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Liouville type theorems for the system of fractional nonlinear equations in R^n_+

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Abstract

In this paper we consider the following system of fractional nonlinear equations in the half space R^n_+:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u_1(x) = x_n^{\gamma} u_1^{\alpha_1}(x) u_2^{\beta_1}(x), & x \in R^n_+, \\ (-\Delta)^{\frac{\alpha}{2}} u_2(x) = x_n^{\gamma} u_1^{\alpha_2}(x) u_2^{\beta_2}(x), & x \in R^n_+, \\ u_1(x) = u_2(x) = 0, & x \notin R^n_+, \end{cases} \tag{1}$$

where $\gamma \geq 0$, $0 < \alpha < 2$, $\alpha_i, \beta_i > 0$, $i = 1, 2$.

First, we use the Kelvin transform and the method of moving planes in integral forms to prove that (1) is equivalent to the following system of integral equations with $1 < \alpha_i + \beta_i \leq \frac{n+\alpha+2\gamma}{n-\alpha}$:

$$\begin{cases} u_1(x) = \int_{R^n_+} G(x,y) y_n^{\gamma} u_1^{\alpha_1}(y) u_2^{\beta_1}(y)\, dy, & x \in R^n_+, \\ u_2(x) = \int_{R^n_+} G(x,y) y_n^{\gamma} u_1^{\alpha_2}(y) u_2^{\beta_2}(y)\, dy, & x \in R^n_+, \end{cases} \tag{2}$$

where $G(x,y)$ is the Green's function associated with $(-\Delta)^{\frac{\alpha}{2}}$ in R^n_+.

Then we continue work on integral systems (2) to establish Liouville type theorems, *i.e.* the nonexistence of positive solutions in the subcritical case and the critical case, $1 < \alpha_i + \beta_i \leq \frac{n+\alpha+2\gamma}{n-\alpha}$.

Keywords: the fractional Laplacian; Green's function; method of moving planes in integral forms; Liouville theorem; Kelvin transform

1 Introduction

In recent years, there has been a great deal of interests in using the fractional Laplacian to model diverse physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars (see [1–6], and the references therein). The fractional Laplacian in R^n is a nonlocal operator, taking the form

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha}\, P.V. \int_{R^n} \frac{u(x) - u(z)}{|x-z|^{n+\alpha}}\, dz, \tag{1.1}$$

where $0 < \alpha < 2$, *P.V.* stands for the Cauchy principal value. This operator is well defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing C^∞ functions in R^n. One can extend this operator to a wider space $\mathcal{L}_\alpha$ of distributions as follows.

Let

$$\mathcal{L}_\alpha = \left\{ u : R^n \to R \;\Big|\; \int_{R^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}}\, dx < \infty \right\}.$$

For $u \in \mathcal{L}_\alpha$, we define $(-\Delta)^{\frac{\alpha}{2}} u(x)$ as a distribution:

$$\left\langle (-\Delta)^{\frac{\alpha}{2}} u(x), \phi \right\rangle = \left\langle u, (-\Delta)^{\frac{\alpha}{2}} \phi \right\rangle, \quad \forall \phi \in \mathcal{S}.$$

Zhang and Cheng [7] considered the positive solutions of the following single equation in R^n:

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = x_n^\gamma u^p(x), \quad u(x) > 0, x \in R^n. \tag{1.2}$$

They showed the following.

Proposition 1.1 ([7]) *Assume* $p > \frac{n}{n-\alpha}$ *and* $\gamma \geq 0$, *if* $u(x) \in L^{\frac{n(p-1)}{\alpha}}(R_+^n)$ *is a non-negative solution of equation* (1.2), *then* $u(x) \equiv 0$.

Proposition 1.2 ([7]) *Assume* $1 < p \leq \frac{n+\alpha+2\gamma}{n-\alpha}$ *and* $\gamma \geq 0$, *if* $u(x)$ *is a locally bounded non-negative solution of the equation* (1.2), *then* $u(x) \equiv 0$. *In particular, when* $\frac{n+\alpha}{n-\alpha} \leq p \leq \frac{n+\alpha+2\gamma}{n-\alpha}$, *we only require* $u(x) \in L^{\frac{n(p-1)}{\alpha}}(R_+^n)$.

Motivated by [7], in this paper we consider the Dirichlet problem for the following pseudo differential system in R_+^n:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u_1 = x_n^\gamma u_1^{\alpha_1}(x) u_2^{\beta_1}(x), & x \in R_+^n, \\ (-\Delta)^{\frac{\alpha}{2}} u_2 = x_n^\gamma u_1^{\alpha_2}(x) u_2^{\beta_2}(x), & x \in R_+^n, \\ u_1(x) = u_2(x) = 0, & x \notin R_+^n, \end{cases} \tag{1.3}$$

where $\gamma \geq 0$, $0 < \alpha < 2$, α_i, $\beta_i > 0$, $i = 1, 2$.

First, we use the maximum principle and the Liouville theorem in R_+^n in [6] to show that the positive solutions of problem (1.3) satisfy the following integral equations under some weak integrability condition:

$$\begin{cases} u_1(x) = c_1 x_n^{\frac{\alpha}{2}} + \int_{R_+^n} G(x,y) y_n^\gamma u_1^{\alpha_1}(y) u_2^{\beta_1}(y)\, dy, & x \in R_+^n, \\ u_2(x) = c_2 x_n^{\frac{\alpha}{2}} + \int_{R_+^n} G(x,y) y_n^\gamma u_1^{\alpha_2}(y) u_2^{\beta_2}(y)\, dy, & x \in R_+^n, \end{cases} \tag{1.4}$$

where

$$G(x,y) = \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} \frac{b^{\frac{\alpha}{2}-1}}{(1+b)^{\frac{n}{2}}}\, db \tag{1.5}$$

is the Green's function associated with $(-\Delta)^{\frac{\alpha}{2}}$ in R_+^n and $A_{n,\alpha}$ is a constant depending on n and α.

Then we use Kelvin transform and the method of moving planes in integral forms to prove that c_1 and c_2 must be 0. We derive that (1.3) is equivalent to the following integral equations under some locally integrable conditions:

$$\begin{cases} u_1(x) = \int_{R_+^n} G(x,y) y_n^\gamma u_1^{\alpha_1}(y) u_2^{\beta_1}(y)\,dy, & x \in R_+^n, \\ u_2(x) = \int_{R_+^n} G(x,y) y_n^\gamma u_1^{\alpha_2}(y) u_2^{\beta_2}(y)\,dy, & x \in R_+^n. \end{cases} \tag{1.6}$$

In the subcritical case and critical case: $1 < \alpha_i + \beta_i \leq \frac{n+\alpha+2\gamma}{n-\alpha}$, we continue work on the integral systems (1.6) to show the nonexistence of positive solutions. That is, we have the following result.

Theorem 1.1 *Assume that $u(x) = (u_1(x), u_2(x))$ is a positive solution of equations* (1.3). *If $|u| \in L_{\mathrm{loc}}^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(R_+^n) \cap L_{\mathrm{loc}}^\infty(R_+^n)$, then in the case $1 < \alpha_i + \beta_i \leq \frac{n+\alpha+2\gamma}{n-\alpha}$, $u(x)$ is also a solution of integral equations* (1.6), *and vice versa.*

Next, we establish the Liouville theorem for the integral equations as follows.

Theorem 1.2 *Assume that $\alpha_i + \beta_i > \frac{n}{n-\alpha}$ and $\gamma \geq 0$, if $|u| \in L^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(R_+^n)$ is a nonnegative solution of the system of the integral equations* (1.6), *then $u(x) \equiv 0$.*

Theorem 1.3 *Assume that $u(x) = (u_1(x), u_2(x))$ is a nonnegative solution of equations* (1.6). *If $|u| \in L_{\mathrm{loc}}^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(R_+^n) \cap L_{\mathrm{loc}}^\infty(R_+^n)$ and $1 < \alpha_i + \beta_i \leq \frac{n+\alpha+2\gamma}{n-\alpha}$, we have $u(x) \equiv 0$.*

Remark 1 In this paper we use the new method in [6] to prove Theorem 1.1, we believe that this new approach can be applied to a variety of other situations.

2 Equivalence between the two systems

The proof of Theorem 1.1 is based on the following maximum principle and the Liouville theorem.

Proposition 2.1 ([8]) *Let Ω be a bounded open set, and let $f(x)$ be a lower-semicontinuous function in $\bar{\Omega}$ such that $(-\Delta)^{\frac{\alpha}{2}} f(x) \geq 0$ in Ω and $f(x) \geq 0$ in $R^n \setminus \Omega$. Then $f(x) \geq 0$ in R^n.*

Proposition 2.2 ([8]) *If $f(x) \in \mathcal{L}_\alpha$ and $(-\Delta)^{\frac{\alpha}{2}} f(x) \geq 0$ in an open set, then $f(x)$ is lower semicontinuous in Ω.*

Theorem 2.1 ([9]) *Let $0 < \alpha < 2$, $u \in \mathcal{L}_\alpha$. Assume u is a nonnegative solution of*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = 0, & x \in R_+^n, \\ u(x) \equiv 0, & x \notin R_+^n. \end{cases} \tag{2.1}$$

Then we have either

$$u(x) \equiv 0 x \in R^n \quad \text{or} \quad u(x) = \begin{cases} C x_n^{\frac{\alpha}{2}}, & x \in R_+^n, \\ 0, & x \notin R_+^n, \end{cases}$$

for some positive constant C.

