assume $0 < \beta < 1$ and $\inf(1, Nf) \in L^p(\mu)$, $1 < p < \infty$, then*

$$\left\|N(f)\right\|_{L^p(\mu)} \le C\left\|M^{\sharp,(\beta)}(f)\right\|_{L^p(\mu)}.$$

Lemma 2.2 ([11, 15]) *For $1 < \rho < \infty$ and $1 \le p < \infty$, if $b \in \mathrm{RBMO}(\mu)$, then for all balls $B \in X$,*

$$\left\{ \frac{1}{\mu(\rho B)} \int_B \left|b_B - m_{\widetilde{B}}(b)\right|^p d\mu(X) \right\}^{1/p} \le C\|b\|_*. \tag{2.1}$$

Lemma 2.3 ([3]) *For $b \in \mathrm{RBMO}(\mu)$,*

$$\left|m_{\widetilde{6^i\frac{6}{5}B}}(b) - m_{\widetilde{B}}(b)\right| \le Ci\|b\|_*.$$

Lemma 2.4 *For $0 < \alpha < 2$, $1 < p_1, p_2, q < \infty$, $1 < r < q$ and $b_1, b_2 \in \mathrm{RBMO}(\mu)$. If I_β is bounded from $L^r(\mu)$ to $L^s(\mu)$, for some $r \in (1, 1/\beta)$ and $1/s = 1/r - \beta$, with $0 < \beta < 1$, then, for every $x \in X$, $g_1 \in L^{p_1}(\mu)$, and $g_2 \in L^{p_2}(\mu)$,*

$$\begin{aligned} &M^{\sharp,(\alpha/2)}[b_1, b_2, I_{\alpha,2}](g_1, g_2)(x) \\ &\quad\le C\big\{\|b_1\|_*\|b_2\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)(x) + \|b_1\|_* M_{r,(6)}\big([b_2, I_{\alpha,2}](g_1,g_2)\big)(x) \\ &\qquad + \|b_2\|_* M_{r,(6)}\big([b_1, I_{\alpha,2}](g_1,g_2)\big)(x) + \|b_1\|_*\|b_2\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x) M_{p_2,(5)}g_2(x)\big\}, \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
&M^{\sharp,(\alpha/2)}[b_1, I_{\alpha,2}](g_1,g_2)(x)\\
&\quad\le C\big\{\|b_1\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)(x) + \|b_1\|_* M^{(\alpha/2)}_{p_1,(5)} g_1(x) M^{(\alpha/2)}_{p_2,(5)} g_2(x)\big\}, \qquad (2.3)\\
&M^{\sharp,(\alpha/2)}[b_2, I_{\alpha,2}](g_1,g_2)(x)\\
&\quad\le C\big\{\|b_2\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)(x) + \|b_2\|_* M^{(\alpha/2)}_{p_1,(5)} g_1(x) M^{(\alpha/2)}_{p_2,(5)} g_2(x)\big\}. \qquad (2.4)
\end{aligned}
$$

Proof Choose $b_1, b_2 \in L^\infty(\mu)$ according to Lemma 3.11 in [14]. As $L^\infty_c(\mu)$ is dense in $L^p(\mu)$ for $1 < p < \infty$, by standard density arguments, we only need to consider the case that $g_1, g_2 \in L^\infty_c(\mu)$.

Similar to Theorem 9.1 in [6], in order to obtain (2.2), we only need to prove that, for every $x \in B$,

$$
\begin{aligned}
&\frac{1}{\mu(6B)}\int_B \big|[b_1,b_2,I_{\alpha,2}](g_1,g_2)(z) - H_B\big|\,d\mu(z)\\
&\quad\le C\big\{\|b_1\|_*\|b_2\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)(x) + \|b_1\|_* M_{r,(6)}\big([b_2,I_{\alpha,2}](g_1,g_2)\big)(x)\\
&\qquad + \|b_2\|_* M_{r,(6)}\big([b_1,I_{\alpha,2}](g_1,g_2)\big)(x) + C\|b_1\|_*\|b_2\|_* M^{(\alpha/2)}_{p_1,(5)} g_1(x) M^{(\alpha/2)}_{p_2,(5)} g_2(x)\big\}, \qquad (2.5)
\end{aligned}
$$

and, for every ball $B \subset R$, with $x \in B$, R is a doubling ball,

$$
\begin{aligned}
|H_B - H_R| \le{}& CK^2_{B,R} K^{(\alpha/2)}_{B,R}\big[\|b_1\|_*\|b_2\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)(x)\\
&+ \|b_1\|_*\|b_2\|_* M^{(\alpha/2)}_{p_1,(5)} g_1(x) M^{(\alpha/2)}_{p_2,(5)} g_2(x)\\
&+ \|b_1\|_* M_{r,(6)}\big([b_2,I_{\alpha,2}](g_1,g_2)\big)(x)\\
&+ \|b_2\|_* M_{r,(6)}\big([b_1,I_{\alpha,2}](g_1,g_2)\big)(x)\big]. \qquad (2.6)
\end{aligned}
$$

For every ball B, let

$$
\begin{aligned}
H_B &:= m_B\big(I_{\alpha,2}\big(\big(b_1 - m_{\widetilde{B}}(b_1)\big)g_1\chi_{X\setminus \frac{6}{5}B}, \big(b_2 - m_{\widetilde{B}}(b_2)\big)g_2\chi_{X\setminus\frac{6}{5}B}\big)\big),\\
H_R &:= m_R\big(I_{\alpha,2}\big(\big(b_1 - m_R(b_1)\big)g_1\chi_{X\setminus \frac{6}{5}R}, \big(b_2 - m_R(b_2)\big)g_2\chi_{X\setminus\frac{6}{5}R}\big)\big).
\end{aligned}
$$

