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# Two new lower bounds for the minimum eigenvalue of $M$-tensors

Jianxing Zhao* and Caili Sang

*Correspondence:
zjx810204@163.com
College of Science, Guizhou Minzu University, Guiyang, Guizhou 550025, P.R. China

**Abstract**

Two new lower bounds for the minimum eigenvalue of an irreducible $M$-tensor are given. It is proved that the new lower bounds improve the corresponding bounds obtained by He and Huang (J. Inequal. Appl. 2014:114, 2014). Numerical examples are given to verify the theoretical results.

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**Keywords:** $M$-tensors; nonnegative tensors; minimum eigenvalue

## 1 Introduction

Let $\mathbb{C}(\mathbb{R})$ be the set of all complex (real) numbers, $n$ be positive integer, $n \geq 2$, and $N = \{1, 2, \ldots, n\}$. We call $\mathcal{A} = (a_{i_1 \cdots i_m})$ a complex (real) tensor of order $m$ and dimension $n$, if

$$a_{i_1 \cdots i_m} \in \mathbb{C}(\mathbb{R}),$$

where $i_j \in N$ for $j = 1, \ldots, m$. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2. We call $\mathcal{A}$ nonnegative if $\mathcal{A}$ is real and each of its entries $a_{i_1 \cdots i_m} \geq 0$. Let $\mathbb{R}^{[m,n]}$ denote the space of real-valued tensors with order $m$ and dimension $n$.

A tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ of order $m$ and dimension $n$ is called reducible if there exists a nonempty proper index subset $\alpha \subset N$ such that

$$a_{i_1 i_2 \cdots i_m} = 0, \quad \forall i_1 \in \alpha, \forall i_2, \ldots, i_m \notin \alpha.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible [2].

For a complex tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ of order $m$ and dimension $n$, if there are a complex number $\lambda$ and a nonzero complex vector $x = (x_1, x_2, \ldots, x_n)^T$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then $\lambda$ is called an eigenvalue of $\mathcal{A}$ and $x$ an eigenvector of $\mathcal{A}$ associated with $\lambda$, where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose $i$th component are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m},$$

and

$$\left(x^{[m-1]}\right)_i = x_i^{m-1},$$

respectively. This definition was introduced by Qi in [3] where he assumed that $\mathcal{A}$ is an order $m$ and dimension $n$ supersymmetric tensor and $m$ is even. Independently, in [4], Lim gave such a definition but restricted $x$ to be a real vector and $\lambda$ to be a real number. In this case, we call $\lambda$ an H-eigenvalue of $\mathcal{A}$ and $x$ an H-eigenvector of $\mathcal{A}$ associated with $\lambda$.

Moreover, the spectral radius $\rho(\mathcal{A})$ of the tensor $\mathcal{A}$ is defined as

$$\rho(\mathcal{A}) = \max\left\{|\lambda| : \lambda \in \sigma(\mathcal{A})\right\},$$

where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$, that is, $\sigma(\mathcal{A}) = \{\lambda : \lambda \text{ is an eigenvalue of } \mathcal{A}\}$; see [2, 5].

The class of $M$-tensors introduced in [6, 7] is related to nonnegative tensors, which is a generalization of $M$-matrices [8].

**Definition 1** ([6, 7]) Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. $\mathcal{A}$ is called:

(i) a $Z$-tensor if all of its off-diagonal entries are non-positive, that is, $a_{i_1\cdots i_m} \leq 0$ for $i_j \in N$, $j = 1, \ldots, m$;

(ii) an $M$-tensor if $\mathcal{A}$ is a $Z$-tensor with the from $\mathcal{A} = c\mathcal{I} - \mathcal{B}$ such that $\mathcal{B}$ is a nonnegative tensor and $c > \rho(\mathcal{B})$, where $\rho(\mathcal{B})$ is the spectral radius of $\mathcal{B}$, and $\mathcal{I}$ is called the unit tensor with its entries

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 1** ([1, 6]) *Let $\mathcal{A}$ be an $M$-tensor and denote by $\tau(\mathcal{A})$ the minimal value of the real part of all eigenvalues of $\mathcal{A}$. Then $\tau(\mathcal{A}) > 0$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector. If $\mathcal{A}$ is irreducible, then $\tau(\mathcal{A})$ is the unique eigenvalue with a positive eigenvector.*

The minimum eigenvalue $\tau(\mathcal{A})$ of $M$-tensors has many applications and these are studied in [1, 6–13]. Very recently, He and Huang [1] provided some inequalities on $\tau(\mathcal{A})$ for an irreducible $M$-tensor $\mathcal{A}$ as follows.