Proof Assume $u(x) \in \mathcal{L}_\alpha$ is a positive solution of the system of the fractional nonlinear PDEs system:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u_i(x) = x_n^\gamma u_1^{\alpha_i}(x) u_2^{\beta_i}(x), & x \in R_+^n, \\ u_i(x) = 0, & x \notin R_+^n, \end{cases} \tag{2.2}$$

where $i = 1, 2$, $\gamma \geq 0$, $0 < \alpha < 2$, α_i, $\beta_i > 0$.

We first show that

$$\int_{R_+^n} G(x,y) y_n^\gamma u_1^{\alpha_i}(y) u_2^{\beta_i}(y)\, dy < \infty. \tag{2.3}$$

Set $P_R := (0, \ldots, 0, R) \in R_+^n$, $B_R^+(P_R) := \{x \in R^n : |x - P_R| < R\}$, the ball of radius R centered at P_R. Let

$$v_i^R(x) = \int_{B_R^+(P_R)} G_R(x,y) y_n^\gamma u_1^{\alpha_i}(y) u_2^{\beta_i}(y)\, dy, \tag{2.4}$$

where $G_R(x,y)$, the Green's function on the ball $B_R^+(P_R)$, was given in [10],

$$G_R(x,y) = \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} \left[1 - \frac{B_{n,\alpha}}{(s_R + t_R)^{\frac{(n-2)}{2}}} \int_0^{\frac{s_R}{t_R}} \frac{(s_R - t_R b)^{\frac{(n-2)}{2}}}{b^{\frac{\alpha}{2}}(1+b)}\, db \right], \quad x, y \in B_R^+(P_R),$$

here $s_R = \frac{|x-y|^2}{R^2}$, $t_R = (1 - \frac{|x-P_R|^2}{R^2})(1 - \frac{|y-P_R|^2}{R^2})$, $A_{n,\alpha}$, and $B_{n,\alpha}$ are constants depending on n and α.

From the local bounded-ness assumption on u, one can see that, for each $R > 0$, $v_i^R(x)$ is well defined and is continuous. Moreover,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} v_i^R(x) = x_n^\gamma u_1^{\alpha_i}(x) u_2^{\beta_i}(x), & x \in B_R^+(P_R), \\ v_i^R(x) = 0, & x \notin B_R^+(P_R). \end{cases} \tag{2.5}$$

Let $w_i^R(x) = u_i(x) - v_i^R(x)$, by (2.2) and (2.5), we derive

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} w_i^R(x) = 0, & x \in B_R^+(P_R), \\ w_i^R(x) \geq 0, & x \notin B_R^+(P_R). \end{cases} \tag{2.6}$$

Applying the maximum principle (see Proposition 2.1), we derive that

$$w_i^R(x) \geq 0, \quad \forall x \in B_R^+(P_R). \tag{2.7}$$

It is easy to prove that

$$v_i^R(x) \to v_i(x) = \int_{R_+^n} G(x,y) y_n^\gamma u_1^{\alpha_i}(y) u_2^{\beta_i}(y)\, dy, \quad \text{as } R \to \infty. \tag{2.8}$$

Obviously,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} v_i(x) = x_n^\gamma u_1^{\alpha_i}(x) u_2^{\beta_i}(x), & x \in R_+^n, \\ v_i(x) \equiv 0, & x \notin R_+^n. \end{cases} \tag{2.9}$$

Denote $w_i(x) = u_i(x) - v_i(x)$. Using (2.2), (2.7), (2.8), and (2.9), we have

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} w_i(x) = 0, & w_i(x) \geq 0, x \in R^n_+, \\ w_i(x) \equiv 0, & x \notin R^n_+. \end{cases}$$

Applying the Liouville theorem (see Theorem 2.1), we deduce that either

$$w_i(x) \equiv 0, \quad x \in R^n \quad \text{or} \quad w_i(x) \equiv c_i x_n^{\frac{\alpha}{2}}, \quad \forall x \in R^n_+, i = 1, 2,$$

for some positive constants $c_i > 0$, then we could write $w_i = c_i x_n^{\frac{\alpha}{2}}$, $c_i \geq 0$. That is, the solutions of (2.2) satisfy

$$u_i(x) = c_i x_n^{\frac{\alpha}{2}} + \int_{R^n_+} G(x,y) y_n^{\gamma} u_1^{\alpha_i}(y) u_2^{\beta_i}(y)\, dy, \quad x \in R^n_+, i = 1, 2, \tag{2.10}$$

where $c_i \geq 0$, $G(x,y)$ is defined in (1.5).

Next we need to prove c_i must be zero for $i = 1, 2$. To this end, we employ a certain type of Kelvin transform and the method of moving planes in integral forms.

For $z^0 = \{z_1^0, \ldots, z_{n-1}^0, 0\} \in \partial R^n_+$, let $\bar{u}_i^{z^0}(x) = \bar{u}_i(x) = \frac{1}{|x-z^0|^{n-\alpha}} u_i(\frac{x-z^0}{|x-z^0|^2} + z^0)$, be the Kelvin transform of $u_i(x)$ centered at z^0.

Through a straightforward calculation by (2.10), we derive

$$\bar{u}_i(x) = \frac{c_i x_n^{\frac{\alpha}{2}}}{|x-z^0|^{n-\alpha}} + \frac{1}{|x-z^0|^{n-\alpha}} \int_{R^n_+} G\left(\frac{x-z^0}{|x-z^0|^2} + z^0, y\right) y_n^{\gamma} u_1^{\alpha_i}(y) u_2^{\beta_i}(y)\, dy.$$

Let $y = \frac{z-z^0}{|z-z^0|^2} + z^0$, then $dy = \frac{1}{|z-z^0|^{2n}}\, dz$,

$$\begin{aligned} \bar{u}_i(x) &= \frac{c_i x_n^{\frac{\alpha}{2}}}{|x-z^0|^{n-\alpha}} + \frac{1}{|x-z^0|^{n-\alpha}} \int_{R^n_+} G(x,z) \left|x-z^0\right|^{n-\alpha} \left|z-z^0\right|^{n-\alpha} \\ &\quad \times \frac{|\frac{z_n}{|z-z^0|^2}|^{\gamma} u_1^{\alpha_i}(\frac{z-z^0}{|z-z^0|^2} + z^0) u_2^{\beta_i}(\frac{z-z^0}{|z-z^0|^2} + z^0)}{|z-z^0|^{2n}}\, dz \\ &= \frac{c_i x_n^{\frac{\alpha}{2}}}{|x-z^0|^{n-\alpha}} + \int_{R^n_+} G(x,z) \frac{z_n^{\gamma}}{|z-z^0|^{n+\alpha+2\gamma}} \frac{u_1^{\alpha_i}(\frac{z-z^0}{|z-z^0|^2} + z^0)}{|z-z^0|^{(n-\alpha)\alpha_i}} \\ &\quad \times \frac{u_2^{\beta_i}(\frac{z-z^0}{|z-z^0|^2} + z^0)}{|z-z^0|^{(n-\alpha)\beta_i}} \left|z-z^0\right|^{(n-\alpha)\alpha_i} \left|z-z^0\right|^{(n-\alpha)\beta_i}\, dz \\ &= \frac{c_i x_n^{\frac{\alpha}{2}}}{|x-z^0|^{n-\alpha}} + \int_{R^n_+} G(x,z) \frac{z_n^{\gamma} \bar{u}_1^{\alpha_i}(z) \bar{u}_2^{\beta_i}(z)}{|z-z^0|^{n+2\gamma+\alpha-(n-\alpha)(\alpha_i+\beta_i)}}\, dz \\ &= \frac{c_i x_n^{\frac{\alpha}{2}}}{|x-z^0|^{n-\alpha}} + \int_{R^n_+} G(x,y) \frac{y_n^{\gamma} \bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y)}{|y-z^0|^{\delta}}\, dy, \quad \forall x \in R^n_+ \backslash B_{\epsilon}\left(z^0\right), \end{aligned} \tag{2.11}$$

where $\epsilon > 0$, $\delta = n + \alpha + 2\gamma - (n-\alpha)(\alpha_i + \beta_i)$.

Then we have $\delta = 0$ *i.e.* $\alpha_i + \beta_i = \frac{n+\alpha+2\gamma}{n-\alpha}$, it is called critical case. When $\delta > 0$, we have $1 < \alpha_i + \beta_i < \frac{n+\alpha+2\gamma}{n-\alpha}$ and it is called the subcritical case. In this section, we consider these two cases $1 < \alpha_i + \beta_i \leq \frac{n+\alpha+2\gamma}{n-\alpha}$, then we have $\delta \geq 0$.

Now we introduce some basic notations in the method of moving planes. For a given real number λ, denote $\Sigma_\lambda = \{x = (x_1, \dots, x_n) \in R^n_+ \mid x_1 < \lambda\}$, $T_\lambda = \{x = (x_1, \dots, x_n) \in R^n_+ \mid x_1 = \lambda\}$. Let $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$ be the reflection of the point $x = (x_1, \dots, x_n)$ about the plane T_λ, and $\bar{u}_i^\lambda(x) = \bar{u}_i(x^\lambda)$, $\bar{w}_i^\lambda(x) = \bar{u}_i^\lambda(x) - \bar{u}_i(x)$.

