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Some remarks on Cîrtoaje's conjecture

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Abstract
In this paper, we give new conditions under which the Cîrtoaje's conjecture is also valid. We also show that a certain generalization of the Cîrtoaje's inequality fulfils an interesting property.

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1 Introduction and preliminaries

The study of inequalities with power-exponential functions is one of the active areas of research in the mathematical analysis. The power-exponential functions have useful applications in mathematical analysis and in other theories like statistics, biology, optimization, ordinary differential equations, and probability [1]. We note that the formulas of inequalities with power-exponential functions look so simple, but their solutions are not as simple as it seems. A lot of interesting results for inequalities with power-exponential functions have been obtained. The history and the literature review of inequalities with power-exponential functions can be found, for example, in [1]. Some other interesting problems concerning inequalities of power-exponential functions can be found in [2]. In this paper, we are studying one inequality conjectured by Cîrtoaje [3]. Cîrtoaje [3] has posted the following conjecture on the inequalities with power-exponential functions.

Conjecture 1.1 *If $a, b \in (0;1]$ and $r \in [0;e]$, then*

$$2\sqrt{a^{ra}b^{rb}} \geq a^{rb} + b^{ra}. \tag{1.1}$$

The conjecture was proved by Matejíčka [4]. Matejíčka [5] also proved (1.1) under other conditions. Now we prove that the conjecture (1.1) is also valid under the following conditions:

$\frac{2}{e} \leq \min\{a,b\} \leq 1$ and $1 \leq \max\{a,b\} \leq e$ for $r \in [0;e]$;

$1 \leq \min\{a,b\} \leq \max\{a,b\} \leq e$ for $r \in [0;e]$.

We also show that a certain generalization of Cîrtoaje's inequality fulfils an interesting property with some applications.

2 Main results

Theorem 2.1 *Let a, b be positive numbers. Then*

$$2\sqrt{a^{ra}b^{rb}} \geq a^{rb} + b^{ra} \tag{2.1}$$

for any $r \in [0, e]$ if one of the following two conditions is satisfied:

$$\frac{2}{e} \leq b \leq 1 \leq a \leq e; \tag{2.2}$$

$$1 \leq b \leq a \leq e. \tag{2.3}$$

Proof According to the proof of the Theorem 2.1 in [5], it suffices to consider the case where $r = e$.

We split the proof into two parts, labeled as (a) and (b) with valid (2.2) and (2.3), respectively.

(a) Let a and b satisfy (2.2). Denote

$$H(x) = 2\sqrt{x^{ex}b^{eb}} - x^{eb} - b^{ex}$$

for $x \in [1, e]$. We have

$$H'(x) = e\bigl(x^{\frac{ex}{2}}b^{\frac{eb}{2}}(\ln x + 1) - bx^{eb-1} - b^{ex}\ln b\bigr) = eb^{ex}F(x),$$

where

$$F(x) = e^{\frac{e}{2}(x\ln x + b\ln b - 2x\ln b)}(1 + \ln x) - be^{(eb-1)\ln x - ex\ln b} - \ln b$$

and

$$F'(x) = e^{\frac{e}{2}(x\ln x + b\ln b - 2x\ln b)}\left(\frac{e}{2}(1 + \ln x)(1 + \ln x - 2\ln b) + \frac{1}{x}\right) - be^{(eb-1)\ln x - ex\ln b}\left(\frac{eb-1}{x} - e\ln b\right).$$

If we show that $H(1) \geq 0$ and $H'(x) \geq 0$ for $x \in [1, e]$, then the proof will be done.

To prove that $H(1) \geq 0$, we consider the function $s : [2/e, 1] \to \mathbf{R}$ defined as

$$s(b) = H(1) = 2b^{\frac{eb}{2}} - 1 - b^{e}.$$

We have that $s(1) = 0$. Now, if we show that $s'(b) \leq 0$ for $b \in [2/e, 1]$, then we can conclude that $H(1) \geq 0$. From

$$s'(b) = e^{\frac{eb}{2}\ln b + 1}(\ln b + 1) - eb^{e-1}$$

we obtain that $s'(b) \leq 0$ is equivalent to

$$\frac{eb\ln b}{2} - (e-1)\ln b + \ln(1 + \ln b) \leq 0 \quad \text{for } \frac{2}{e} \leq b \leq 1.$$

Using

$$\ln(1+\ln b) \le \ln b - \frac{\ln^2 b}{2},$$

it suffices to show that

$$v(b) = \frac{eb}{2} + 2 - e - \frac{\ln b}{2} \ge 0.$$

But the latter follows from $v(2/e) = 3 - e + \frac{1}{2}\ln(\frac{e}{2}) > 0$ and from $v'(b) = (eb-1)/(2b) > 0$.