It is easy to see that

$$
[b_1,b_2,I_{\alpha,2}] = I_{\alpha,2}\big(\big(b_1 - b_1(z)\big)g_1, \big(b_2 - b_2(z)\big)g_2\big)
$$

and

$$
\begin{aligned}
&I_{\alpha,2}\big(\big(b_1 - m_{\widetilde{B}}(b_1)\big)g_1, \big(b_2 - m_{\widetilde{B}}(b_2)\big)g_2\big)\\
&\quad= I_{\alpha,2}\big(\big(b_1 - b_1(z) + b_1(z) - m_{\widetilde{B}}(b_1)\big)g_1, \big(b_2 - b_2(z) + b_2(z) - m_{\widetilde{B}}(b_2)\big)g_2\big)\\
&\quad= \big(b_1(z) - m_{\widetilde{B}}(b_1)\big)\big(b_2(z) - m_{\widetilde{B}}(b_2)\big)I_{\alpha,2}(g_1,g_2)\\
&\qquad - \big(b_1(z) - m_{\widetilde{B}}(b_1)\big)I_{\alpha,2}\big(g_1, \big(b_2 - b_2(z)\big)g_2\big)\\
&\qquad - \big(b_2(z) - m_{\widetilde{B}}(b_2)\big)I_{\alpha,2}\big(\big(b_1 - b_1(z)\big)g_1, g_2\big)\\
&\qquad + I_{\alpha,2}\big(\big(b_1 - b_1(z)\big)g_1, \big(b_2 - b_2(z)\big)g_2\big).
\end{aligned}
$$

Thus

$$\begin{aligned}
&\frac{1}{\mu(6B)}\int_B \big|[b_1,b_2,I_{\alpha,2}](g_1,g_2)(z)-H_B\big|\,d\mu(z)\\
&\quad\le C\bigg(\frac{1}{\mu(6B)}\int_B \big|\big(b_1(z)-m_{\widetilde{B}}(b_1)\big)\big(b_2(z)-m_{\widetilde{B}}(b_2)\big)I_{\alpha,2}(g_1,g_2)(z)\big|\,d\mu(z)\bigg)\\
&\qquad + C\bigg(\frac{1}{\mu(6B)}\int_B \big|\big(b_1(z)-m_{\widetilde{B}}(b_1)\big)I_{\alpha,2}\big(g_1,\big(b_2-b_2(z)\big)g_2\big)(z)\big|\,d\mu(z)\bigg)\\
&\qquad + C\bigg(\frac{1}{\mu(6B)}\int_B \big|\big(b_2(z)-m_{\widetilde{B}}(b_2)\big)I_{\alpha,2}\big(\big(b_1-b_1(z)\big)g_1,g_2\big)(z)\big|\,d\mu(z)\bigg)\\
&\qquad + C\bigg(\frac{1}{\mu(6B)}\int_B \big|I_{\alpha,2}\big(\big(b_1-m_{\widetilde{B}}(b_1)\big)g_1,\big(b_2-m_{\widetilde{B}}(b_2)\big)g_2\big)(z)-H_B\big|\,d\mu(z)\bigg)\\
&\quad =: F_1+F_2+F_3+F_4. \qquad (2.7)
\end{aligned}$$

For F_1, choose $r_1, r_2 > 1$, such that $\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}=1$. It follows from Hölder's inequality that

$$\begin{aligned}
F_1 &\le C\bigg(\frac{1}{\mu(6B)}\int_B \big|b_1(z)-m_{\widetilde{B}}b_1\big|^{r_1}\,d\mu(z)\bigg)^{1/r_1}\\
&\quad\times\bigg(\frac{1}{\mu(6B)}\int_B \big|b_2(z)-m_{\widetilde{B}}b_2\big|^{r_2}\,d\mu(z)\bigg)^{1/r_2}\\
&\quad\times\bigg(\frac{1}{\mu(6B)}\int_B \big|I_{\alpha,2}(g_1,g_2)\big|^{r}\,d\mu(z)\bigg)^{1/r}\\
&\le C\|b_1\|_*\|b_2\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)(x).
\end{aligned}$$

For F_2, choose $s>1$ such that $\frac{1}{s}+\frac{1}{r}=1$, it follows that

$$\begin{aligned}
F_2 &\le C\bigg(\frac{1}{\mu(6B)}\int_B \big|b_1(z)-m_{\widetilde{B}}b_1\big|^{s}\,d\mu(z)\bigg)^{1/s}\\
&\quad\times\bigg(\frac{1}{\mu(6B)}\int_B \big|[b_2,I_{\alpha,2}](g_1,g_2)\big|^{r}\,d\mu(z)\bigg)^{1/r}\\
&\le C\|b_1\|_* M_{r,(6)}\big([b_2,I_{\alpha,2}](g_1,g_2)\big)(x).
\end{aligned}$$

For F_3, in the same way, one obtains

$$F_3 \le C\|b_2\|_* M_{r,(6)}\big([b_1,I_{\alpha,2}](g_1,g_2)\big)(x).$$

For F_4, let $g_k^1 = g_k\chi_{\frac{6}{5}B}$ and $g_k^2 = g_k - g_k^1$ for $k=1,2$. Therefore,

$$\begin{aligned}
F_4 &\le C\bigg(\frac{1}{\mu(6B)}\int_B \big|I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1^1(z),(b_2-m_{\widetilde{B}}b_2)g_2^1\big)(z)\big|\,d\mu(z)\bigg)\\
&\quad + C\bigg(\frac{1}{\mu(6B)}\int_B \big|I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1^1(z),(b_2-m_{\widetilde{B}}b_2)g_2^2\big)(z)\big|\,d\mu(z)\bigg)\\
&\quad + C\bigg(\frac{1}{\mu(6B)}\int_B \big|I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1^2(z),(b_2-m_{\widetilde{B}}b_2)g_2^1\big)(z)\big|\,d\mu(z)\bigg)
\end{aligned}$$

$$+ C\left(\frac{1}{\mu(6B)}\int_B \left|I_{\alpha,2}\big((b_1 - m_{\widetilde{B}}b_1)g_1^2(z),(b_2 - m_{\widetilde{B}}b_2)g_2^2\big)(z) - H_B\right| d\mu(z)\right)$$
$$=: F_{41} + F_{42} + F_{43} + F_{44}.$$