For $x, y \in \Sigma_\lambda$, $x \neq y$, by [5], we have

$$G(x,y) = G\big(x^\lambda, y^\lambda\big) > G\big(x, y^\lambda\big) = G\big(x^\lambda, y\big). \tag{2.12}$$

Obviously, we have

$$\begin{aligned}
\bar{u}_i(x) &= \frac{c_i x_n^{\frac{\alpha}{2}}}{|x - z^0|^{n-\alpha}} + \int_{R^n_+} G(x,y) \frac{y_n^\gamma \bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y)}{|y - z^0|^\delta} \, dy \\
&= \frac{c_i x_n^{\frac{\alpha}{2}}}{|x - z^0|^{n-\alpha}} + \int_{\Sigma_\lambda} G(x,y) \frac{y_n^\gamma \bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y)}{|y - z^0|^\delta} \, dy \\
&\quad + \int_{\Sigma_\lambda} G\big(x^\lambda, y\big) \frac{y_n^\gamma \bar{u}_1^{\alpha_i}(y^\lambda) \bar{u}_2^{\beta_i}(y^\lambda)}{|y^\lambda - z^0|^\delta} \, dy, \\
\bar{u}_i^\lambda(x) &= \frac{c_i x_n^{\frac{\alpha}{2}}}{|x^\lambda - z^0|^{n-\alpha}} + \int_{\Sigma_\lambda} G\big(x^\lambda, y\big) \frac{y_n^\gamma \bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y)}{|y - z^0|^\delta} \, dy \\
&\quad + \int_{\Sigma_\lambda} G(x,y) \frac{y_n^\gamma \bar{u}_1^{\alpha_i}(y^\lambda) \bar{u}_2^{\beta_i}(y^\lambda)}{|y^\lambda - z^0|^\delta} \, dy.
\end{aligned}$$

By an elementary calculation, we derive

$$\begin{aligned}
\bar{u}_i(x) - \bar{u}_i^\lambda(x) &= \frac{c_i x_n^{\frac{\alpha}{2}}}{|x - z^0|^{n-\alpha}} - \frac{c_i x_n^{\frac{\alpha}{2}}}{|x^\lambda - z^0|^{n-\alpha}} \\
&\quad + \int_{\Sigma_\lambda} \big[G(x,y) - G\big(x^\lambda, y\big)\big] y_n^\gamma \left[\frac{\bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y)}{|y - z^0|^\delta} - \frac{\bar{u}_1^{\alpha_i}(y^\lambda) \bar{u}_2^{\beta_i}(y^\lambda)}{|y^\lambda - z^0|^\delta} \right] dy \\
&\leq \int_{\Sigma_\lambda} \big[G(x,y) - G\big(x^\lambda, y\big)\big] y_n^\gamma \left[\frac{\bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y)}{|y - z^0|^\delta} - \frac{\bar{u}_1^{\alpha_i}(y^\lambda) \bar{u}_2^{\beta_i}(y^\lambda)}{|y^\lambda - z^0|^\delta} \right] dy \\
&\leq \int_{\Sigma_\lambda} \big[G(x,y) - G\big(x^\lambda, y\big)\big] \frac{y_n^\gamma [\bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y) - \bar{u}_1^{\alpha_i}(y^\lambda) \bar{u}_2^{\beta_i}(y^\lambda)]}{|y - z^0|^\delta} \, dy.
\end{aligned} \tag{2.13}$$

The proof consists of two steps. In step 1, we will show that, for λ sufficiently negative,

$$\bar{w}_i^\lambda(x) = \bar{u}_i^\lambda(x) - \bar{u}_i(x) \geq 0, \quad \text{a.e. } \forall x \in \Sigma_\lambda.$$

In step 2, we deduce that T_λ can be moved to the right all the way to z_1^0. Furthermore, we obtain $\bar{w}_{z_1^0} \equiv 0$, $\forall x \in \Sigma_{z_1^0}$.

Step 1. (Prepare to move the plane from near $x_1 = -\infty$.) In this step, we will show that, for λ sufficiently negative, $\epsilon > 0$ sufficiently small

$$\bar{u}_i^\lambda(x) \geq \bar{u}_i(x), \quad \text{a.e. } \forall x \in \Sigma_\lambda \backslash B_\epsilon\big(\big(z^0\big)^\lambda\big), \tag{2.14}$$

where $(z^0)^\lambda$ is the reflection of z^0 about the plane T_λ. Define $\Gamma_i^\lambda = \{x \in \Sigma_\lambda \backslash B_\epsilon((z^0)^\lambda) \mid \bar{u}_i^\lambda(x) < \bar{u}_i(x)\}$, the sets where the inequalities (2.14) are violated. We will prove that Γ_i^λ are empty, where $i = 1, 2$.

Without loss of generality, we consider $\bar{u}_1$. Denote $\Sigma_i^\lambda = \{x \in \Sigma_\lambda \backslash B_\epsilon((z^0)^\lambda) \mid \bar{u}_1^{\alpha_i}(x^\lambda) \times \bar{u}_2^{\beta_i}(x^\lambda) < \bar{u}_1^{\alpha_i}(x)\bar{u}_2^{\beta_i}(x)\}$, for $y \in \Sigma_1^\lambda$, we may assume that $\bar{u}_1(y) > \bar{u}_1^\lambda(y)$ and $\bar{u}_2(y) \le \bar{u}_2^\lambda(y)$. Define

$$\bar{w}_i^\lambda(y) = \begin{cases} 0, & \text{for } \bar{u}_i(y) < \bar{u}_i^\lambda(y), \\ \bar{u}_i(y) - \bar{u}_i^\lambda(y), & \text{for } \bar{u}_i(y) > \bar{u}_i^\lambda(y), \end{cases}$$

and $\bar{w}^\lambda(y) = (\bar{w}_1^\lambda(y), \bar{w}_2^\lambda(y))$. By the expression of $G(x,y)$, it is easy to see

$$G(x,y) \le \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}}. \tag{2.15}$$

Applying the mean value theorem, combining (2.13) and (2.15), we have, for $x \in \Gamma_i^\lambda$,

$$\begin{aligned} \bar{u}_i(x) - \bar{u}_i^\lambda(x) &\le \int_{\Sigma_\lambda} G(x,y) \frac{y_n^\gamma [\bar{u}_1^{\alpha_i}(y)\bar{u}_2^{\beta_i}(y) - \bar{u}_1^{\alpha_i}(y^\lambda)\bar{u}_2^{\beta_i}(y^\lambda)]}{|y-z^0|^\delta}\, dy \\ &\le \int_{\Sigma_i^\lambda} G(x,y) \frac{y_n^\gamma [\bar{u}_1^{\alpha_i}(y)\bar{u}_2^{\beta_i}(y) - \bar{u}_1^{\alpha_i}(y^\lambda)\bar{u}_2^{\beta_i}(y^\lambda)]}{|y-z^0|^\delta}\, dy \\ &= \int_{\Sigma_i^\lambda} G(x,y) \frac{y_n^\gamma \{[\bar{u}_1^{\alpha_i}(y) - \bar{u}_1^{\alpha_i}(y^\lambda)]\bar{u}_2^{\beta_i}(y) + \bar{u}_1^{\alpha_i}(y)[\bar{u}_2^{\beta_i}(y) - u_2^{\beta_i}(y^\lambda)]\}}{|y-z^0|^\delta}\, dy \\ &\le c\int_{\Sigma_i^\lambda} G(x,y) \frac{y_n^\gamma [\bar{u}_1^{\alpha_i}(y) - \bar{u}_1^{\alpha_i}(y^\lambda)]\bar{u}_2^{\beta_i}(y)}{|y-z^0|^\delta}\, dy \\ &\le c\int_{\Sigma_i^\lambda} G(x,y) \frac{y_n^\gamma \psi_1^{\alpha_i-1}(y)[\bar{u}_1(y) - \bar{u}_1(y^\lambda)]\bar{u}_2^{\beta_i}(y)}{|y-z^0|^\delta}\, dy \\ &\le c\int_{\Sigma_i^\lambda} \frac{1}{|x-y|^{n-\alpha}} \frac{y_n^\gamma \bar{u}_1^{\alpha_i-1}(y)[\bar{u}_1(y) - \bar{u}_1(y^\lambda)]\bar{u}_2^{\beta_i}(y)}{|y-z^0|^\delta}\, dy. \end{aligned} \tag{2.16}$$