Now we show $F'(x) \ge 0$ for $x \in [1,e]$. This implies $H'(x) \ge 0$ for $x \in [1,e]$. Indeed, if we show that $F(1) \ge 0$ and $F'(x) \ge 0$, then $F(x) \ge 0$, so that $H'(x) \ge 0$.

We have

$$F(1) = e^{\frac{eb}{2}\ln b - e\ln b} - be^{-e\ln b} - \ln b.$$

Because of $(\frac{eb}{2} - e) < -\frac{e}{2}$, it suffices to show that

$$y(b) = b^{-\frac{e}{2}} - b^{1-e} - \ln b \ge 0$$

for $\frac{2}{e} \le b \le 1$.

Since $y(1) = 0$, it suffices to show that

$$y' = -\frac{e}{2}b^{-\frac{e}{2}-1} - (1-e)b^{-e} - \frac{1}{b} \le 0.$$

This is equivalent to

$$g = -\frac{e}{2}b^{-\frac{e}{2}} + (e-1)b^{1-e} \le 1,$$

which follows from $g(2/e) = 0.8488$ and from

$$g' = \frac{e^2}{4}b^{-\frac{e}{2}-1} - (e-1)^2 b^{-e} \le 0.$$

Indeed, $g' < 0$ follows from $b \le 1 < e/2$.

Next, we have that $F'(x) \ge 0$ is equivalent to

$$e^{\frac{e}{2}(x\ln x + b\ln b - 2x\ln b)} \ge \frac{be^{(eb-1)\ln x - ex\ln b}(eb-1-ex\ln b)}{\frac{ex}{2}(\ln x+1)(\ln x+1-2\ln b)+1}. \tag{2.4}$$

This can be rewritten as

$$\begin{aligned}&\frac{e}{2}(x\ln x + b\ln b) - (eb-1)\ln x \\ &\quad \ge \ln\bigl(2b(eb-1-ex\ln b)\bigr) - \ln\bigl(ex(\ln x+1)(\ln x+1-2\ln b)+2\bigr). \end{aligned} \tag{2.5}$$

Evidently,

$$ex(\ln x+1)(\ln x+1-2\ln b)+2 \ge ex+2.$$

So, to prove $F'(x) \geq 0$, it suffices to show that

$$\frac{e}{2}(x\ln x + b\ln b) - (eb-1)\ln x \\ \geq \ln\bigl(2b(eb-1-ex\ln b)\bigr) - \ln(ex+2).$$

Using $\ln b > (b-1)/b$, we obtain

$$2b(eb-1-ex\ln b) < 2eb^2 - 2b + 2ex(1-b).$$

So we need to show that

$$\frac{e}{2}(x\ln x + b\ln b) - (eb-1)\ln x \\ \geq \ln\bigl(2eb^2 - 2b + 2ex(1-b)\bigr) - \ln(ex+2).$$

Using again $\ln b > (b-1)/b$ and $\ln x > (x-1)/x$, it suffices to show that

$$r(x) = \frac{e}{2}(x+b-2) - (eb-1)\ln x \\ - \ln\bigl(2eb^2 - 2b + 2ex(1-b)\bigr) + \ln(ex+2) \geq 0.$$

Because of $\ln x < x - 1$, it suffices to prove that

$$r^*(x) = \frac{e}{2}(x+b-2) - (eb-1)(x-1) \\ - \ln\bigl(2eb^2 - 2b + 2ex(1-b)\bigr) + \ln(ex+2) \geq 0.$$

It will be done if we show that $r^{*\prime\prime}(x) \leq 0$, $r^*(1) \geq 0$, and $r^*(e) \geq 0$.