For $1 < p_i < \infty$, $i = 1, 2$, choose $s_1 = \sqrt{p_1}$, $s_2 = \sqrt{p_2}$, $\frac{1}{\nu} = \frac{1}{s_1} + \frac{1}{s_2} - \alpha$, $\frac{1}{s_1} = \frac{1}{p_1} + \frac{1}{\nu_1}$ and $\frac{1}{s_2} = \frac{1}{p_2} + \frac{1}{\nu_2}$. It follows from Hölder's inequality and Theorem 1.9 that

$$\begin{aligned}
F_{41} &\le C\frac{\mu(B)^{1-1/\nu}}{\mu(6B)}\left\|I_{\alpha,2}\big((b_1 - m_{\widetilde{B}}b_1)g_1^1,(b_2 - m_{\widetilde{B}}b_2)g_2^1\big)\right\|_{L^\nu(\mu)}\\
&\le C\frac{1}{\mu(6B)^{1/\nu}}\left\|(b_1 - m_{\widetilde{B}}b_1)g_1^1\right\|_{L^{s_1}(\mu)}\left\|(b_2 - m_{\widetilde{B}}b_2)g_2^1\right\|_{L^{s_2}(\mu)}\\
&\le C\frac{1}{\mu(6B)^{1/\nu}}\left(\int_{\frac{6}{5}B}|b_1 - m_{\widetilde{B}}b_1|^{\nu_1}\,d\mu(z)\right)^{1/\nu_1}\left(\int_{\frac{6}{5}B}\left|g_1(z)\right|^{p_1}\,d\mu(z)\right)^{1/p_1}\\
&\quad\times\left(\int_{\frac{6}{5}B}|b_2 - m_{\widetilde{B}}b_2|^{\nu_2}\,d\mu(z)\right)^{1/\nu_2}\left(\int_{\frac{6}{5}B}\left|g_2(z)\right|^{p_2}\,d\mu(z)\right)^{1/p_2}\\
&\le C\prod_{i=1}^{2}\left(\frac{\int_{\frac{6}{5}B}|b_i - m_{\widetilde{B}}b_i|^{\nu_i}\,d\mu(z)}{\mu(6B)}\right)^{1/\nu_i}\left(\frac{\int_{\frac{6}{5}B}|g_i(z)|^{p_i}\,d\mu(z)}{\mu(6B)^{1-\alpha p_i/2}}\right)^{1/p_i}\\
&\le C\|b_1\|_*\|b_2\|_*M_{p_1,(5)}^{(\alpha/2)}g_1(x)M_{p_2,(5)}^{(\alpha/2)}g_2(x).
\end{aligned}$$

For F_{42}, it follows from (i) of Definition 1.5, Lemmas 2.2-2.3, the condition of $\lambda(x,al) \ge a^m\lambda(x,l)$, and Hölder's inequality that

$$\begin{aligned}
F_{42} &\le C\frac{1}{\mu(6B)}\int_B\int_X\int_X \frac{|b_1(y_1) - m_{\widetilde{B}}b_1||g_1^1(y_1)|}{[\lambda(z,d(z,y_1)) + \lambda(z,d(z,y_2))]^{2-\alpha}}\\
&\quad\times\left|b_2(y_2) - m_{\widetilde{B}}b_2\right|\left|g_2^2(y_2)\right|d\mu(y_1)\,d\mu(y_2)\,d\mu(z)\\
&\le C\frac{1}{\mu(6B)}\int_B\int_{\frac{6}{5}B}\left|b_1(y_1) - m_{\widetilde{B}}b_1\right|\left|g_1(y_1)\right|d\mu(y_1)\\
&\quad\times\int_{X\setminus\frac{6}{5}B}\frac{|b_2(y_2) - m_{\widetilde{B}}b_2||g_2(y_2)|\,d\mu(y_2)}{[\lambda(z,d(z,y_2))]^{2-\alpha}}\,d\mu(z)\\
&\le C\left(\frac{\int_{\frac{6}{5}B}|b_1(y_1) - m_{\widetilde{B}}b_1|^{p_1'}\,d\mu(y_1)}{\mu(6B)}\right)^{1/p_1'}\left(\frac{\int_{\frac{6}{5}B}|g_1(y_1)|^{p_1}\,d\mu(y_1)}{\mu(6B)^{1-\alpha p_1/2}}\right)^{1/p_1}\\
&\quad\times\mu(6B)^{-\alpha/2}\mu(B)\sum_{i=1}^{\infty}\int_{6^i\frac{6}{5}B\setminus 6^{i-1}\frac{6}{5}B}\frac{|b_2(y_2) - m_{\widetilde{B}}b_2||g_2(y_2)|}{[\lambda(x,6^{i-1}\frac{6}{5}l_B)]^{2-\alpha}}\,d\mu(y_2)\\
&\le C\|b_1\|_*M_{p_1,(5)}^{(\alpha/2)}g_1(x)\sum_{i=1}^{\infty}6^{-km(1-\alpha/2)}\left[\frac{\mu(B)}{\mu(\frac{6}{5}B)}\right]^{1-\alpha/2}\left[\frac{\mu(\frac{6}{5}B)}{\lambda(x,\frac{6}{5}l_B)}\right]^{1-\alpha/2}\\
&\quad\times\frac{1}{[\lambda(x,5\times 6^i\frac{6}{5}l_B)]^{1-\alpha/2}}\int_{6^i\frac{6}{5}B}\left|b_2(y_2) - m_{\widetilde{B}}b_2\right|\left|g_2(y_2)\right|d\mu(y_2)\\
&\le C\|b_1\|_*M_{p_1,(5)}^{(\alpha/2)}g_1(x)\sum_{i=1}^{\infty}6^{-im(1-\alpha/2)}\frac{1}{[\mu(5\times 6^i\frac{6}{5}B)]^{1-\alpha/2}}\\
&\quad\times\int_{6^i\frac{6}{5}B}\left|b_2(y_2) - m_{\widetilde{6^i\frac{6}{5}B}}(b_2) + m_{\widetilde{6^i\frac{6}{5}B}}(b_2) - m_{\widetilde{B}}b_2\right|\left|g_2(y_2)\right|d\mu(y_2)
\end{aligned}$$