Noticing $\Sigma_i^\lambda \subseteq \Gamma_j^\lambda$ for some j, applying the Hardy-Littlewood-Sobolev inequality and the Hölder inequality we obtain for any $q > \frac{n}{n-\alpha}$,

$$\|\bar{w}_\lambda\|_{L^q(\Gamma_\lambda)} \le c\left\| \frac{y_n^\gamma |\bar{u}|^{\alpha_i+\beta_i-1}}{|y-z^0|^\delta} \bar{w}_\lambda \right\|_{L^{\frac{nq}{n+\alpha q}}(\Gamma_\lambda)} \le c\left\| \frac{y_n^\gamma |\bar{u}|^{\alpha_i+\beta_i-1}}{|y-z^0|^\delta} \right\|_{L^{\frac{n}{\alpha}}(\Gamma_\lambda)} \|\bar{w}_\lambda\|_{L^q(\Gamma_\lambda)}, \tag{2.17}$$

where $\Gamma_\lambda = \Gamma_1^\lambda \cup \Gamma_2^\lambda$. Since $\gamma \ge 0$, we can easily see that y_n^γ is bounded in each bounded domain $\Omega \subset R_+^n$. Therefore, by our assumption $|u| \in L_{\mathrm{loc}}^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(R_+^n)$, *i.e.* $|u|^{\alpha_i+\beta_i-1} \in L_{\mathrm{loc}}^{\frac{n}{\alpha}}(R_+^n)$, we derive

$$y_n^\gamma |u|^{\alpha_i+\beta_i-1} \in L_{\mathrm{loc}}^{\frac{n}{\alpha}}(R_+^n). \tag{2.18}$$

Hence, we obtain

$$\int_{\hat{\Omega}} \left[\frac{y_n^\gamma |\bar{u}|^{\alpha_i+\beta_i-1}(y)}{|y-z^0|^\delta} \right]^{\frac{n}{\alpha}} dy = \int_\Omega \left(y_n^\gamma |u|^{\alpha_i+\beta_i-1}(y) \right)^{\frac{n}{\alpha}} dy < \infty \tag{2.19}$$

for any domain Ω that is a positive distance away from the z^0 and $\hat{\Omega}$ is the image of Ω about the Kelvin transform. By (2.19), for λ sufficiently negative and for $\epsilon > 0$ sufficiently small, $c\|\frac{y_n^\gamma|\bar{u}|^{\alpha_i+\beta_i-1}}{|y-z^0|^\delta}\|_{L^{\frac{n}{\alpha}}(\Gamma_\lambda)}$ can be made very small. Combining this with (2.17), we arrive at $\|\bar{w}_\lambda\|_{L^q(\Gamma_\lambda)} = 0$, and Γ_i^λ must be of measure zero. Hence (2.14) holds.

Step 2. (Move the plane to the limiting position to derive symmetry.)

Inequality (2.14) provides a starting point to move the plane T_λ. Now we start to move the plane T_λ along the x_1 direction as long as (2.14) holds. Define

$$\lambda_0 = \sup\bigl\{\lambda \le z_1^0 \mid \bar{u}_i^\mu(x) \ge \bar{u}_i(x),\ \text{a.e. } \forall x \in \Sigma_\mu, \mu \le \lambda\bigr\}.$$

We prove that $\lambda_0 = z_1^0$. On the contrary, suppose that $\lambda_0 < z_1^0$. We will show that $\bar{u}(x)$ is symmetric about the plane T_{λ_0}, that is,

$$\bar{u}(x) \equiv \bar{u}^{\lambda_0}(x), \quad \text{a.e. } \forall x \in \Sigma_{\lambda_0} \backslash B_\epsilon\bigl(\bigl(z^0\bigr)^{\lambda_0}\bigr). \tag{2.20}$$

Suppose (2.20) is not true, then, for such $\lambda_0 < z_1^0$, for all $i = 1, 2$, we have

$$\bar{u}_i^{\lambda_0}(x) > \bar{u}_i(x) \quad \text{a.e. } x \in \Sigma_{\lambda_0} \backslash B_\epsilon\bigl(\bigl(z^0\bigr)^{\lambda_0}\bigr). \tag{2.21}$$

In fact, by (2.14), we have two cases for all $i = 1, 2$; one is

$$\bar{u}_i^{\lambda_0}(y) > \bar{u}_i(y) \quad \text{on a set of positive measure, } i = 1 \text{ and } 2. \tag{2.22}$$

For the other case, without loss of generality, we may assume that $\bar{u}_1^{\lambda_0}(z_1) > \bar{u}_1(z_1)$ and

$$\bar{u}_2^{\lambda_0}(z_2) = \bar{u}_2(z_2). \tag{2.23}$$

For the first case, (2.21) is proved. For the other case, we have

$$\bar{u}_1^{\alpha_i}(y)\bar{u}_2^{\beta_i}(y) - \bar{u}_1^{\alpha_i}\bigl(y^{\lambda_0}\bigr)\bar{u}_2^{\beta_i}\bigl(y^{\lambda_0}\bigr) < 0. \tag{2.24}$$

Combining (2.23) with (2.24), we obtain

$$0 = \bar{u}_i(z_i) - \bar{u}_i^{\lambda_0}(z_i) < \int_{\Sigma_{\lambda_0}} \bigl[G(z_i, y) - G\bigl(z_i{}^\lambda, y\bigr)\bigr] \frac{y_n^\gamma[\bar{u}_1^{\alpha_i}(y)\bar{u}_2^{\beta_i}(y) - \bar{u}_1^{\alpha_i}(y^{\lambda_0})\bar{u}_2^{\beta_i}(y^{\lambda_0})]}{|y - z^0|^\delta}\,dy < 0.$$

This is impossible. Hence (2.21) holds. Next based on (2.21), we will verify that the plane can be moved further to the right. More precisely, there exists a $\zeta > 0$ such that, for all $\lambda \in [\lambda_0, \lambda_0 + \zeta)$ $\bar{u}_i^\lambda(x) \ge \bar{u}_i(x)$, a.e. $\forall x \in \Sigma_\lambda \backslash B_\epsilon(z^0)^{\lambda_0}$. By inequality (2.17), we have

$$\|\bar{w}_\lambda\|_{L^q(\Gamma_\lambda)} \le \biggl\{\int_{\Gamma_\lambda}\biggl(\frac{y_n^\gamma|\bar{u}|^{\alpha_i+\beta_i-1}(y)}{|y - z^0|^\delta}\biggr)^{\frac{n}{\alpha}}dy\biggr\}^{\frac{\alpha}{n}}\|\bar{w}_\lambda\|_{L^q(\Gamma_\lambda)}. \tag{2.25}$$

Equation (2.19) ensures that one can choose η sufficiently small so that, for all λ in $[\lambda_0, \lambda_0 + \eta)$,

$$c\biggl\{\int_{\Gamma_\lambda}\biggl(\frac{y_n^\gamma|\bar{u}|^{\alpha_i+\beta_i-1}(y)}{|y - z^0|^\delta}\biggr)^{\frac{n}{\alpha}}dy\biggr\}^{\frac{\alpha}{n}} \le \frac{1}{2}. \tag{2.26}$$

We postpone the proof of this inequality (2.26) for a moment.

Now combining (2.25) and (2.26), we have $\|\bar{w}_\lambda\|_{L^q(\Gamma_\lambda)} = 0$, and Γ_λ must be of measure zero. Hence, for these values of $\lambda > \lambda_0$, we have $\bar{w}_\lambda(x) \geq 0$, a.e. $\forall x \in \Sigma_{\lambda_0} \backslash B_\epsilon((z^0)^\lambda)$, $\epsilon > 0$. This contradicts the definition of λ_0. Therefore (2.20) must hold. That is, if $\lambda_0 < z_1^0$, then we must have

$$\bar{u}_i(x) \equiv \bar{u}_i^\lambda(x), \quad \text{a.e. } \forall x \in \Sigma_{\lambda_0} \backslash B_\epsilon\big(\big(z^0\big)^{\lambda_0}\big). \tag{2.27}$$

Recall that, by our assumption, $c_{i_0} > 0$ and

$$\bar{u}_{i_0}(x) = \frac{c_{i_0} x_n^{\frac{\alpha}{2}}}{|x - z^0|^{n-\alpha}} + \int_{R_+^n} G(x,y) \frac{y_n^\gamma \bar{u}_1^{\alpha_1}(y) \bar{u}_2^{\beta_1}(y)}{|y - z^0|^\delta}\, dy. \tag{2.28}$$

It follows that $\bar{u}_{i_0}$ is singular at z^0, hence by (2.27), $\bar{u}_{i_0}$ must also be singular at $(z^0)^\lambda$. This is impossible, because z^0 is the only singularity of $\bar{u}$. Hence, we must have $\lambda_0 = z_1^0$. Since ϵ is an arbitrary positive number, we have actually derived that

$$\bar{u}_i^{\lambda_0}(x) \geq \bar{u}_i(x), \quad \text{a.e. } \forall x \in \Sigma_{\lambda_0}, \lambda_0 = z_1^0.$$

Entirely similarly, we can move the plane from near $x_1 = +\infty$ to the left and obtain $\bar{u}_i^{\lambda_0}(x) \leq \bar{u}_i(x)$, a.e. $\forall x \in \Sigma_{\lambda_0}$, $\lambda_0 = z_1^0$. Therefore we have $\bar{w}_{\lambda_0}(x) \equiv 0$, a.e. $\forall x \in \Sigma_{\lambda_0}$, $\lambda_0 = z_1^0$.