**Theorem 2** ([1]) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible $M$-tensor. Then*

$$0 < \tau(\mathcal{A}) \leq \min_{i\in N} a_{ii\cdots i}, \quad \textit{and} \quad \tau(\mathcal{A}) \geq \min_{i\in N} R_i(\mathcal{A}),$$

*where $R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m\in N} a_{ii_2\cdots i_m}$.*

**Theorem 3** ([1]) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible $M$-tensor. Then*

$$\tau(\mathcal{A}) \geq \min_{\substack{i,j\in N,\\ j\neq i}} \frac{1}{2}\left\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \left[\left(a_{i\cdots i} - a_{j\cdots j} + r_i^j(\mathcal{A})\right)^2 - 4a_{ij\cdots j}r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\},$$

*where*

$$r_i(\mathcal{A}) = \sum_{\substack{i_2,\dots,i_m \in N, \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|, \qquad r_i^j(\mathcal{A}) = r_i(\mathcal{A}) - |a_{ij\cdots j}| = \sum_{\substack{\delta_{ii_2\cdots i_m}=0, \\ \delta_{ji_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|.$$

In this paper, we continue to research the problem of estimating the minimum eigenvalue of $M$-tensors, give two new lower bounds for the minimum eigenvalue, and prove that the two new lower bounds are better than that in Theorem 2 and one of the two bounds is the correction of Theorem 3. Finally, some numerical examples are given to verify the results obtained.

## 2 Main results

In this section, we give two new lower bounds for the minimum eigenvalue of an irreducible $M$-tensor.

**Theorem 4** *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible $M$-tensor. Then*

$$\tau(\mathcal{A}) \geq \min_{\substack{i,j\in N, \\ j\neq i}} L_{ij}(\mathcal{A}),$$

*where*

$$L_{ij}(\mathcal{A}) = \frac{1}{2}\left\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \left[\left(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\right)^2 - 4a_{ij\cdots j}r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\}.$$

*Proof* Because $\tau(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$, from Theorem 2.1 in [14], there are $i, j \in N$, $j \neq i$, such that

$$\left(\left|\tau(\mathcal{A}) - a_{i\cdots i}\right| - r_i^j(\mathcal{A})\right)\left(\left|\tau(\mathcal{A}) - a_{j\cdots j}\right|\right) \leq |a_{ij\cdots j}| r_j(\mathcal{A}).$$

From Theorem 2, we can get

$$\left(a_{i\cdots i} - \tau(\mathcal{A}) - r_i^j(\mathcal{A})\right)\left(a_{j\cdots j} - \tau(\mathcal{A})\right) \leq -a_{ij\cdots j} r_j(\mathcal{A}),$$

equivalently,

$$\tau(\mathcal{A})^2 - \left(a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A})\right)\tau(\mathcal{A}) + a_{j\cdots j}\left(a_{i\cdots i} - r_i^j(\mathcal{A})\right) + a_{ij\cdots j} r_j(\mathcal{A}) \leq 0. \tag{1}$$

Solving for $\tau(\mathcal{A})$ gives

$$\begin{aligned}
\tau(\mathcal{A}) &\geq \frac{1}{2}\left\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \left[\left(a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A})\right)^2 \right.\right. \\
&\qquad \left.\left. - 4\left(a_{j\cdots j}\left(a_{i\cdots i} - r_i^j(\mathcal{A})\right) + a_{ij\cdots j} r_j(\mathcal{A})\right)\right]^{\frac{1}{2}}\right\} \\
&= \frac{1}{2}\left\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \left[\left(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\right)^2 - 4a_{ij\cdots j} r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\} \\
&\geq \min_{\substack{i,j\in N, \\ j\neq i}} \frac{1}{2}\left\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \left[\left(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\right)^2 - 4a_{ij\cdots j} r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\}.
\end{aligned}$$

The proof is completed. □

**Remark 1** Note here that the bound in Theorem 4 is the correction of the bound in Theorem 3. Because the bound in Theorem 3 is obtained by solving for $\tau(\mathcal{A})$ from inequality (1); for details, see the proof of Theorem 2.2 in [1]. However, solving for $\tau(\mathcal{A})$ by inequality (1) gives the bound in Theorem 4.