$$
\begin{aligned}
&\le C\|b_1\|_* M^{(\alpha/2)}_{p_1,(5)} g_1(x) \sum_{i=1}^{\infty} 6^{-im(1-\alpha/2)} \\
&\quad\times \Bigg[\Bigg(\frac{\int_{6^i\frac{6}{5}B} |b_2(y_2) - m_{\widetilde{6^i\frac{6}{5}B}}(b_2)|^{p_2'}\, d\mu(y_2)}{\mu(5\times 6^i\frac{6}{5}B)} \Bigg)^{1/p_2'} \Bigg(\frac{\int_{6^i\frac{6}{5}B} |g_2(y_2)|^{p_2}\, d\mu(y_2)}{\mu(5\times 6^i\frac{6}{5}B)^{1-\alpha p_2/2}} \Bigg)^{1/p_2} \\
&\quad + Ck\|b_2\|_* \Bigg(\frac{\int_{6^k\frac{6}{5}B} |g_2(y_2)|^{p_2}\, d\mu(y_2)}{\mu(5\times 6^i\frac{6}{5}B)^{1-\alpha p_2/2}} \Bigg)^{1/p_2} \Bigg(\frac{\int_{6^i\frac{6}{5}B} d\mu(y_2)}{\mu(5\times 6^i\frac{6}{5}B)} \Bigg)^{1/p_2'} \Bigg] \\
&\le C\|b_1\|_*\|b_2\|_* M^{(\alpha/2)}_{p_1,(5)} g_1(x) M^{(\alpha/2)}_{p_2,(5)} g_2(x).
\end{aligned}
$$

In the same way, one obtains

$$
F_{43} \le C\|b_1\|_*\|b_2\|_* M^{(\alpha/2)}_{p_1,(5)} g_1(x) M^{(\alpha/2)}_{p_2,(5)} g_2(x).
$$

For F_{44}, let $z, z_0 \in B$, it follows from (ii) of Definition 1.5, Lemmas 2.2-2.3, the condition of λ, and Hölder's inequality that

$$
\begin{aligned}
&\big|I_{\alpha,2}\big((b_1 - m_{\widetilde{B}}b_1)g_2^2, (b_2 - m_{\widetilde{B}}b_2)g_2^2\big)(z) \\
&\qquad - I_{\alpha,2}\big((b_1 - m_{\widetilde{B}}b_1)g_2^2, (b_2 - m_{\widetilde{B}}b_2)g_2^2\big)(z_0)\big| \\
&\le C\int_{X\setminus\frac{6}{5}B}\int_{X\setminus\frac{6}{5}B} \big|K(z,y_1,y_2) - K(z_0,y_1,y_2)\big| \\
&\qquad\times \prod_{j=1}^{2} \big|\big(b_j(y_j) - m_{\widetilde{B}}b_j\big)g_j(y_j)\big|\, d\mu(y_j) \\
&\le C\int_{X\setminus\frac{6}{5}B}\int_{X\setminus\frac{6}{5}B} \frac{d(z,z_0)^{\delta}\prod_{j=1}^{2}|(b_j(y_j) - m_{\widetilde{B}}b_j)g_j(y_j)|\, d\mu(y_j)}{(d(z,y_1)+d(z,y_2))^{\delta}[\sum_{j=1}^{2}\lambda(x,d(x,y_j))]^{2-\alpha}} \\
&\le C\prod_{j=1}^{2}\int_{X\setminus\frac{6}{5}B} \frac{d(z,z_0)^{\delta_i}|b_j(y_j) - m_{\widetilde{B}}b_j||g_j(y_j)|\, d\mu(y_j)}{d(z,y_j)^{\delta_j}[\lambda(z,d(z,y_j))]^{1-\alpha/2}} \\
&\le C\prod_{j=1}^{2}\sum_{k=1}^{\infty}\int_{6^k\frac{6}{5}B\setminus 6^{k-1}\frac{6}{5}B} 6^{-k\delta_j}\Bigg[\frac{\mu(5\times 6^k\frac{6}{5}B)}{\lambda(z,5\times 6^k\frac{6}{5}l_B)}\Bigg]^{1-\alpha/2} \\
&\qquad\times \frac{|b_j(y_j) - m_{\widetilde{B}}b_j||g_j(y_j)|\, d\mu(y_j)}{[\mu(5\times 6^k\frac{6}{5}B)]^{1-\alpha/2}} \\
&\le C\prod_{j=1}^{2}\sum_{k=1}^{\infty} 6^{-k\delta_j}\Bigg(\frac{\int_{6^k\frac{6}{5}B}|b_j(y_j) - m_{\widetilde{B}}b_j|^{p_j'}\, d\mu(y_j)}{[\mu(5\times 6^k\frac{6}{5}B)]^{1-\alpha p_j/2}}\Bigg)^{1/p_j'} \\
&\qquad\times \Bigg(\frac{\int_{6^k\frac{6}{5}B}|g_j(y_j)|^{p_j}\, d\mu(y_j)}{\mu(5\times 6^k\frac{6}{5}B)}\Bigg)^{1/p_i} \\
&\le C\prod_{j=1}^{2}\sum_{k=1}^{\infty} 6^{-k\delta_j} M_{p_j,(6)}g_j(x)\Bigg(\frac{1}{[\mu(5\times 6^k\frac{6}{5}B)]^{1-\alpha p_j/2}} \\
&\qquad\times \int_{6^k\frac{6}{5}B}\big|b_j(y_j) - m_{\widetilde{6^k\frac{6}{5}B}} + m_{\widetilde{6^k\frac{6}{5}B}} - m_{\widetilde{B}}b_j\big|^{p_j'}\, d\mu(y_j)\Bigg)^{1/p_j'}
\end{aligned}
$$

$$\begin{aligned}&\le C\prod_{j=1}^{2}\sum_{k=1}^{\infty}6^{-k\delta_j}k\|b_j\|_*M_{p_j,(6)}g_j(x)\\&\le C\|b_1\|_*\|b_2\|_*M^{(\alpha/2)}_{p_1,(6)}g_1(x)M^{(\alpha/2)}_{p_2,(6)}g_2(x),\end{aligned}$$

where $\delta=\delta_1+\delta_2$, $\delta_1,\delta_2>0$.