We have

$$r^{*\prime\prime}(x) = \frac{4e^2(1-b)^2}{(2eb^2 - 2b + 2ex(1-b))^2} - \frac{e^2}{(ex+2)^2}.$$

Because of $r^{*\prime\prime}(x) = 0$ only for one real root $x_1 = (eb^2 - 3b + 2)/(2be - 2e) < 0$, we obtain $r^{*\prime\prime}(x) \leq 0$ for $2/e \leq b < 1$ and $1 \leq x \leq e$.

Now we show that $r^*(e) \geq 0$. We have

$$r^*(e) = u(b) \\ = \frac{e^2}{2} - 1 + b\left(\frac{3}{2}e - e^2\right) - \ln\bigl(2e^2 - 2b\bigl(1+e^2\bigr) + 2eb^2\bigr) + \ln\bigl(e^2+2\bigr) \geq 0.$$

First, we show that $u(1) \geq 0$ and then $u'(b) < 0$. We have

$$u(1) = \frac{e^2}{2} - 1 + \frac{3}{2}e - e^2 - \ln(2e-2) + \ln\bigl(e^2+2\bigr) \doteq 0.388 > 0.$$

Since

$$u'(b) = \frac{3e}{2} - e^2 - \frac{4eb - 2 - 2e^2}{2e^2 - 2b(1+e^2) + 2eb^2},$$

we obtain that $u'(b) < 0$ is equivalent to

$$k(b) = 2 + 2e^2 + 3e^3 - 2e^4 - 2b\big(1 + e^2\big)\left(\frac{3e}{2} - e^2\right) - 4eb + 2e\left(\frac{3e}{2} - e^2\right)b^2 \leq 0.$$

It is evident that $k(b)$ is a concave function. We show that $k' = 0$ only for $m > 1$ and $k(1) < 0$. This implies that $k(b) < 0$ for $2/e \leq b \leq 1$. So $u'(b) < 0$. Indeed, if $k' = 0$, then

$$m = \frac{(1 + e^2)(2e^2 - 3e) - 4e}{2e(2e^2 - 3e)} = \frac{2e^3 - 3e^2 + 2e - 7}{4e^2 - 6e} \doteq 1.2411.$$

We also have

$$k(1) = 2 + 2e^2 + 3e^3 - 2e^4 - 2b\big(1 + e^2\big)\left(3\frac{e}{2} - e^2\right) - 4eb + 2e\left(3\frac{e}{2} - e^2\right)b^2 \doteq -5.4757 < 0.$$

Now we show that $r^*(1) \geq 0$. It will be done if we prove

$$t(b) = \frac{e}{2}(b - 1) - \ln\big(2eb^2 - 2b + 2e - 2eb\big) + \ln(e + 2) \geq 0.$$

But this follows from $t'(b) \geq 0$ and $t(2/e) \geq 0$. We have

$$t\left(\frac{2}{e}\right) = 1 - \frac{e}{2} - \ln\left(\frac{4}{e} + 2e - 4\right) + \ln(e + 2) \doteq 0.1248 \geq 0$$

and

$$t'(b) = \frac{e}{2} - \frac{4eb - 2e - 2}{2eb^2 - 2b + 2e - 2eb}.$$

The inequality $t'(b) \geq 0$ is equivalent to

$$n(b) = e^2b^2 - b\big(5e + e^2\big) + e^2 + 2e + 2 \geq 0$$

since $o(b) = 2eb^2 - 2b + 2e - 2eb \geq 0$, which is evident ($o''(b) > 0$, $o'(b) = 0$ for $b = (1 + e)/(2e) < 2/e, o(2/e) > 0$). Now $n(b) \geq 0$ follows from $n''(b) \geq 0$, $n'(b) = 0$ for $b = (5 + e)/(2e) > 1$, and $n(1) = e^2 - 3e + 2 \doteq 1.2342 \geq 0$.

(b) We assume that a and b satisfy (2.3).

We show again that $H'(x) \geq 0$ but now for $1 \leq b \leq x \leq e$. Because of $H(b) = 0$, the proof will be done.