In the following, a counterexample is given to show that the result in Theorem 3 is false. Consider the tensor $\mathcal{A} = (a_{ijkl})$ of order 4 and dimension 2 with entries defined as follows:

$$\mathcal{A}(1,1,:,:) = \begin{pmatrix} 21 & -4 \\ -3 & -3 \end{pmatrix}, \qquad \mathcal{A}(1,2,:,:) = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix},$$
$$\mathcal{A}(2,1,:,:) = \begin{pmatrix} -3 & -3 \\ -1 & -1 \end{pmatrix}, \qquad \mathcal{A}(2,2,:,:) = \begin{pmatrix} -3 & -1 \\ -1 & 27 \end{pmatrix}.$$

By Theorem 3, we have $\tau(\mathcal{A}) \geq 8$. By Theorem 4, we have $\tau(\mathcal{A}) \geq 2.4700$. In fact, $\tau(\mathcal{A}) = 6.9711$.

We now give the following comparison theorem for Theorem 2 and Theorem 4.

**Theorem 5** *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible M-tensor. Then*

$$\min_{\substack{i,j\in N,\\ j\neq i}} L_{ij}(\mathcal{A}) \geq \min_{i\in N} R_i(\mathcal{A}).$$

*Proof* (i) For any $i,j \in N$, $j \neq i$, if $R_i(\mathcal{A}) \leq R_j(\mathcal{A})$, i.e., $a_{ii\cdots i} + a_{ij\cdots j} - r_i^j(\mathcal{A}) \leq a_{jj\cdots j} - r_j(\mathcal{A})$, then

$$0 \leq r_j(\mathcal{A}) \leq -a_{ij\cdots j} - \big(a_{ii\cdots i} - a_{jj\cdots j} - r_i^j(\mathcal{A})\big). \tag{2}$$

Hence,

$$\begin{aligned}
&\big[a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\big]^2 - 4a_{ij\cdots j}r_j(\mathcal{A}) \\
&\quad\leq \big[a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\big]^2 - 4a_{ij\cdots j}\big[-a_{ij\cdots j} - \big(a_{ii\cdots i} - a_{jj\cdots j} - r_i^j(\mathcal{A})\big)\big] \\
&\quad= \big[a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\big]^2 + 4a_{ij\cdots j}\big[a_{ii\cdots i} - a_{jj\cdots j} - r_i^j(\mathcal{A})\big] + 4a_{ij\cdots j}^2 \\
&\quad= \big[-2a_{ij\cdots j} - \big(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\big)\big]^2.
\end{aligned}$$

From (2), we have

$$-2a_{ij\cdots j} - \big(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\big) \geq r_j(\mathcal{A}) \geq 0.$$

Thus,

$$\begin{aligned}
L_{ij}(\mathcal{A}) &= \frac{1}{2}\Big\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \big[\big(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\big)^2 - 4a_{ij\cdots j}r_j(\mathcal{A})\big]^{\frac{1}{2}}\Big\} \\
&\geq \frac{1}{2}\Big\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \big[-2a_{ij\cdots j} - \big(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\big)\big]\Big\} \\
&= \frac{1}{2}\big\{2a_{i\cdots i} + 2a_{ij\cdots j} - 2r_i^j(\mathcal{A})\big\} \\
&= R_i(\mathcal{A}),
\end{aligned}$$

which implies

$$\min_{\substack{i,j\in N,\\ j\neq i}} L_{ij}(\mathcal{A}) \geq \min_{i\in N} R_i(\mathcal{A}).$$

(ii) For any $i,j \in N$, $j \neq i$, if $R_j(\mathcal{A}) \leq R_i(\mathcal{A})$, *i.e.*, $a_{jj\cdots j} - r_j(\mathcal{A}) \leq a_{ii\cdots i} + a_{ij\cdots j} - r_i^j(\mathcal{A})$, then

$$0 \leq -a_{ij\cdots j} \leq r_j(\mathcal{A}) - \bigl(a_{jj\cdots j} - a_{ii\cdots i} + r_i^j(\mathcal{A})\bigr).$$

Similar to the proof of (i), we have $L_{ij}(\mathcal{A}) \geq R_j(\mathcal{A})$. Hence,

$$\min_{\substack{i,j\in N,\\ j\neq i}} L_{ij}(\mathcal{A}) \geq \min_{j\in N} R_j(\mathcal{A}).$$

The conclusion follows. □

Theorem 5 shows the lower bound in Theorem 4 is better than that in Theorem 2. To obtain a better lower bound, we give the following theorem by breaking $N$ into disjoint subsets $S$ and its complement $\overline{S}$.