Now we prove inequality (2.26). For any small $\eta > 0$, $\forall \varepsilon > 0$, we can choose R sufficiently large so that

$$\left(\int_{(R_+^n \backslash B_\varepsilon(z^0)) \backslash B_R} \left[\frac{y_n^\gamma |\bar{u}|^{\alpha_i + \beta_i - 1}(y)}{|y - z^0|^\delta}\, dy \right]^{\frac{n}{\alpha}} \right)^{\frac{\alpha}{n}} \leq \eta. \tag{2.29}$$

For any $\tau > 0$, define $E_i^\tau = \{x \in (\Sigma_{\lambda_0} \backslash B_\varepsilon((z^0)^{\lambda_0})) \cap B_R(0) \mid \bar{u}_i^{\lambda_0}(x) - \bar{u}_i(x) > \tau\}$, and $F_i^\tau = \{(\Sigma_{\lambda_0} \backslash B_\varepsilon((z^0)^{\lambda_0})) \cap B_R(0)\} \backslash E_i^\tau$. Obviously, $\lim_{\tau \to 0} \mu(F_i^\tau) = 0$.

For $\lambda > \lambda_0$, let $D_\lambda = \{(\Sigma_\lambda \backslash B_\varepsilon((z^0)^\lambda)) \backslash (\Sigma_{\lambda_0} \backslash B_\varepsilon((z^0)^{\lambda_0})) \cap B_R(0)\}$.

It is easy to see that

$$\big\{\Gamma_i^\lambda \cap B_R(0)\big\} \subset \big(\Gamma_i^\lambda \cap E_i^\tau\big) \cup F_i^\tau \cup D_\lambda. \tag{2.30}$$

For λ sufficiently close to λ_0, $\mu(D_\lambda)$ is very small. We will show that $\mu(\Gamma_i^\lambda \cap E_i^\tau)$ is sufficiently small as λ close to λ_0.

In fact, $\bar{w}_i^\lambda(x) = \bar{u}_i^\lambda(x) - \bar{u}_i(x) = \bar{u}_i^\lambda(x) - \bar{u}_i^{\lambda_0}(x) + \bar{u}_i^{\lambda_0}(x) - \bar{u}_i(x) < 0$, $\forall x \in (\Gamma_i^\lambda \cap E_i^\tau)$. Therefore, $\bar{u}_i^{\lambda_0}(x) - \bar{u}_i^\lambda(x) > \bar{u}_i^\lambda(x) - \bar{u}_i^\lambda(x) > \tau$, $\forall x \in (\Gamma_i^\lambda \cap E_i^\tau)$. It follows that

$$\big(\Gamma_i^\lambda \cap E_i^\tau\big) \subset H_i^\tau = \big\{x \in B_R(0) \mid \bar{u}_i^{\lambda_0}(x) - \bar{u}_i^\lambda(x) > \tau\big\}. \tag{2.31}$$

By the well-known Chebyshev inequality, for fixed τ, as λ is close to λ_0, $\mu(E_i^\tau)$ can be sufficiently small. By (2.30) and (2.31), we derive that $\mu(\Gamma_i^\lambda \cap B_R(0))$ can be made as small as we wish. Combining this with (2.29), we deduce that (2.26) holds.

Since we can choose any direction that is perpendicular to the x_n-axis as the x_1 direction, we have actually shown that the Kelvin transform of the solution $\bar{u}(x)$ is rotationally symmetric about the line parallel to the x_n-axis and passing through z^0. Now we take any two points X^1 and X^2, with $X^l = (x'^l, x_n) \in R^{n-1} \times [0, \infty)$, $l = 1, 2$. Let z^0 be the projection of $\bar{X} = \frac{X^1+X^2}{2}$ on ∂R^n_+. Set $Y^l = \frac{X^l - z^0}{|X^l - z^0|^2} + z^0$, $l = 1, 2$. From the above arguments, it is easy to see $\bar{u}(Y^1) = \bar{u}(Y^2)$, hence $u(X^1) = u(X^2)$. This implies that u_i is independent of $x' = (x_1, \ldots, x_{n-1})$. That is, $u_i = u_i(x_n)$, and we will show that this will contradict the finiteness of the integral $\int_{R^n_+} G(x,y) y_n^\gamma u_1^{\alpha_i}(y) u_2^{\beta_i}(y)\, dy$. To continue, we need the following lemma.

Lemma 2.1 ([5]) *If $\frac{t}{s}$ is sufficiently small, then $\forall x = (x', x_n)$, $y = (y', y_n) \in R^n_+$, one can derive that*

$$\frac{c_{n,\alpha}}{s^{\frac{n-\alpha}{2}}} \cdot \frac{t^{\frac{\alpha}{2}}}{s^{\frac{\alpha}{2}}} \le G(x,y) \le \frac{C_{n,\alpha}}{s^{\frac{n-\alpha}{2}}} \cdot \frac{t^{\frac{\alpha}{2}}}{s^{\frac{\alpha}{2}}}, \quad \text{i.e. } G(x,y) \sim \frac{t^{\frac{\alpha}{2}}}{s^{\frac{n}{2}}}.$$

Here $s = |x-y|^2$, $t = 4x_n y_n$, $c_{n,\alpha}$, and $C_{n,\alpha}$ stand for different positive constants that only depend on n and α.

Set $x = (x', x_n)$, $y = (y', y_n) \in R^{n-1} \times (0, +\infty)$, $r^2 = |x' - y'|^2$ and $a^2 = |x_n - y_n|^2$. If $u_i = u_i(x_n)$ is a solution of

$$u_i(x) = \int_{R^n_+} G(x,y) y_n^\gamma u_1^{\alpha_i}(y) u_2^{\beta_i}(y)\, dy, \tag{2.32}$$

then, for each fixed $x \in R^n_+$, letting R be large enough, by elementary calculations, we have

$$\begin{aligned} +\infty > u_i(x_n) &= \int_0^\infty y_n^\gamma u_1^{\alpha_i}(y_n) u_2^{\beta_i}(y_n) \int_{R^{n-1}} G(x,y)\, dy'\, dy_n \\ &\ge C \int_R^\infty y_n^\gamma u_1^{\alpha_i}(y_n) u_2^{\beta_i}(y_n) y_n^{\frac{\alpha}{2}} \int_{R^{n-1} \setminus B_R(0)} \frac{1}{|x-y|^n}\, dy'\, dy_n \\ &\ge C \int_R^\infty y_n^\gamma u_1^{\alpha_i}(y_n) u_2^{\beta_i}(y_n) y_n^{\frac{\alpha}{n-2}} \int_R^\infty \frac{r^2}{(r^2+a^2)^{\frac{n}{2}}}\, dr\, dy_n \\ &\ge C \int_R^\infty y_n^\gamma u_1^{\alpha_i}(y_n) u_2^{\beta_i}(y_n) y_n^{\frac{\alpha}{2}} \frac{1}{|x_n - y_n|} \int_{\frac{R}{a}}^\infty \frac{\tau^{n-2}}{(\tau^2+1)^{\frac{n}{2}}}\, d\tau\, dy_n \\ &\ge C \int_R^\infty u_1^{\alpha_i}(y_n) u_2^{\beta_i}(y_n) y_n^{\gamma + \frac{\alpha}{2} - 1}\, dy_n. \end{aligned} \tag{2.33}$$

Equation (2.33) implies that there exists a sequence $\{y_n^k\} \to \infty$ as $k \to \infty$, such that

$$u_1^{\alpha_i}(y_n^k) u_2^{\beta_i}(y_n^k) (y_n^k)^{\gamma + \frac{\alpha}{2}} \to 0. \tag{2.34}$$

Similarly to (2.33), for any $x = (0, x_n) \in R^n_+$, we derive that

$$+\infty > u_i(x_n) \ge C_0 \int_0^\infty y_n^\gamma u_1^{\alpha_i}(y_n) u_2^{\beta_i}(y_n) y_n^{\frac{\alpha}{2}} \frac{1}{|x_n - y_n|}\, dy_n x_n^{\frac{\alpha}{2}}. \tag{2.35}$$

Let $x_n = 2R$ be sufficiently large. By (2.35), we deduce that

$$\begin{aligned} +\infty > u_i(x_n) &\ge C_0 \int_0^1 y_n^\gamma u_1^{\alpha_i}(y_n) u_2^{\beta_i}(y_n) y_n^{\frac{\alpha}{2}} \frac{1}{|x_n - y_n|}\, dy_n x_n^{\frac{\alpha}{2}} \\ &\ge \frac{C_0}{2R} (2R)^{\frac{\alpha}{2}} \int_0^1 y_n^\gamma u_1^{\alpha_i}(y_n) u_2^{\beta_i}(y_n) y_n^{\frac{\alpha}{2}}\, dy_n \ge C_1 (2R)^{\frac{\alpha}{2}-1} = C_1 x_n^{\frac{\alpha}{2}-1}. \end{aligned} \tag{2.36}$$

Then by (2.35) and (2.36), for $x_n = 2R$ sufficiently large, we also obtain

$$\begin{aligned} u_i(x_n) &\geq C_0 \int_{\frac{R}{2}}^{R} y_n^{\gamma} u_1^{\alpha_i}(y_n) u_2^{\beta_i}(y_n) y_n^{\frac{\alpha}{2}} \frac{1}{|x_n - y_n|} \, dy_n x_n^{\frac{\alpha}{2}} \\ &\geq C_0 \int_{\frac{R}{2}}^{R} y_n^{\gamma} C_1^{\alpha_i+\beta_i} y_n^{(\alpha_i+\beta_i)(\frac{\alpha}{2}-1)} y_n^{\frac{\alpha}{2}} \frac{1}{|x_n - y_n|} \, dy_n x_n^{\frac{\alpha}{2}} \\ &\geq C_0 C_1^{\alpha_i+\beta_i} R^{(\alpha_i+\beta_i)(\frac{\alpha}{2}-1)+\gamma} (2R)^{\frac{\alpha}{2}} \frac{2}{3R} \int_{\frac{R}{2}}^{R} y_n^{\frac{\alpha}{2}} \, dy_n \\ &:= A R^{(\alpha_i+\beta_i)(\frac{\alpha}{2}-1)+\gamma+\alpha} := A_1 x_n^{(\alpha_i+\beta_i)(\frac{\alpha}{2}-1)+\gamma+\alpha}. \end{aligned}$$