From (2.4) we have that if $(eb - 1 - ex\ln b) \leq 0$, then $F'(x) \geq 0$. So we need to show that $F'(x) \geq 0$ for $s(x, b) = (eb - 1 - ex\ln b) > 0$.

Let $s(x, b) = (eb - 1 - ex\ln b) > 0$ for $1 \leq b \leq x \leq e$. Then $F'(x) \geq 0$ if only if

$$\begin{aligned} f(x, b) = {}& \frac{e}{2}(x\ln x + b\ln b) - (eb - 1)\ln x - \ln\big(2b(eb - 1 - ex\ln b)\big) \\ & + \ln\big(ex(\ln x + 1)(\ln x + 1 - 2\ln b) + 2\big) \geq 0. \end{aligned} \tag{2.6}$$

If $(eb - 1 - ex\ln b) > 0$, then $x < \frac{eb-1}{e\ln b}$. Because of $x > b$, we have $be\ln b < eb - 1$. Put $t = eb$ and $v(t) = t\ln t - 2t + 1$ for $e \leq t \leq e^2$. Then we have $v(e) = 1 - e < 0$, $v(e^2) = 1$, $v'(t) = \ln t - 1 >$

0. This implies that there is only one t^* such that $e < t^* < e^2$ and $v(t^*) = 0$. Because of $v(6.3055) = 8.8113e - 005 > 0$, we get $t^* < 6.3055$. So $b < b^* < 2.3196$. This implies that it suffices to show $f(x,b) \geq 0$ for $1 < b < 2.3196$.

The mean value theorem gives

$$\ln x - \ln b + 1 - \ln b > \frac{1}{x}(x-b) + \frac{1}{e}(e-b), \qquad \ln x + 1 \geq \frac{2x-1}{x}.$$

This implies

$$\begin{aligned}&\ln\bigl(ex(\ln x + 1)(\ln x + 1 - 2\ln b) + 2\bigr)\\ &\quad \geq \ln\bigl(e(2x-1)(2ex - eb - xb) + 2ex\bigr) - \ln x - 1.\end{aligned}$$

Similarly,

$$\begin{aligned}x\ln x + b\ln b - 2b\ln x &= (x-b)\ln x + b(\ln b - \ln x) \geq (x-b)(\ln x - \ln e)\\ &> -\frac{(x-b)(e-x)}{x}.\end{aligned}$$

So

$$\frac{e}{2}(x\ln x + b\ln b) - eb\ln x \geq -\frac{e}{2}\frac{(x-b)(e-x)}{x}.$$

From $\ln b > \frac{2(b-1)}{b+1}$ we have $f(x,b) \geq G(x,b)$, where

$$\begin{aligned}G(x,b) = {}& -\frac{e}{2}\frac{(x-b)(e-x)}{x} - \ln\left(2b\left(eb - 1 - \frac{2ex(b-1)}{1+b}\right)\right)\\ &+ \ln\bigl((2x-1)(2ex - eb - xb) + 2x\bigr).\end{aligned}$$

We show that $G(b,b) \geq 0$ and $G'_x(x,b) \geq 0$, and the proof will be done.

We have

$$\begin{aligned}G(b,b) &= -\ln\left(2b\left(eb - 1 - \frac{2eb(b-1)}{1+b}\right)\right) + \ln\bigl((2b-1)\bigl(eb - b^2\bigr) + 2b\bigr) =\\ L(b) &= \ln\left(\frac{1+b}{2}\right) - \ln\bigl(-eb^2 + 3eb - b - 1\bigr) + \ln\bigl(-2b^2 + 2eb + b - e + 2\bigr).\end{aligned}$$

If we show that

$$\frac{-2b^2 + 2eb + b - e + 2}{-eb^2 + 3eb - b - 1} \geq 1, \tag{2.7}$$

then $L(b) \geq 0$, so $G(b,b) \geq 0$.

Inequality (2.7) is equivalent to

$$s(b) = (e-2)b^2 + (2-e)b + 3 - e \geq 0.$$

From $s'(b) = 2(e-2)b + (2-e)$, $s'(b) = 0$ if $b = 0.5$, $s(1) = 3 - e > 0$ we have $s(b) > 0$, so $G(b,b) \geq 0$.