**Theorem 6** *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible M-tensor, $S$ be a nonempty proper subset of $N$, $\overline{S}$ be the complement of $S$ in $N$. Then*

$$\tau(\mathcal{A}) \geq \min\Bigl\{\min_{i\in S}\max_{j\in\overline{S}} L_{ij}(\mathcal{A}), \min_{i\in\overline{S}}\max_{j\in S} L_{ij}(\mathcal{A})\Bigr\}.$$

*Proof* Let $x = (x_1, x_2, \ldots, x_n)^T$ be an associated positive eigenvector of $\mathcal{A}$ corresponding to $\tau(\mathcal{A})$, *i.e.*,

$$\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}. \tag{3}$$

Let $x_p = \max\{x_i : i \in S\}$ and $x_q = \max\{x_j : j \in \overline{S}\}$. We next distinguish two cases to prove.

Case I: If $x_p \geq x_q$, then $x_p = \max\{x_i : i \in N\}$. For $p \in S$ and any $j \in \overline{S}$, we have by (3)

$$\tau(\mathcal{A})x_p^{m-1} = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{pi_2\cdots i_m}x_{i_2}\cdots x_{i_m} + a_{p\cdots p}x_p^{m-1} + a_{pj\cdots j}x_j^{m-1}$$

and

$$\tau(\mathcal{A})x_j^{m-1} = \sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m} + a_{j\cdots j}x_j^{m-1} + a_{jp\cdots p}x_p^{m-1},$$

equivalently,

$$\bigl(\tau(\mathcal{A}) - a_{p\cdots p}\bigr)x_p^{m-1} - a_{pj\cdots j}x_j^{m-1} = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{pi_2\cdots i_m}x_{i_2}\cdots x_{i_m} \tag{4}$$

and

$$\bigl(\tau(\mathcal{A})-a_{j\cdots j}\bigr)x_j^{m-1}-a_{jp\cdots p}x_p^{m-1}=\sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m}. \tag{5}$$

Solving $x_p^{m-1}$ by (4) and (5), we obtain

$$\begin{aligned}&\bigl(\bigl(\tau(\mathcal{A})-a_{p\cdots p}\bigr)\bigl(\tau(\mathcal{A})-a_{j\cdots j}\bigr)-a_{pj\cdots j}a_{jp\cdots p}\bigr)x_p^{m-1}\\ &\quad=\bigl(\tau(\mathcal{A})-a_{j\cdots j}\bigr)\sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{pi_2\cdots i_m}x_{i_2}\cdots x_{i_m}+a_{pj\cdots j}\sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m}.\end{aligned}$$

Since $\tau(\mathcal{A})\le\min_{i\in N}a_{i\cdots i}$ by Theorem 2 and $\mathcal{A}$ is a $Z$-tensor, we have

$$\begin{aligned}&\bigl(\bigl(a_{p\cdots p}-\tau(\mathcal{A})\bigr)\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)-a_{pj\cdots j}a_{jp\cdots p}\bigr)x_p^{m-1}\\ &\quad=\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)\sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} |a_{pi_2\cdots i_m}|x_{i_2}\cdots x_{i_m}+|a_{pj\cdots j}|\sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} |a_{ji_2\cdots i_m}|x_{i_2}\cdots x_{i_m}.\end{aligned}$$

Hence,

$$\begin{aligned}&\bigl(\bigl(a_{p\cdots p}-\tau(\mathcal{A})\bigr)\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)-|a_{pj\cdots j}||a_{jp\cdots p}|\bigr)x_p^{m-1}\\ &\quad\le\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)\sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} |a_{pi_2\cdots i_m}|x_p^{m-1}+|a_{pj\cdots j}|\sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} |a_{ji_2\cdots i_m}|x_p^{m-1}.\end{aligned}$$