It follows from taking the mean over $z_0\in B$ that

$$F_{44}\le C\|b_1\|_*\|b_2\|_*M^{(\alpha/2)}_{p_1,(6)}g_1(x)M^{(\alpha/2)}_{p_2,(6)}g_2(x). \tag{2.8}$$

Thus (2.5) is obtained from (2.7) to (2.8).

Now we turn to the proof of (2.6). Set $N=N_{B,R}+1$. For two balls $B\subset R$ with $x\in B$, here R is a doubling ball and B is an every ball,

$$\begin{aligned}&\big|\big|m_B\big[I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1^2,(b_2-m_{\widetilde{B}}b_2)g_2^2\big)\big]\big|-\big|m_R\big[I_{\alpha,2}\big((b_1-m_Rb_1)g_1^2,(b_2-m_Rb_2)g_2^2\big)\big]\big|\big|\\&\quad\le\big|m_B\big[I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1\chi_{X\setminus6^NB},(b_2-m_{\widetilde{B}}b_2)g_2\chi_{X\setminus6^NB}\big)\big]\\&\qquad-m_R\big[I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1\chi_{X\setminus6^NB},(b_2-m_{\widetilde{B}}b_2)g_2\chi_{X\setminus6^NB}\big)\big]\big|\\&\qquad+\big|m_R\big[I_{\alpha,2}\big((b_1-m_Rb_1)g_1\chi_{X\setminus6^NB},(b_2-m_Rb_2)g_2\chi_{X\setminus6^NB}\big)\big]\\&\qquad-m_R\big[I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1\chi_{X\setminus6^NB},(b_2-m_{\widetilde{B}}b_2)g_2\chi_{X\setminus6^NB}\big)\big]\big|\\&\qquad+\big|m_B\big[I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1\chi_{6^NB\setminus\frac{6}{5}B},(b_2-m_{\widetilde{B}}b_2)g_2\chi_{X\setminus\frac{6}{5}B}\big)\big]\big|\\&\qquad+\big|m_B\big[I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1\chi_{X\setminus\frac{6}{5}B},(b_2-m_{\widetilde{B}}b_2)g_2\chi_{6^NB\setminus\frac{6}{5}B}\big)\big]\big|\\&\qquad+\big|m_R\big[I_{\alpha,2}\big((b_1-m_Rb_1)g_1\chi_{6^NB\setminus\frac{6}{5}R},(b_2-m_Rb_2)g_2\chi_{X\setminus6^NB}\big)\big]\big|\\&\qquad+\big|m_R\big[I_{\alpha,2}\big((b_1-m_Rb_1)g_1\chi_{X\setminus\frac{6}{5}R},(b_2-m_Rb_2)g_2\chi_{6^NB\setminus\frac{6}{5}R}\big)\big]\big|\\&\quad=:G_1+G_2+G_3+G_4+G_5+G_6.\end{aligned}\tag{2.9}$$

Similar to the estimate of F_{44},

$$G_1\le C\|b_1\|_*\|b_2\|_*M^{(\alpha/2)}_{p_1,(6)}g_1(x)M^{(\alpha/2)}_{p_2,(6)}g_2(x).$$

For G_2, it is easy to see that

$$\begin{aligned}&I_{\alpha,2}\big((b_1-m_Rb_1)g_1\chi_{X\setminus6^NB},(b_2-m_Rb_2)g_2\chi_{X\setminus6^NB}\big)(z)\\&\qquad-I_{\alpha,2}\big((b_1-m_{\widetilde{B}}b_1)g_1\chi_{X\setminus6^NB},(b_2-m_{\widetilde{B}}b_2)g_2\chi_{X\setminus6^NB}\big)(z)\\&\quad=(m_Rb_2-m_{\widetilde{B}}b_2)I_{\alpha,2}\big((b_1-m_Rb_1)g_1\chi_{X\setminus6^NB},g_2\chi_{X\setminus6^NB}\big)(z)\\&\qquad+(m_Rb_1-m_{\widetilde{B}}b_1)I_{\alpha,2}\big(g_1\chi_{X\setminus6^NB},(b_2-m_Rb_2)g_2\chi_{X\setminus6^NB}\big)(z)\\&\qquad+(m_Rb_1-m_{\widetilde{B}}b_1)(m_Rb_2-m_{\widetilde{B}}b_2)I_{\alpha,2}(g_1\chi_{X\setminus6^NB},g_2\chi_{X\setminus6^NB})(z).\end{aligned}$$

Thus

$$\begin{aligned}G_2\le&\left|(m_Rb_2-m_{\widetilde{B}}b_2)\frac{1}{\mu(R)}\int_R I_{\alpha,2}\big((b_1-m_Rb_1)g_1\chi_{X\setminus6^NB},g_2\chi_{X\setminus6^NB}\big)(z)\,d\mu(z)\right|\\&+\left|(m_Rb_1-m_{\widetilde{B}}b_1)\frac{1}{\mu(R)}\int_R I_{\alpha,2}\big(g_1\chi_{X\setminus6^NB},(b_2-m_Rb_2)g_2\chi_{X\setminus6^NB}\big)(z)\,d\mu(z)\right|\end{aligned}$$

$$+\left|(m_R b_1 - m_{\widetilde{B}} b_1)(m_R b_2 - m_{\widetilde{B}} b_2)\frac{1}{\mu(R)}\int_R I_{\alpha,2}(g_1\chi_{X\setminus 6^{N}B}, g_2\chi_{X\setminus 6^{N}B})(z)\,d\mu(z)\right|$$
$$=: G_{21} + G_{22} + G_{23}.$$