Now we show $G'_x(x,b) \geq 0$ for $1 < b < x < \min\{e, (eb-1)/(e\ln b)\}$ and $1 < b < b^*$. Because of $eb - 1 - \frac{2ex(b-1)}{1+b} > eb - 1 - ex\ln(b) > 0$, we have

$$G'_x(x,b) \geq -\frac{e}{2}\left(\frac{be-x^2}{x^2}\right) + \frac{2e(b-1)}{b((eb-1)(1+b)-2ex(b-1))} + \frac{8ex-4bx-2be-2e+b+2}{(2x-1)(2ex-eb-xb)+2x}.$$

(We omitted a positive term of derivation $G'_x(x,b)$.)

Since

$$\frac{be-x^2}{x^2} < \frac{x(e-x)}{x^2}, \qquad \frac{2e(b-1)}{b((eb-1)(1+b)-2ex(b-1))} > 0,$$

it suffices to show that

$$\frac{xe-e^2}{2x} + \frac{8ex-4bx-2be-2e+b+2}{(2x-1)(2ex-eb-xb)+2x} \geq 0. \tag{2.8}$$

To prove (2.8), it suffices to show that (we used $x > b$)

$$x^2\big((4e-2b)\big(eb-e^2\big)+16e-8b\big) + x\big(\big(eb-e^2\big)(2+b-2eb-2e)-4eb-4e+2b+4\big)+eb\big(eb-e^2\big) \geq 0.$$

This can be rewritten as

$$T(x,b) = x^2u(b) + xv(b) + w(b) \geq 0,$$

where

$$u(b) = -2eb^2 + b\big(6e^2-8\big) + e\big(16-4e^2\big),$$
$$v(b) = b^2\big(e-2e^2\big) + b\big(2-2e-3e^2+2e^3\big) + 2e^3 - 2e^2 - 4e + 4,$$
$$w(b) = e^2b^2 - be^3.$$

From this we obtain that the roots of u are $b_1 = 1.2468$ and $b_2 = 5.4366$. We have that $u < 0$ on $(1, b_1)$ and $u > 0$ on (b_1, b^*). If we show that $T(b,b) \geq 0$, $T(e,b) \geq 0$ for $b \in (1, b_1)$ ($T(x)$ is a concave function), and $T'_x(x,b) \geq 0$, $T(b,b) \geq 0$ for $b \in (b_1, b^*)$, then the proof will be complete. Because of $T'_x(x,b) = 2xu + v$, it suffices to prove that $P(b) = 2bu(b) + v(b) \geq 0$ for $b \in (b_1, b^*)$.

First, we show $T(b,b) \geq 0$, $T(e,b) \geq 0$ for $b \in (1, b_1)$. We have

$$T(b,b) = b\big(-2eb^3 + b^2\big(4e^2+e-8\big) + b\big(2+14e-2e^2-2e^3\big) + 4 - 4e - 2e^2 + e^3\big).$$

The roots of $T(b,b) = 0$ are $r_1 = -0.1913$, $r_2 = 0.8517$, $r_3 = 3.7046$, $r_4 = 0$. This implies that $T(b,b) \geq 0$ for $b \in (1, e)$.

Next, we have

$$T(e,b) = 2e\big(b^2\big(-2e^2 + e\big) + b\big(4e^3 - 2e^2 - 5e + 1\big) - 2e^4 + e^3 + 7e^2 - 2e + 2\big).$$

The roots of $T(e,b) = 0$ are $r_1 = 0.9969$, $r_2 = 3.3956$, and $T(e,0) < 0$ implies $T(e,b) \geq 0$ for $b \in (1,e)$.

Now we show that $P(b) \geq 0$ for $b \in (b_1, b^*)$. We have

$$P(b) = -4eb^3 + b^2\big(10e^2 + e - 16\big) + b\big(-6e^3 - 3e^2 + 30e + 2\big) + 2e^3 - 2e^2 - 4e + 4.$$

Because of $P(b) = 0$ has only one real root $r_1 = 4.4344$, $P(0) = 18.5198$, and $T(e,0) < 0$, we obtain that $p(b) \geq 0$ for $b \in (b_1, b^*)$.