Note that $x_p>0$, then

$$\begin{aligned}&\bigl(a_{p\cdots p}-\tau(\mathcal{A})\bigr)\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)-|a_{pj\cdots j}||a_{jp\cdots p}|\\ &\quad\le\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)\sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} |a_{pi_2\cdots i_m}|+|a_{pj\cdots j}|\sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} |a_{ji_2\cdots i_m}|,\end{aligned}$$

equivalently,

$$\bigl(a_{p\cdots p}-\tau(\mathcal{A})\bigr)\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)-|a_{pj\cdots j}||a_{jp\cdots p}|\le\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)r_p^j(\mathcal{A})+|a_{pj\cdots j}|r_j^p(\mathcal{A}).$$

This implies

$$\bigl(a_{p\cdots p}-\tau(\mathcal{A})\bigr)\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)-\bigl(a_{j\cdots j}-\tau(\mathcal{A})\bigr)r_p^j(\mathcal{A})-|a_{pj\cdots j}|r_j(\mathcal{A})\le 0,$$

that is,

$$\tau(\mathcal{A})^2-\bigl(a_{p\cdots p}+a_{j\cdots j}-r_p^j(\mathcal{A})\bigr)\tau(\mathcal{A})+a_{p\cdots p}a_{j\cdots j}-a_{j\cdots j}r_p^j(\mathcal{A})-|a_{pj\cdots j}|r_j(\mathcal{A})\le 0.$$

Solving for $\tau(\mathcal{A})$ gives

$$\tau(\mathcal{A})\ge\frac{1}{2}\bigl\{a_{p\cdots p}+a_{j\cdots j}-r_p^j(\mathcal{A})-\bigl[\bigl(a_{p\cdots p}-a_{j\cdots j}-r_p^j(\mathcal{A})\bigr)^2+4|a_{pj\cdots j}|r_j(\mathcal{A})\bigr]^{\frac{1}{2}}\bigr\}.$$

This must also be true for any $j \in \overline{S}$. Therefore,

$$\begin{aligned}\tau(\mathcal{A}) &\geq \max_{j\in\overline{S}} \frac{1}{2}\left\{a_{p\cdots p} + a_{j\cdots j} - r_p^j(\mathcal{A}) - \left[\left(a_{p\cdots p} - a_{j\cdots j} - r_p^j(\mathcal{A})\right)^2 + 4|a_{pj\cdots j}|r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\} \\ &= \max_{j\in\overline{S}} \frac{1}{2}\left\{a_{p\cdots p} + a_{j\cdots j} - r_p^j(\mathcal{A}) - \left[\left(a_{p\cdots p} - a_{j\cdots j} - r_p^j(\mathcal{A})\right)^2 - 4a_{pj\cdots j}r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\}.\end{aligned}$$

Since this could be true for some $p \in S$, we finally have

$$\tau(\mathcal{A}) \geq \min_{i\in S}\max_{j\in\overline{S}} \frac{1}{2}\left\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \left[\left(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\right)^2 - 4a_{ij\cdots j}r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\}.$$

Case II: If $x_q \geq x_p$, then $x_q = \max\{x_i : i \in N\}$. Similar to the proof of Case I, we can easily prove that

$$\tau(\mathcal{A}) \geq \min_{i\in\overline{S}}\max_{j\in S} \frac{1}{2}\left\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \left[\left(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\right)^2 - 4a_{ij\cdots j}r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\}.$$

The conclusion follows from Cases I and II. □

By Theorem 4, Theorem 5, and Theorem 6, the following comparison theorem is obtained easily.

**Theorem 7** *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible M-tensor. Then*

$$\min\left\{\min_{i\in S}\max_{j\in\overline{S}} L_{ij}(\mathcal{A}), \min_{i\in\overline{S}}\max_{j\in S} L_{ij}(\mathcal{A})\right\} \geq \min_{\substack{i,j\in N,\\ j\neq i}} L_{ij}(\mathcal{A}) \geq \min_{i\in N} R_i(\mathcal{A}).$$

**Remark 2**

(i) Theorem 7 shows that the bound in Theorem 6 is better than those in Theorem 2 and Theorem 4, respectively.

(ii) For an $M$-tensor $\mathcal{A}$ of order $m$ and dimension $n$, as regards Theorem 4 and Theorem 6 we need to compute $n(n-1)$ and $2|S|(n-|S|)$ $L_{ij}(\mathcal{A})$ to obtain their lower bound for $\tau(\mathcal{A})$, respectively, where $|S|$ is the cardinality of $S$. When $n$ is very large, one needs more computations to obtain these lower bounds by Theorem 4 and Theorem 6 than Theorem 2.