For G_{21},

$$\begin{aligned}
&I_{\alpha,2}\big((b_1 - m_R b_1)g_1\chi_{X\setminus 6^{N}R}, g_2\chi_{X\setminus 6^{N}R}\big)(z)\\
&\quad= I_{\alpha,2}\big((b_1 - m_R b_1)g_1, g_2\big)(z) - T\big((b_1 - m_R b_1)g_1\chi_{6^{N}B}\chi_{\frac{6}{5}R}, g_2\chi_{\frac{6}{5}R}\big)(z)\\
&\qquad- I_{\alpha,2}\big((b_1 - m_R b_1)g_1\chi_{\frac{6}{5}R}, g_2\chi_{6^{N}B}\chi_{\frac{6}{5}R}\big)(z)\\
&\qquad+ I_{\alpha,2}\big((b_1 - m_R b_1)g_1\chi_{6^{N}B}\chi_{\frac{6}{5}R}, g_2\chi_{6^{N}B}\chi_{\frac{6}{5}R}\big)(z)\\
&\qquad- I_{\alpha,2}\big((b_1 - m_R b_1)g_1\chi_{X\setminus\frac{6}{5}R}, g_2\chi_{6^{N}B}\big)(z)\\
&\qquad- I_{\alpha,2}\big((b_1 - m_R b_1)g_1\chi_{6^{N}B}, g_2\chi_{X\setminus\frac{6}{5}R}\big)(z)\\
&\qquad+ I_{\alpha,2}\big((b_1 - m_R b_1)g_1\chi_{6^{N}B\setminus\frac{6}{5}R}, g_2\chi_{6^{N}B\setminus\frac{6}{5}R}\big)(z)\\
&\quad=: E_1(z) + E_2(z) + E_3(z) + E_4(z) + E_5(z) + E_6(z) + E_7(z).
\end{aligned}$$

For $E_1(z)$, it is easy to see that

$$\frac{1}{\mu(R)}\int_R \big|I_{\alpha,2}\big(b_1 - b_1(z)g_1, g_2\big)(z)\big|\,d\mu(z) \le CM_{r,(6)}\big([b_1, I_{\alpha,2}]g_1, g_2\big)(x).$$

It follows from Hölder's inequality that

$$\frac{1}{\mu(R)}\int_R \big|\big(b_1(z) - m_R(b_1)\big)I_{\alpha,2}(g_1, g_2)(z)\big|\,d\mu(z) \le C\|b_1\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1, g_2)\big)(x).$$

Therefore

$$\begin{aligned}
\big|m_R(E_1)\big| &\le \big|m_R\big(I_{\alpha,2}\big(b_1 - b_1(z)g_1, g_2\big)\big)\big| + \big|m_R\big(\big(b_1(z) - m_R(b_1)\big)I_{\alpha,2}(g_1, g_2)\big)\big|\\
&\le C\big\{M_{r,(6)}\big([b_1, I_{\alpha,2}]g_1, g_2\big)(x) + \|b_1\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1, g_2)\big)(x)\big\}.
\end{aligned}$$

For $E_2(z)$, denote $s_1 = \sqrt{p_1}$, $s_2 = p_2$, $\frac{1}{\nu} = \frac{1}{s_1} + \frac{1}{s_2} - \alpha$ and $\frac{1}{s_1} = \frac{1}{p_1} + \frac{1}{\nu_1}$. Noting that R is a doubling ball, by Theorem 1.9, one obtains

$$\begin{aligned}
\big|m_R(E_2)\big| &\le C\frac{\mu(R)^{1-1/\nu}}{\mu(6R)}\big\|I_{\alpha,2}\big((b_1 - m_R b_1)g_1\chi_{6^{N}B}\chi_{6/5R}, g_2\chi_{6/5R}\big)\big\|_{L^{\nu}(\mu)}\\
&\le C\mu(6R)^{-1/\nu}\big\|(b_1 - m_R b_1)g_1\chi_{6^{N}B}\chi_{6/5R}\big\|_{L^{s_1}(\mu)}\|g_2\chi_{6/5R}\|_{L^{s_2}(\mu)}\\
&\le C\frac{1}{\mu(6R)^{1/\nu}}\left(\int_{\frac{6}{5}R}|b_1 - m_R b_1|^{\nu_1}\,d\mu(z)\right)^{1/\nu_1}\left(\int_{\frac{6}{5}R}\big|g_1(z)\big|^{p_1}\,d\mu(z)\right)^{1/p_1}\\
&\quad\times\left(\int_{\frac{6}{5}R}\big|g_2(z)\big|^{p_2}\,d\mu(z)\right)^{1/p_2}\\
&\le C\left(\frac{1}{\mu(6R)}\int_{\frac{6}{5}R}|b_1 - m_R b_1|^{\nu_1}\,d\mu(z)\right)^{1/\nu_1}
\end{aligned}$$

$$\times \prod_{j=1}^{2}\Bigg(\frac{1}{\mu(6R)^{1-\alpha p_j/2}}\int_{\frac{6}{5}R}\big|g_j(z)\big|^{p_j}\,d\mu(z)\Bigg)^{1/p_j}$$
$$\le C\|b_1\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)M^{(\alpha/2)}_{p_2,(5)}g_2(x).$$

Also one deduces

$$\big|m_R(E_3)\big| + \big|m_R(E_4)\big| \le C\|b_1\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)M^{(\alpha/2)}_{p_2,(5)}g_2(x).$$