So the proof is complete. □

3 Some generalizations of Conjecture 1.1

Denote $M^* = \{(a,b); (0 < a, b \leq e) \vee (0 < b, a \leq e) \vee (a \geq e^2, b \leq \sqrt{a}) \vee (b \geq e^2, a \leq \sqrt{b}) \vee (0 < a = b)\}$ (see Figure 1) and $M(n,r) = \{(x_1,\dots,x_n); x_i > 0, r \geq 0 \wedge x_1,\dots,x_n$ are solutions of the inequality (3.3)$\}$.

We have:

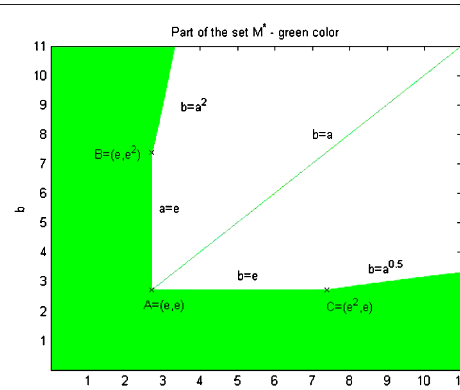
- $\{(a,b); 0 < a, b \leq e\} \subset M(2,e)$ (see [5]).
- $\forall s > e$, $\exists a, b < 1$ such that $(a,b) \not\subset M(2,s)$ (see [4]).
- $M^* \subset M(2,e)$.
- $(5,10) \not\subset M(2,e)$ (see [5]).
- $(1/3, 1/9, 2/3) \not\subset M(3, 5/2)$ (see [5]).
- If $0 < r < s$, then $M(n,s) \subset M(n,r)$ (Note 3.4).
- $\forall x_1,\dots,x_n > 0$, $\exists s > 0$ such that $(x_1,\dots,x_n) \in M(n,r)$ for $0 \leq r \leq s$ (Note 3.4).

Lemma 3.1 ([6], the log-sum inequality) *Let $n \in \mathbf{N}$, $x_1,\dots,x_n, y_1,\dots,y_n$ be positive numbers. Then*

$$\sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{y_i}\right) \geq \left(\sum_{i=1}^{n} x_i\right) \ln\left(\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i}\right) \tag{3.1}$$

with equality only for $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$.

Figure 1 Part of the set M^*.



Lemma 3.2 *Let*

$$F(r)=\ln n+\frac{r}{n}\left(\sum_{i=1}^{n}x_i\ln x_i\right)-\ln\left(e^{rx_1\ln x_n}+\sum_{i=1}^{n-1}e^{rx_{i+1}\ln x_i}\right),\tag{3.2}$$

where $r\geq 0$, $n\in\mathbf{N}$, $n\geq 2$, $x_1,\dots,x_n>0\wedge\exists i\neq j$ *such that* $x_i\neq x_j$. *Then* $F(0)=0$, $F'(0)>0$, $F''(r)<0$.

Note 3.3 *We note that* $F(r)\geq 0$ *is equivalent to*

$$n\sqrt[n]{\prod_{i=1}^{n}x_i^{rx_i}}\geq x_n^{rx_1}+\sum_{i=1}^{n-1}x_i^{rx_{i+1}}.\tag{3.3}$$

Proof It is evident that $F(0)=0$. Next, we have

$$F'(r)=\frac{1}{n}\left(\sum_{i=1}^{n}x_i\ln x_i\right)-\frac{e^{rx_1\ln x_n}x_1\ln x_n+\sum_{i=1}^{n-1}e^{rx_{i+1}\ln x_i}x_{i+1}\ln x_i}{e^{rx_1\ln x_n}+\sum_{i=1}^{n-1}e^{rx_{i+1}\ln x_i}}.$$

The inequality $F'(0)>0$ is equivalent to

$$\sum_{i=1}^{n}x_i\ln x_i-x_1\ln x_n-\sum_{i=1}^{n-1}x_{i+1}\ln x_i>0,$$

which can be rewritten as

$$\sum_{i=2}^{n}x_i\ln\left(\frac{x_i}{x_{i-1}}\right)+x_1\ln\left(\frac{x_1}{x_n}\right)>0.\tag{3.4}$$

To prove (3.4), we use the Jensens log-sum inequality (Lemma 3.1).