(iii) Note that $|S| < n$. When $n = 2$, then $|S| = 1$ and $n(n-1) = 2|S|(n-|S|) = 2$, which implies that

$$\min\left\{\min_{i\in S}\max_{j\in\overline{S}} L_{ij}(\mathcal{A}), \min_{i\in\overline{S}}\max_{j\in S} L_{ij}(\mathcal{A})\right\} = \min_{\substack{i,j\in N,\\ j\neq i}} L_{ij}(\mathcal{A}).$$

When $n \geq 3$, then $2|S|(n-|S|) < n(n-1)$ and

$$\min\left\{\min_{i\in S}\max_{j\in\overline{S}} L_{ij}(\mathcal{A}), \min_{i\in\overline{S}}\max_{j\in S} L_{ij}(\mathcal{A})\right\} \geq \min_{\substack{i,j\in N,\\ j\neq i}} L_{ij}(\mathcal{A}).$$

## 3 Numerical examples

In this section, two numerical examples are given to verify the theoretical results.

**Example 1** Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,4]}$ be an irreducible $M$-tensor with elements defined as follows:

$$\mathcal{A}(1,:,:) = \begin{pmatrix} 37 & -2 & -1 & -4 \\ -1 & -3 & -3 & -2 \\ -1 & -1 & -3 & -2 \\ -2 & -3 & -3 & -3 \end{pmatrix}, \qquad \mathcal{A}(2,:,:) = \begin{pmatrix} -2 & -4 & -2 & -3 \\ -1 & 39 & -2 & -1 \\ -3 & -3 & -4 & -2 \\ -2 & -3 & -1 & -4 \end{pmatrix},$$

$$\mathcal{A}(3,:,:) = \begin{pmatrix} -4 & -1 & -1 & -1 \\ -1 & -2 & -2 & -3 \\ -1 & -1 & 62 & -1 \\ -2 & -2 & -4 & -3 \end{pmatrix}, \qquad \mathcal{A}(4,:,:) = \begin{pmatrix} -2 & -4 & -3 & -1 \\ -4 & -4 & -2 & -4 \\ -3 & -3 & -3 & -3 \\ -3 & -3 & -4 & 55 \end{pmatrix}.$$

Let $S = \{1,2\}$. Obviously $\overline{S} = \{3,4\}$. By Theorem 2, we have

$$\tau(\mathcal{A}) \geq 2.$$

By Theorem 4, we have

$$\tau(\mathcal{A}) \geq 2.0541.$$

By Theorem 6, we have

$$\tau(\mathcal{A}) \geq 4.$$

In fact, $\tau(\mathcal{A}) = 9.9363$. Hence, this example verifies Theorem 7, that is, the bound in Theorem 6 is better than those in Theorem 2 and Theorem 4, respectively.

**Example 2** Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be an irreducible $M$-tensor with elements defined as follows:

$$a_{1111} = 5, \qquad a_{1222} = -1, \qquad a_{2111} = -2, \qquad a_{2222} = 4,$$

the other $a_{ijkl} = 0$. By Theorem 2, we have

$$\tau(\mathcal{A}) \geq 2.$$

By Theorem 4, we have

$$\tau(\mathcal{A}) \geq 3.$$

In fact, $\tau(\mathcal{A}) = 3$. Hence, the lower bound in Theorem 4 is tight and sharper than that in Theorem 2.

## 4 Further work

In this paper, we give an $S$-type lower bound

$$\Delta^S(\mathcal{A}) = \min\left\{\min_{i\in S}\max_{j\in\overline{S}} L_{ij}(\mathcal{A}), \min_{i\in\overline{S}}\max_{j\in S} L_{ij}(\mathcal{A})\right\}$$

for the minimum eigenvalue of an irreducible $M$-tensor $\mathcal{A}$ by breaking $N$ into disjoint subsets $S$ and its complement $\overline{S}$. Then an interesting problem is how to pick $S$ to make $\Delta^S(\mathcal{A})$ as big as possible. But it is difficult when the dimension of the tensor $\mathcal{A}$ is large. We will continue to study this problem in the future.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

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