Continuing this way m times, for $x_n = 2R$, we have

$$u_i(x_n) \geq A(m, \alpha_i + \beta_i, \alpha, \gamma) x_n^{(\alpha_i+\beta_i)^m(\frac{\alpha}{2}-1)+\frac{(\alpha_i+\beta_i)^m-1}{\alpha_i+\beta_i-1}(\gamma+\alpha)}. \tag{2.37}$$

For any fixed α and γ in their respective domain, we choose m to be an integer greater than $\frac{-\alpha^2-\alpha\gamma+\gamma+3}{\alpha+\gamma}$ and 1. That is,

$$m \geq \max\left\{ \left\lceil \frac{-\alpha^2 - \alpha\gamma + \gamma + 3}{\alpha + \gamma} \right\rfloor + 1, 1 \right\}, \tag{2.38}$$

where $\lceil a \rfloor$ is the integer part of a.

We claim that, for such a choice of m, we have

$$\tau(\alpha_i + \beta_i) := \left[(\alpha_i + \beta_i)^m \left(\frac{\alpha}{2} - 1 \right) + \frac{(\alpha_i + \beta_i)^m - 1}{\alpha_i + \beta_i - 1} (\alpha + \gamma) \right] (\alpha_i + \beta_i) + \frac{\alpha}{2} + \gamma \geq 0. \tag{2.39}$$

We postpone the proof of (2.39) for a moment. Now by (2.37) and (2.39), we derive that

$$u_i^{\alpha_i+\beta_i}(x_n) x_n^{\frac{\alpha}{2}+\gamma} \geq A(m, \alpha_i + \beta_i, \alpha, \gamma) x_n^{\tau(\alpha_i+\beta_i)} \geq A(m, \alpha_i + \beta_i, \alpha, \gamma) > 0,$$

for all x_n sufficiently large. This contradicts (2.34). So there is no positive solution of (2.32). This implies that $u(x)$ must be constant. By our positive assumption on u, we have $u_i(x) = b_i > 0$, $i = 1, 2$. Taking u_i into (2.2), we have $0 = (-\Delta)^{\frac{\alpha}{2}} u_i(x) = x_n^{\gamma} u_1^{\alpha_i}(x) u_2^{\beta_i}(x) > 0$. This is impossible. Hence, in (2.10), c_i must be zero, $i = 1, 2$.

Now it is left to verify (2.39). In fact, if we let

$$\begin{aligned} f(\alpha_i + \beta_i) &:= \tau(\alpha_i + \beta_i)(\alpha_i + \beta_i - 1) \\ &= (\alpha_i + \beta_i)^{m+2} \left(\frac{\alpha}{2} - 1 \right) + (\alpha_i + \beta_i)^{m+1} \left(\frac{\alpha}{2} + \gamma + 1 \right) - \frac{\alpha}{2}(\alpha_i + \beta_i) - \frac{\alpha}{2} - \gamma, \end{aligned}$$

then

$$f'(\alpha_i + \beta_i) = (\alpha_i + \beta_i)^m \left[(m+2) \left(\frac{\alpha}{2} - 1 \right) (\alpha_i + \beta_i) + (m+1) \left(\frac{\alpha}{2} + \gamma + 1 \right) \right] - \frac{\alpha}{2}.$$

We show that $f'(\alpha_i + \beta_i) > 0$, for $1 < \alpha_i + \beta_i \leq \frac{n+\alpha+2\gamma}{n-\alpha}$. Since $\alpha_i + \beta_i > 1$, it suffices to show $(m+2)(\frac{\alpha}{2} - 1)(\alpha_i + \beta_i) + (m+1)(\frac{\alpha}{2} + \gamma + 1) \geq \frac{\alpha}{2}$. Due to the fact $\frac{\alpha}{2} - 1 < 0$, $n \geq 3$, and $\alpha_i + \beta_i \leq$

$\frac{n+\alpha+2\gamma}{n-\alpha}$, we only need to verify that

$$(m+2)\left(\frac{\alpha}{2}-1\right)\frac{3+\alpha+2\gamma}{3-\alpha}+(m+1)\left(\frac{\alpha}{2}+\gamma+1\right)\geq\frac{\alpha}{2},$$

which can be derived directly from (2.38).

On the other hand, assume that $u(x)$ is a solution of the integral equation (1.6). Then, for any $\phi\in C_0^\infty(R_+^n)$, we have

$$\begin{aligned}
\langle(-\Delta)^{\frac{\alpha}{2}}u_i,\phi\rangle &= \left\langle\int_{R_+^n}G_\infty(x,y)y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\,dy,(-\Delta)^{\frac{\alpha}{2}}\phi(x)\right\rangle \\
&= \int_{R_+^n}\left\{\int_{R_+^n}G_\infty(x,y)y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\,dy\right\}(-\Delta)^{\frac{\alpha}{2}}\phi(x)\,dx \\
&= \int_{R_+^n}\left\{\int_{R_+^n}G_\infty(x,y)(-\Delta)^{\frac{\alpha}{2}}\phi(x)\,dx\right\}y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\,dy \\
&= \int_{R_+^n}\left\{\int_{R_+^n}\delta(x-y)\phi(x)\,dx\right\}y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\,dy \\
&= \int_{R_+^n}y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\phi(y)\,dy=\langle y_n^\gamma u_1^{\alpha_i}u_2^{\beta_i},\phi\rangle.
\end{aligned}$$

Hence u also satisfies equations (1.3).

This completes the proof of Theorem 1.1. □

3 Liouville theorems

3.1 The proof of Theorem 1.2

In this section, we will establish the nonexistence of the solutions to (1.6) by using the method of moving planes base up in the positive x_n direction.

For a given positive real number λ, define $\hat{\Sigma}_\lambda=\{x=(x_1,\dots,x_{n-1},x_n)\in R_+^n\mid 0<x_n<\lambda\}$, $\hat{T}_\lambda=\{x=(x_1,\dots,x_{n-1},x_n)\in R_+^n\mid x_n=\lambda\}$.

Let $x^\lambda=(x_1,x_2,\dots,x_{n-1},2\lambda-x_n)$ be the reflection of the point $x=(x_1,\dots,x_n)$ about the plane $\hat{T}_\lambda$, set $\Sigma_\lambda^c=R_+^n\backslash\hat{\Sigma}_\lambda$ the complement of $\hat{\Sigma}_\lambda$, and write $u_i^\lambda(x)=u_i(x^\lambda)$ and $w_i^\lambda(x)=u_i^\lambda(x)-u_i(x)$.

The following two lemmas are the key ingredient in our integral estimate.

Lemma 3.1 ([5]) (i) *For any $x,y\in\hat{\Sigma}_\lambda$, $x\neq y$, we have*

$$\begin{aligned}
&G(x^\lambda,y^\lambda)>\max\{G(x^\lambda,y),G(x,y^\lambda)\}, \\
&G(x^\lambda,y^\lambda)-G(x,y)>\left|G(x^\lambda,y)-G(x,y^\lambda)\right|.
\end{aligned}$$

(ii) *For any $x\in\hat{\Sigma}_\lambda$, $y\in\Sigma_\lambda^C$, we have $G(x^\lambda,y)>G(x,y)$.*

Lemma 3.2 *For any $x\in\hat{\Sigma}_\lambda$, u_i are the positive solution of* (1.6), *we have*

$$u_i(x)-u_i^\lambda(x)\leq\int_{\hat{\Sigma}_\lambda}\left[G(x^\lambda,y^\lambda)-G(x,y^\lambda)\right]y_n^\gamma\left[u_1^{\alpha_i}(y)u_2^{\beta_i}(y)-u_1^{\alpha_i}(y^\lambda)u_2^{\beta_i}(y^\lambda)\right]dy.$$

Proof Let $\tilde{\Sigma}_\lambda$ be the reflection of $\hat{\Sigma}_\lambda$ about the plane $\hat{T}_\lambda$. we have

$$\begin{aligned}
u_i(x) &= \int_{R^n_+} G(x,y)y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\,dy \\
&= \int_{\hat{\Sigma}_\lambda} G(x,y)y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\,dy \\
&\quad + \int_{\hat{\Sigma}_\lambda} G\bigl(x,y^\lambda\bigr)\bigl(y_n^\lambda\bigr)^\gamma u_1^{\alpha_i}\bigl(y^\lambda\bigr)u_2^{\beta_i}\bigl(y^\lambda\bigr)\,dy + \int_{\Sigma^c_\lambda\setminus\tilde{\Sigma_\lambda}} G(x,y)y_n^\gamma u_\lambda^{\alpha_i}(y)u_\lambda^{\beta_i}(y)\,dy, \\
u_i\bigl(x^\lambda\bigr) &= \int_{\hat{\Sigma}_\lambda} G\bigl(x^\lambda,y\bigr)y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\,dy \\
&\quad + \int_{\hat{\Sigma}_\lambda} G\bigl(x^\lambda,y^\lambda\bigr)\bigl(y_n^\lambda\bigr)^\gamma u_1^{\alpha_i}\bigl(y^\lambda\bigr)u_2^{\beta_i}\bigl(y^\lambda\bigr)\,dy + \int_{\Sigma^c_\lambda\setminus\tilde{\Sigma}_\lambda} G\bigl(x^\lambda,y\bigr)y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\,dy.
\end{aligned}$$