Put $y_1=x_n$, $y_2=x_1,\dots,y_n=x_{n-1}$ in (3.1). We obtain

$$\sum_{i=2}^{n}x_i\ln\left(\frac{x_i}{x_{i-1}}\right)+x_1\ln\left(\frac{x_1}{x_n}\right)\geq\nu=\left(\sum_{i=2}^{n}x_i\right)\ln\left(\frac{\sum_{i=2}^{n}x_i}{\sum_{i=1}^{n-1}x_i}\right)+x_1\ln\left(\frac{x_1}{x_n}\right).\tag{3.5}$$

We show that $\nu=\nu(y,x_1,x_n)>0$, where $y=\sum_{i=2}^{n-1}x_i$. We have

$$\nu(y,x_1,x_n)=(y+x_n)\ln\left(\frac{y+x_n}{y+x_1}\right)+x_1\ln\left(\frac{x_1}{x_n}\right).$$

It is evident that $\nu(0,x_1,x_n)=(x_n-x_1)\ln(\frac{x_n}{x_1})>0$ and

$$\nu'_y(y,x_1,x_n)=\ln\left(\frac{y+x_n}{y+x_1}\right)+\frac{x_1-x_n}{y+x_1}.\tag{3.6}$$

If we show $\nu'_y(y,x_1,x_n)\leq 0$, then $\lim_{y\to+\infty}\nu(y,x_1,x_n)=x_n-x_1+x_1\ln\frac{x_1}{x_n}\geq 0$ (if we put $t=x_1/x_n$, then $g=1-t+t\ln t\geq 0$) implies $\nu(y,x_1,x_n)\geq 0$.

Put $t=\frac{y+x_n}{y+x_1}$ in (3.6). Then $\nu'_y(y,x_1,x_n)=\ln t+1-t$. This implies $\nu'_y(y,x_1,x_n)<0$.

Now we prove $F''(r) < 0$. We have

$$F''(r) = \frac{-L(r)}{(\exp(rx_1 \ln x_n) + \sum_{i=1}^{n-1} \exp(rx_{i+1} \ln x_i))^2} < 0,$$

where

$$\begin{aligned} L(r) = {} & \left(\exp(rx_1 \ln x_n) x_1^2 \ln^2 x_n + \sum_{i=1}^{n-1} \exp(rx_{i+1} \ln x_i) x_{i+1}^2 \ln^2 x_i\right) \\ & \times \left(\exp(rx_1 \ln x_n) + \sum_{i=1}^{n-1} \exp(rx_{i+1} \ln x_i)\right) \\ & - \left(\exp(rx_1 \ln x_n) x_1 \ln x_n + \sum_{i=1}^{n-1} \exp(rx_{i+1} \ln x_i) x_{i+1} \ln x_i\right)^2 \geq 0. \end{aligned}$$

The equality $L(r) \geq 0$ can be rewritten as

$$\begin{aligned} L(r) = A_n + B_n = {} & \sum_{i=1}^{n-1} \exp\bigl(r(x_{i+1} \ln x_i + x_1 \ln x_n)\bigr)(x_{i+1} \ln x_i - x_1 \ln x_n)^2 \\ & + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \exp\bigl(r(x_{i+1} \ln x_i + x_{j+1} \ln x_j)\bigr)\bigl(x_{i+1}^2 \ln^2 x_i - x_{i+1} x_{j+1} (\ln x_i) \ln x_j\bigr) \geq 0. \end{aligned}$$

From $B_2 \geq 0$ and

$$B_{n+1} = B_n + \sum_{i=1}^{n-1} \exp\bigl(r(x_{i+1} \ln x_i + x_{n+1} \ln x_n)\bigr)(x_{i+1} \ln x_i - x_{n+1} \ln x_n)^2$$

we have $A_n + B_n \geq 0$ for $n \geq 2$. So, $F(r)$ is a concave function for $r \geq 0$. □

Note 3.4 *We note that Lemma* 3.2 *implies*: *if* $F(s) \geq 0$ *for some* $s > 0$ *and for positive numbers* $x_1, \ldots, x_n \in M(n,s)$, *then* $F(r) \geq 0$ *for* $r \in [0,s]$ *on* $M(n,s)$.