For E_5, as $z \in R$, noting that R is a doubling ball, it follows from (i) of Definition 1.5, Lemmas 2.2-2.3, and the conditions of λ that

$$\begin{aligned}
\big|E_5(z)\big| &\le C\int_{6^N B}\int_{X\setminus\frac{6}{5}R}\frac{|b_1(y_1)-m_R b_1||g_1(y_1)||g_2(y_2)|\,d\mu(y_1)\,d\mu(y_2)}{[\sum_{i=1}^{2}\lambda(x,d(x,y_i))]^{2-\alpha}}\\
&\le C\int_{6^N B}\big|g_2(y_2)\big|\,d\mu(y_2)\sum_{j=1}^{\infty}\int_{6^j\frac{6}{5}R\setminus 6^{j-1}\frac{6}{5}R}\frac{|b_1(y_1)-m_R b_1||g_1(y_1)|}{(\lambda(z,5\times 6^j\frac{6}{5}l_R))^{2-\alpha}}\,d\mu(y_1)\\
&\le C\int_{6^N B}\big|g_2(y_2)\big|\,d\mu(y_2)\sum_{j=1}^{\infty}6^{-jm(1-\alpha/2)}\\
&\quad\times\int_{6^j\frac{6}{5}R}\frac{1}{[\lambda(z,5\times\frac{6}{5}l_R)]^{1-\alpha/2}}\frac{|b_1(y_1)-m_R b_1||g_1(y_1)|\,d\mu(y_1)}{[\lambda(z,5\times 6^j\frac{6}{5}l_R)]^{1-\alpha/2}}\\
&\le C\frac{1}{[\lambda(z,6l_R)]^{1-\alpha/2}}\int_{6^N B}\big|g_2(y_2)\big|\,d\mu(y_2)\sum_{j=1}^{\infty}6^{-jm(1-\alpha/2)}\\
&\quad\times\frac{1}{[\lambda(z,5\times 6^j\frac{6}{5}l_R)]^{1-\alpha/2}}\times\Bigg[\int_{6^j\frac{6}{5}R}\big|b_1(y_1)-m_{6^j\frac{6}{5}R}(b_1)\big|\big|g_1(y_1)\big|\,d\mu(y_1)\\
&\quad+\int_{6^j\frac{6}{5}R}\big|m_{6^j\frac{6}{5}R}(b_1)-m_R b_1\big|\big|g_1(y_1)\big|\,d\mu(y_1)\Bigg]\\
&\le C\frac{1}{[\lambda(z,6l_R)]^{1-\alpha/2}}\int_{6^N B}\big|g_2(y_2)\big|\,d\mu(y_2)\sum_{j=1}^{\infty}6^{-jm(1-\alpha/2)}\\
&\quad\times\Bigg[\Bigg(\frac{1}{\lambda(z,5\times 6^j\frac{6}{5}l_R)}\int_{6^j\frac{6}{5}R}\big|b_1(y_1)-m_{6^j\frac{6}{5}R}(b_1)\big|^{p_1'}\,d\mu(y_1)\Bigg)^{1/p_1'}\\
&\quad\times\Bigg(\frac{1}{[\lambda(z,6^{j+1}\frac{6}{5}l_R)]^{1-\alpha p_1/2}}\int_{6^j\frac{6}{5}R}\big|g_1(y_1)\big|^{p_1}\,d\mu(y_1)\Bigg)^{1/p_1}\\
&\quad+j\|b_1\|_*\frac{1}{[\lambda(z,5\times 6^j\frac{6}{5}l_R)]^{1-\alpha/2}}\int_{6^j\frac{6}{5}R}\big|g_1(y_1)\big|\,d\mu(y_1)\Bigg]\\
&\le C\|b_1\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)\sum_{j=1}^{N_{B,R}}\frac{1}{[\lambda(z,6l_R)]^{1-\alpha/2}}\\
&\quad\times\Bigg[\int_{6^{j+1}B\setminus 6^j B}\big|g_2(y_2)\big|\,d\mu(y_2)+\int_{6B}\big|g_2(y_2)\big|\,d\mu(y_2)\Bigg]\\
&\le C\|b_1\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)\sum_{j=1}^{N_{B,R}}\frac{\int_{6^{j+1}B}|g_2(y_2)|\,d\mu(y_2)}{[\mu(5\times 6^{j+1}B)]^{1-\alpha/2}}
\end{aligned}$$

$$\times \left[\frac{\mu(5\times 6^{j+1}B)}{\lambda(z,5\times 6^{j+1}l_B)}\right]^{1-\alpha/2}\left[\frac{\lambda(z,5\times 6^{j}l_B)}{\lambda(z,6l_R)}\right]^{1-\alpha/2}$$
$$+ C\|b_1\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)\frac{1}{[\lambda(z,6l_R)]^{1-\alpha/2}}\int_{6B}\left|g_2(y_2)\right|d\mu(y_2)$$
$$\le CK^{(\alpha/2)}_{B,R}\|b_1\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)M^{(\alpha/2)}_{p_2,(5)}g_2(x).$$

Therefore

$$|m_R(E_5)| \le CK^{(\alpha/2)}_{B,R}\|b_1\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)M^{(\alpha/2)}_{p_2,(5)}g_2(x).$$

Similar to the estimate of $m_R(E_5)$,

$$\left|m_R(E_6)\right| + \left|m_R(E_7)\right| \le CK^{(\alpha/2)}_{B,R}\|b_1\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)M^{(\alpha/2)}_{p_2,(5)}g_2(x).$$

By (1.10), it yields

$$G_{21} \le C\big[\|b_1\|_*\|b_2\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)(x) + \|b_1\|_* M_{r,(6)}\big([b_2,I_{\alpha,2}](g_1,g_2)\big)(x)$$
$$+ \|b_2\|_* M_{r,(6)}\big([b_1,I_{\alpha,2}](g_1,g_2)\big)(x) + \|b_1\|_*\|b_2\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)M^{(\alpha/2)}_{p_2,(5)}g_2(x)\big].$$

For G_{22} and G_{23}, they are similar to G_{21}, thus

$$G_2 \le C\big[\|b_1\|_*\|b_2\|_* M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)(x) + \|b_1\|_* M_{r,(6)}\big([b_2,I_{\alpha,2}](g_1,g_2)\big)(x)$$
$$+ \|b_2\|_* M_{r,(6)}\big([b_1,I_{\alpha,2}](g_1,g_2)\big)(x) + \|b_1\|_*\|b_2\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)M^{(\alpha/2)}_{p_2,(5)}g_2(x)\big].$$