By Lemma 3.1, we arrive at

$$\begin{aligned}
u_i(x) - u_i\bigl(x^\lambda\bigr) &\le \int_{\hat{\Sigma}_\lambda} \bigl[G(x,y) - G\bigl(x^\lambda,y\bigr)\bigr]y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y)\,dy \\
&\quad - \int_{\hat{\Sigma}_\lambda} \bigl[G\bigl(x^\lambda,y^\lambda\bigr) - G\bigl(x,y^\lambda\bigr)\bigr]\bigl(y_n^\lambda\bigr)^\gamma u_1^{\alpha_i}\bigl(y^\lambda\bigr)u_2^{\beta_i}\bigl(y^\lambda\bigr)\,dy \\
&= \int_{\hat{\Sigma}_\lambda} \bigl[G\bigl(x^\lambda,y^\lambda\bigr) - G\bigl(x,y^\lambda\bigr)\bigr]\bigl[y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y) - \bigl(y_n^\lambda\bigr)^\gamma u_1^{\alpha_i}\bigl(y^\lambda\bigr)u_2^{\beta_i}\bigl(y^\lambda\bigr)\bigr]\,dy \\
&\le \int_{\hat{\Sigma}_\lambda} \bigl[G\bigl(x^\lambda,y^\lambda\bigr) - G\bigl(x,y^\lambda\bigr)\bigr]\bigl[y_n^\gamma u_1^{\alpha_i}(y)u_2^{\beta_i}(y) - \bigl(y_n^\lambda\bigr)^\gamma u_1^{\alpha_i}\bigl(y^\lambda\bigr)u_2^{\beta_i}\bigl(y^\lambda\bigr)\bigr]\,dy \\
&\le \int_{\hat{\Sigma}_\lambda} \bigl[G\bigl(x^\lambda,y^\lambda\bigr) - G\bigl(x,y^\lambda\bigr)\bigr]y_n^\gamma\bigl[u_1^{\alpha_i}(y)u_2^{\beta_i}(y) - u_1^{\alpha_i}\bigl(y^\lambda\bigr)u_2^{\beta_i}\bigl(y^\lambda\bigr)\bigr]\,dy.
\end{aligned}$$

This completes the proof of Lemma 3.2. □

In order to prove the Liouville theorem, Theorem 1.2, we carry out the method of moving planes in integral form in the positive x_n direction.

The proof consists of two steps. In the first step, we start from the very low end of our region R^n_+, *i.e.* near $x_n = 0$ and show that, for λ sufficiently small,

$$w_i^\lambda(x) = u_i^\lambda(x) - u_i(x) \ge 0, \quad \text{a.e. } \forall x \in \hat{\Sigma}_\lambda. \tag{3.1}$$

In the second step, we will move our plane $\hat{T}_\lambda$ up in the positive x_n direction as long as the inequality (3.1) holds and show that $u(x)$ is monotone increasing in x_n and thus derive a contradiction.

Step 1. Define $\Gamma_i^\lambda = \{x \in \hat{\Sigma}_\lambda \mid u_i^\lambda(x) < u_i(x)\}$, $i = 1,2$ and $\Sigma_i^\lambda = \{x \in \hat{\Sigma}_\lambda \mid u_1^{\alpha_i}(x^\lambda)u_2^{\beta_i}(x^\lambda) < u_1^{\alpha_i}(x)u_2^{\beta_i}(x)\}$. We show that, for λ sufficiently small, Γ_i^λ must be measure zero. In fact, for any $x \in \Gamma_i^\lambda$, by the mean value theorem similar to (2.16) and Lemma 3.2, we obtain

$$\begin{aligned}
u_i(x) - u_i^\lambda(x) &\le \int_{\Sigma_i^\lambda} G\bigl(x^\lambda,y^\lambda\bigr)y_n^\gamma\bigl[u_1^{\alpha_i}(y)u_2^{\beta_i}(y) - u_1^{\alpha_i}\bigl(y^\lambda\bigr)u_2^{\beta_i}\bigl(y^\lambda\bigr)\bigr]\,dy \\
&= \int_{\Sigma_i^\lambda} G\bigl(x^\lambda,y^\lambda\bigr)y_n^\gamma\bigl[\bigl(u_1^{\alpha_i}(y) - u_1^{\alpha_i}\bigl(y^\lambda\bigr)\bigr)u_2^{\beta_i}(y) + u_1^{\alpha_i}(y)\bigl(u_2^{\beta_i}(y) - u_2^{\beta_i}\bigl(y^\lambda\bigr)\bigr)\bigr]\,dy
\end{aligned}$$

$$\begin{aligned} &\leq \int_{\Sigma_i^\lambda} G(x^\lambda, y^\lambda) y_n^\gamma \big[u_1^{\alpha_i}(y) - u_1^{\alpha_i}(y^\lambda)\big] u_2^{\beta_i}(y)\,dy \\ &\leq c\int_{\Sigma_i^\lambda} G(x^\lambda, y^\lambda) y_n^\gamma u_1^{\alpha_i-1}(y)\big[u_1(y) - u_1(y^\lambda)\big] u_2^{\beta_i}(y)\,dy. \end{aligned} \tag{3.2}$$

By the expression of $G(x,y)$, it is easy to see $G(x,y) \leq \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}}$. From (3.2), we have

$$1u_i(x) - u_i^\lambda(x) \leq c\int_{\Sigma_i^\lambda} \frac{1}{|x-y|^{n-\alpha}} \big|y_n^\gamma u^{\alpha_i+\beta_i-1}(y)\big|\big|w^\lambda(y)\big| \tag{3.3}$$

$$\leq c\int_{\Sigma_i^\lambda} \frac{1}{|x-y|^{n-\alpha}} \big|u^{\alpha_i+\beta_i-1}(y)\big|\big|w^\lambda(y)\big|. \tag{3.4}$$

Notice that now γ is only a little larger than 0, so y_n is bounded within Σ_i^λ, and since $\gamma \geq 0$, we get $|y_n^\gamma| \leq C$, hence we derive (3.4) from (3.3).

Noticing $\Sigma_i^\lambda \subseteq \Gamma_j^\lambda$ for some j, applying Hardy-Littlewood-Sobolev inequality and the Hölder inequality we obtain, for any $q > \frac{n}{n-\alpha}$,

$$\|w_\lambda\|_{L^q(\Gamma_\lambda)} \leq c\big\||u|^{\alpha_i+\beta_i-1} w_\lambda\big\|_{L^{\frac{nq}{n+\alpha q}}(\Gamma_\lambda)} \leq c\big\||u|^{\alpha_i+\beta_i-1}\big\|_{L^{\frac{n}{\alpha}}(\Gamma_\lambda)} \|w_\lambda\|_{L^q(\Gamma_\lambda)}, \tag{3.5}$$

where $\Gamma_\lambda = \Gamma_1^\lambda \cup \Gamma_2^\lambda$.

Since $|u| \in L^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(R_+^n)$, we can choose sufficiently small positive λ such that

$$c\big\|u^{\alpha_i+\beta_i-1}\big\|_{L^{\frac{n}{\alpha}}(\Gamma_\lambda)} = c\left\{\int_{\Gamma_\lambda} |u|^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(y)\,dy\right\}^{\frac{\alpha}{n}} \leq \frac{1}{2}. \tag{3.6}$$

By (3.5) and (3.6), we derive $\|w_\lambda\|_{L^q(\Gamma_\lambda)} = 0$, and Γ_i^λ must be of measure zero, hence (3.1) holds. This provides us a starting point for moving the plane.

Step 2. Now we start from such small λ and move the plane $\hat{T}_\lambda$ up as long as (3.1) holds. Define

$$\lambda_0 = \sup\big\{\lambda \mid w_\rho(x) \geq 0, \rho \leq \lambda, \forall x \in \hat{\Sigma}_\rho\big\}.$$

We will prove

$$\lambda_0 = +\infty. \tag{3.7}$$

Suppose to the contrary that $\lambda_0 < +\infty$, we will show that $u_i(x)$ is symmetric about the plane $\hat{T}_{\lambda_0}$, *i.e.*

$$u_i^{\lambda_0}(x) \equiv u_i(x), \quad \text{a.e. } \forall x \in \hat{\Sigma}_{\lambda_0}. \tag{3.8}$$

This will contradict the strict positivity of $u_i(x)$. Suppose (3.8) does not hold. Then, for such a λ_0, we have $u_i^{\lambda_0}(x) \geq u_i(x)$, but $u_i^{\lambda_0}(x) \not\equiv u_i(x)$ a.e. on $\hat{\Sigma}_{\lambda_0}$. We show that the plane can be moved further up. More precisely, there exists an $\epsilon > 0$ such that for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$

$$u_i^\lambda(x) \geq u_i(x), \quad \text{a.e. on } \hat{\Sigma}_\lambda. \tag{3.9}$$

To verify this, we will again resort to inequality (3.5). If one can show that, for ϵ sufficiently small so that, for all λ in $[\lambda_0, \lambda_0 + \epsilon)$, we have

$$c\left\{\int_{\Gamma_\lambda} |u|^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(y)\,dy\right\}^{\frac{\alpha}{n}} \le \frac{1}{2}, \tag{3.10}$$

then by (3.3) and (3.10), we have $\|w_{\lambda_0}\|_{L^q(\Gamma_\lambda)} = 0$, and therefore Γ^λ must be of measure zero. Hence, for these values of $\lambda > \lambda_0$, we have (3.9). This contradicts the definition of λ_0. Therefore (3.8) must hold.