3.1 Other applications of Lemma 3.2

- For each $A \in R_+^n = \{(x_1, \ldots, x_n), x_i > 0, i = 1, \ldots, n\}$, $n \in \mathbf{N}$, there is a finite limit $L_A = \lim_{r\to+\infty} F'(r) = \frac{1}{n} \sum_{i=1}^n x_i \log(x_i) - m_x$, where $m_x = \max_{1\leq m\leq n}\{x_{m+1} \log(x_m)\}$, $x_{n+1} = x_1$.
- Denote by r_A the positive root of $F(r) = 0$ (if the root exists) for $A \in R_+^n$-S^n where $S^n = \{(x_1, \ldots, x_n), x_i = x_j, i, j = 1, \ldots, n\}$. Then
  - (a) $L_A \geq 0 \Leftrightarrow$ there is no $r_A > 0$ such that $F(r_A) = 0$.
  - (b) $L_A < 0 \Leftrightarrow$ there is $r_A > 0$ such that $F(r_A) = 0$.

Let $\emptyset \neq M \subset R_+^n - S^n$. Put $r_M = \inf_{A\in M}\{r_A\}$ and $R_M = \sup_{A\in M}\{r_A\}$. Then there are seven cases:

(a) $r_M = R_M = 0$,
(b) $0 = r_M < R_M < \infty$,
(c) $r_M = 0, R_M = \infty$,
(d) $0 < r_M = R_M < \infty$,

(e) $0 < r_M < R_M < \infty$,
(f) $0 < r_M < R_M = \infty$,
(g) $r_M = R_M = \infty$.

From this we have:

- Case (a) is not possible (Lemma 3.2).
- In case (b), inequality (3.3) is not valid for $r > 0$ on M, but the reverse inequality to (3.3) is valid for $r > R_M$ on M.
- In case (c), inequality (3.3) and the reverse inequality to (3.3) are not valid for $r > 0$ on M.
- In case (d), inequality (3.3) is valid for $0 \leq r \leq r_M$ on M, and the reverse inequality to (3.3) is valid for $r > r_M$ on M.
- In case (e), inequality (3.3) is valid for $0 \leq r \leq r_M$ on M, but the reverse inequality to (3.3) is valid for $r > R_M > r_M$ on M.
- In case (f), inequality (3.3) is valid for $0 \leq r \leq r_M < \infty$ on M, but the reverse inequality to (3.3) is not valid for any $r > 0$ on M.
- In case (g), inequality (3.3) is valid for all $r \geq 0$ on M.

3.2 Example

Let $n = 2$. Denote $a = x_2$, $b = x_1$. Then (1.1) is equivalent to $F(r) \geq 0$.

We have three cases:

$$\begin{cases} b\log(a) > a\log(b); & \text{then } L_A = \lim_{r\to+\infty} F'(r) = (\frac{a-b}{2})\log(a) + \frac{b}{2}\log(\frac{b}{a}); \\ a\log(b) > b\log(a); & \text{then } L_A = \lim_{r\to+\infty} F'(r) = (\frac{b-a}{2})\log(b) + \frac{a}{2}\log(\frac{a}{b}); \\ b\log(a) = a\log(b); & \text{then } L_A = \lim_{r\to+\infty} F'(r) = (\frac{a-b}{2})\log(\frac{a}{b}) \geq 0. \end{cases}$$

Let

$$M = \{(a,b); 0 < b < a \leq 1\}.$$

From $b < a$ we have $a\log(b) < b\log(a)$, so $(\frac{a-b}{2})\log(a) + \frac{b}{2}\log(\frac{b}{a}) < 0$. Lemma 2.2 in [4] gives that $r_M = e$. $\lim_{a\to1,b\to0} F(r) = \log 2$ implies that $R_M = \infty$. So, we have that the reverse inequality to (3.3) cannot be valid for any $r > 0$ on M.

Competing interests
The author declares that he has no competing interests.

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