For G_3 to G_6, in the same way as $E_5(z)$, it yields

$$G_3 + G_4 + G_5 + G_6 \le C\|b_1\|_*\|b_2\|_* M^{(\alpha/2)}_{p_1,(5)}g_1(x)M^{(\alpha/2)}_{p_2,(5)}g_2(x). \tag{2.10}$$

Thus (2.6) is obtained from (2.9) to (2.10) and (2.2) is proved. Also, one obtains (2.3) and (2.4) in a similar way to (2.2). Here the details is omitted. Thus Lemma 2.4 is proved. □

Proof of Theorem 1.10 Set $0<\alpha<2$, $1<p_1,p_2<\infty$, $0<\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\alpha<1$, $1<r<q$, $f_1\in L^{p_1}(\mu)$, $g_2\in L^{p_2}(\mu)$, $b_1,b_2\in \mathrm{RBMO}(\mu)$. Noticing that $|g(x)|\le Ng(x)$, recalling the boundedness of $M^{(\alpha/2)}_{r,(\rho)}$ and $M_{r,(\rho)}$, for $\rho\ge 5$ and $r<q$, and using Hölder's inequality, it follows from Lemmas 2.1-2.4 and Theorem 1.9 that

$$\begin{aligned}
&\big\|[b_1,b_2,I_{\alpha,2}](g_1,g_2)\big\|_{L^q(\mu)}\\
&\quad\le \big\|N\big([b_1,b_2,I_{\alpha,2}](g_1,g_2)\big)\big\|_{L^q(\mu)}\\
&\quad\le C\big\|M^{\sharp,(\alpha/2)}\big([b_1,b_2,I_{\alpha,2}](g_1,g_2)\big)\big\|_{L^q(\mu)}\\
&\quad\le C\|b_1\|_*\|b_2\|_*\big\|M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)\big\|_{L^q(\mu)}\\
&\qquad + C\|b_1\|_*\big\|M_{r,(6)}\big([b_2,I_{\alpha,2}](g_1,g_2)\big)\big\|_{L^q(\mu)}\\
&\qquad + C\|b_2\|_*\big\|M_{r,(6)}\big([b_1,I_{\alpha,2}](g_1,g_2)\big)\big\|_{L^q(\mu)}
\end{aligned}$$

$$
\begin{aligned}
&+ C\|b_1\|_*\|b_2\|_*\big\|M^{(\alpha/2)}_{p_1,(5)}g_1M^{(\alpha/2)}_{p_2,(5)}g_2\big\|_{L^q(\mu)}\\
&\le C\|b_1\|_*\|b_2\|_*\|g_1\|_{L^{p_1}(\mu)}\|g_2\|_{L^{p_2}(\mu)}\\
&\quad + C\|b_1\|_*\big\|[b_2,I_{\alpha,2}](g_1,g_2)\big\|_{L^q(\mu)}\\
&\quad + C\|b_2\|_*\big\|[b_1,I_{\alpha,2}](g_1,g_2)\big\|_{L^q(\mu)}\\
&\le C\|b_1\|_*\|b_2\|_*\|g_1\|_{L^{p_1}(\mu)}\|g_2\|_{L^{p_2}(\mu)}\\
&\quad + C\|b_1\|_*\big\|M^{\sharp,(\alpha/2)}\big([b_2,I_{\alpha,2}](g_1,g_2)\big)\big\|_{L^q(\mu)}\\
&\quad + C\|b_2\|_*\big\|M^{\sharp,(\alpha/2)}\big([b_1,I_{\alpha,2}](g_1,g_2)\big)\big\|_{L^q(\mu)}\\
&\le \|b_1\|_*\|b_2\|_*\|g_1\|_{L^{p_1}(\mu)}\|g_2\|_{L^{p_2}(\mu)}\\
&\quad + C\|b_1\|_*\big\|M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)\big\|_{L^q(\mu)}\\
&\quad + C\|b_1\|_*\big\|M^{(\alpha/2)}_{p_1,(5)}g_1M^{(\alpha/2)}_{p_2,(5)}g_2\big\|_{L^q(\mu)}\\
&\quad + C\|b_2\|_*\big\|M_{r,(6)}\big(I_{\alpha,2}(g_1,g_2)\big)\big\|_{L^q(\mu)}\\
&\quad + C\|b_2\|_*\big\|M^{(\alpha/2)}_{p_1,(5)}g_1M^{(\alpha/2)}_{p_2,(5)}g_2\big\|_{L^q(\mu)}\\
&\le C\|b_1\|_*\|b_2\|_*\|g_1\|_{L^{p_1}(\mu)}\|g_2\|_{L^{p_2}(\mu)}.
\end{aligned}
$$

This proves Theorem 1.10. □

3 Applications

In this section, we apply Theorem 1.9 and Theorem 1.10 to the study of a fractional integral operator.

Lemma 3.1 ([15]) *Suppose* $\operatorname{diam}(X) = \infty$, $\alpha \in (0,1)$, $p \in (1, 1/\alpha)$, *and* $1/q = 1/p - \alpha$. *If* λ *satisfies the* ϵ*-weak reverse doubling condition, for some* $\epsilon \in (0, \min\{\alpha, 1-\alpha, 1/q\})$, *then*

$$\|T_\alpha f\|_{L^q(\mu)} \le C\|f\|_{L^p(\mu)},$$

where T_α *is defined by*

$$T_\alpha f(x) := \int_X \frac{f(y)}{[\lambda(y, d(x,y))]^{1-\alpha}}\, d\mu(y).$$

Theorem 3.2 *Under the same conditions as that in Lemma* 3.1, *the results of Theorem* 1.9 *and Theorem* 1.10 *hold true, on replacing* I_β *there by* T_α.

4 Conclusion

In this paper, we prove that multilinear fractional integral operators and commutators, generated by multilinear fractional integrals, with an RBMO(μ) function on non-homogeneous metric measure spaces, are bounded in Lebesgue spaces. The results are established for both the homogeneous spaces and the non-doubling measure spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HG and RX proposed this problem and finished the proof of Theorem 1.9 and Theorem 1.10 together. CX finished the proof of Lemma 2.4. All authors read and approved the final manuscript.

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