The proof of the inequality (3.10) is similar to the argument of the inequality (2.26) in Section 2 and the proof is standard.

By (3.8), we derive that $u_i(x) = 0$ on the plane $x_n = 2\lambda_0$, the symmetric image of the boundary ∂R^n_+ with respect to the plane $\hat{T}_{\lambda_0}$. This contradicts our assumption $u_i(x) > 0$ in R^n_+. Therefore, (3.7) must be valid. Now we have proved that the positive solution of (1.6) is monotone increasing with respect to x_n, and this contradicts $u^{\alpha_i+\beta_i-1} \in L^{\frac{n}{\alpha}}(R^n_+)$. Therefore the positive solutions of (1.6) do not exist.

This completes the proof of Theorem 1.2.

3.2 The proof of Theorem 1.3

In this section, we will use a proper Kelvin type transforms and derive the nonexistence of positive solutions for (1.6) in R^n_+ under much weaker conditions, *i.e.* the solution u of (1.6) is only locally integrable and locally bounded.

With no explicit global integrability assumptions on the solution u, we cannot directly carry out the method of moving planes on u. To overcome this difficulty, we employ Kelvin type transforms.

For $z^0 \in R^n_+$, let

$$\bar{u}_i(x) = \frac{1}{|x - z^0|^{n-\alpha}} u_i\left(\frac{x - z^0}{|x - z^0|^2} + z^0\right) \tag{3.11}$$

be the Kelvin transform of $u_i(x)$ centered at z^0.

Through a straightforward calculation, we have $\bar{u}_i(x) = \int_{R^n_+} G(x,y)\frac{y_n^\gamma \bar{u}_1^{\alpha_i}(y)\bar{u}_2^{\beta_i}(y)}{|y-z^0|^\delta}\,dy$, $\forall x \in R^n_+ \backslash B_\epsilon(z^0)$, where $\delta = n + \alpha + 2\gamma - (n - \alpha)(\alpha_i + \beta_i)$, $\epsilon > 0$, $i = 1, 2$.

Proof of Theorem 1.3 *in the subcritical case* $1 < \alpha_i + \beta_i < \frac{n+\alpha+2\gamma}{n-\alpha}$:

$$\bar{u}_i(x) = \int_{R^n_+} G(x,y)\frac{y_n^\gamma \bar{u}_1^{\alpha_i}(y)\bar{u}_2^{\beta_i}(y)}{|y - z^0|^\delta}\,dy, \quad \forall x \in R^n_+ \backslash B_\epsilon(z^0), \tag{3.12}$$

where $\delta = n + \alpha + 2\gamma - (n - \alpha)(\alpha_i + \beta_i) > 0$, $\epsilon > 0$.

This specific proof is the same as the proof of Theorem 1.1 and we omit here. □

Remark When we carry out the method of moving planes on equation (1.4), we derive the fact $c_i = 0$ and consequently obtain the equivalence. While applying the same method on equation (1.6), surprisingly, we arrive at a Liouville type theorem for it.

Proof of Theorem 1.3 *in the critical case* $1 < \alpha_i + \beta_i = \frac{n+\alpha+2\gamma}{n-\alpha}$:

$$u_i(x) = \int_{R_+^n} G(x,y) y_n^\gamma u_1^{\alpha_i}(y) u_2^{\beta_i}(y)\, dy. \tag{3.13}$$

By the Kelvin transform of $u_i(x)$ we derive

$$\bar{u}_i(x) = \int_{R_+^n} G(x,y) y_n^\gamma \bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y)\, dy. \tag{3.14}$$

If $u(x)$ is a solution of (3.13), then $\bar{u}(x)$ is also a solution of (3.14). Therefore, by our assumption $|u| \in L_{\mathrm{loc}}^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(R_+^n)$, we derive $y_n^\gamma |u|^{\alpha_i+\beta_i-1} \in L_{\mathrm{loc}}^{\frac{n}{\alpha}}(R_+^n)$. then

$$\int_{\hat{\Omega}} \left[y_n^\gamma \bar{u}^{\alpha_i+\beta_i-1}(y)\right]^{\frac{n}{\alpha}} dy = \int_{\Omega} \left[y_n^\gamma u^{\alpha_i+\beta_i-1}(y)\right]^{\frac{n}{\alpha}} dy < \infty, \tag{3.15}$$

where $\hat{\Omega}$ is the image of Ω about the Kelvin transform. Now we consider two possibilities.

Possibility 1. If there is a $z^0 = (z_1^0, \dots, z_{n-1}^0, 0) \in \partial R_+^n$ such that $\bar{u}_i(x)$ is bounded near z^0, then by (3.11), we obtain

$$u_i(y) = \frac{1}{|y - z^0|^{n-\alpha}} \bar{u}_i\left(\frac{y - z^0}{|y - z^0|^2} + z^0\right).$$

And we further deduce

$$u_i(y) = O\left(\frac{1}{|y|^{n-\alpha}}\right), \quad \text{as } |y| \to \infty. \tag{3.16}$$

Since $\alpha_i + \beta_i = \frac{n+\alpha+2\gamma}{n-\alpha} > \frac{n}{n-\alpha}$ and $|u| \in L_{\mathrm{loc}}^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(R_+^n)$, together with (3.16), we have

$$\int_{R_+^n} u^{\frac{n(\alpha_i+\beta_i-1)}{\alpha}}(y)\, dy \le c \int_{R_+^n} \frac{1}{|y|^{\frac{n(\alpha_i+\beta_i-1)(n-\alpha)}{\alpha}}}\, dy < \infty. \tag{3.17}$$

In this situation, we still carry on the moving planes on u. Going through exactly the same arguments as in the proof of Theorem 1.2, we obtain the nonexistence of positive solutions for (3.13).

Possibility 2. For all $z^0 = (z_1^0, \dots, z_{n-1}^0, 0) \in \partial R_+^n$, $\bar{u}_i(x)$ is unbounded near z^0, we will carry out the method of moving planes on $\bar{u}(x)$ in R^{n-1} to prove that it is rotationally symmetric about the line passing through z^0 and parallel to the x_n-axis. From this, we will deduce that u is independent of the first $n-1$ variables $x_1, \dots, x_{n-1}$. That is $u = u(x_n)$, and we will derive a contradiction with the finiteness of $\int_{R_+^n} G(x,y) y_n^\gamma u_1^{\alpha_i}(y) u_2^{\beta_i}(y)\, dy$.

For a given real number λ, the notations such as Σ_λ, T_λ are the same as the ones in Section 2. By (2.12), obviously we have

$$\bar{u}_i(x) = \int_{\Sigma_\lambda} G(x,y) y_n^\gamma \bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y)\, dy + \int_{\Sigma_\lambda} G\left(x^\lambda, y\right) y_n^\gamma \bar{u}_1^{\alpha_i}\left(y^\lambda\right) \bar{u}_2^{\beta_i}\left(y^\lambda\right) dy,$$

$$\bar{u}_i^\lambda(x) = \int_{\Sigma_\lambda} G\left(x^\lambda, y\right) y_n^\gamma \bar{u}_1^{\alpha_i}(y) \bar{u}_2^{\beta_i}(y)\, dy + \int_{\Sigma_\lambda} G(x,y) y_n^\gamma \bar{u}_1^{\alpha_i}\left(y^\lambda\right) \bar{u}_2^{\beta_i}\left(y^\lambda\right) dy.$$

By elementary calculation we derive

$$\bar{u}_i(x) - \bar{u}_i^{\lambda}(x) = \int_{\Sigma_\lambda} \left[G(x,y) - G\left(x^\lambda, y\right)\right] y_n^\gamma \left[\bar{u}_1^{\alpha_i}(y)\bar{u}_2^{\beta_i}(y) - \bar{u}_1^{\alpha_i}\left(y^\lambda\right)\bar{u}_2^{\beta_i}\left(y^\lambda\right)\right] dy. \tag{3.18}$$

We will move the plane T_λ along the direction of the x_1-axis to show that the solution is rotationally symmetric about the line passing through z^0 and parallel to the x_n-axis. The proof is the same as the proof of $c_i = 0$ in Section 2, in fact we only need to apply arguments of the inequality (2.13) to equation (3.18). Similarly, we derive that $u_i = u_i(x_n)$ and any positive solution u of (3.13) must be $u(x) \equiv 0$. This implies that there is no positive solution of (3.13) in the critical case.

This completes the proof of Theorem 1.3. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZD participated in the method of moving plane studies in the paper and drafted the manuscript; LC carried out the Liouville type theorem, and PW carried out the evaluation of inequalities. All authors read and approved the final manuscript.

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