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A generalized Lyapunov inequality for a higher-order fractional boundary value problem

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Abstract
In the paper, we establish a Lyapunov inequality and two Lyapunov-type inequalities for a higher-order fractional boundary value problem with a controllable nonlinear term. Two applications are discussed. One concerns an eigenvalue problem, the other a Mittag-Leffler function.

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1 Introduction

For the following boundary value problem (BVP for short):

$$\begin{cases} y''(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.1}$$

where q is a real and continuous function, Lyapunov [1] proved that if (1.1) has a nontrivial solution, then

$$\int_a^b \left|q(s)\right| ds > \frac{4}{b-a}. \tag{1.2}$$

The expression in (1.2) is called the Lyapunov inequality. The Lyapunov inequality proved to be very useful in various problems related with differential equations. Many improvements and generalizations of the inequality (1.2) for integer-order BVP have appeared in the literature, and here we omit these detailed conclusions but only refer the reader to a summary reference [2] given by Tiryaki in 2010, in which research results about Lyapunov inequality were summarized. After that, new results for the integer-order boundary value problem appeared continuously; see [3–9].

Only in recent years, the research of Lyapunov inequality for fractional BVP has begun in which a fractional derivative (Riemann-Liouville derivative and Caputo derivative) is used instead of the classical ordinary derivative in differential equation. We refer the reader to Ferreira [10–12], Jleli and Samet [13], Rong and Bai [14], Arifi *et al.* [15], and so on.

About the basics for fractional calculus, we refer the reader to [16–19] and the references therein.

Most of the above cited work deals with lower-order fractional BVP. Then in 2015, O'Regan and Samet gave the following result for a higher-order fractional BVP.

Theorem 1.1 ([20]) *If the fractional BVP*

$$\begin{cases} ({}_a^R D^\nu y)(t) + q(t)y(t) = 0, & a < t < b, 3 < \nu \leq 4, \\ y(a) = y'(a) = y''(a) = y''(b) = 0, \end{cases}$$

has a nontrivial solution, where ${}_a^R D^\nu$ is the Riemann-Liouville derivative of order ν, then the following Lyapunov inequality holds:

$$\int_a^b |q(s)|\, ds > \frac{\Gamma(\nu)(\nu-2)^{\nu-2}}{2(\nu-3)^{\nu-3}(b-a)^{\nu-1}}.$$

In a recent paper [21] in 2016, Cabrera, Sadarangani, and Samet gave the following result for a higher-order fractional BVP.

Theorem 1.2 ([21]) *If the fractional BVP*

$$\begin{cases} ({}_a^R D^\nu y)(t) + q(t)y(t) = 0, & a < t < b, 2 < \nu \leq 3, \\ y(a) = y'(a) = 0, & y'(b) = \beta y'(\xi), \end{cases}$$

has a nontrivial solution, where ${}_a^R D^\nu$ is the Riemann-Liouville derivative of order ν, then the following Lyapunov inequality holds:

$$\int_a^b |q(s)|\, ds > \frac{\Gamma(\nu)(\nu-1)^{\nu-1}}{(\nu-2)^{\nu-2}(b-a)^{\nu-1}(1 + \frac{\beta(b-a)^{\nu-1}}{(\nu-1)(b-a)^{\nu-2}-\beta(\xi-a)^{\nu-1}})}.$$

Whether for lower-order fractional BVP in [10–14], or for higher-order fractional BVP in [20, 21], we see the BVP of concern is all with a linear term $q(t)y(t)$. The BVP in [15] is with a half-linear term $q(t)|y(t)|^{p-2}y(t)$. In a recent paper [22], Chidouh and Torres extended the linear term $q(t)y(t)$ to a nonlinear term $q(t)f(y(t))$. The following result has been established.

Theorem 1.3 ([22]) *Let q be a real nontrival Lebesgue integral function. Assume that $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a concave and nondecreasing function. If the fractional BVP*

$$\begin{cases} ({}_a^R D^\nu y)(t) + q(t)f(y(t)) = 0, & a < t < b, 1 < \nu \leq 2, \\ y(a) = y(b) = 0, \end{cases}$$

has a nontrivial solution $y \in C[a,b]$, where ${}_a^R D^\nu$ is the Riemann-Liouville derivative of order ν, then the following Lyapunov inequality holds:

$$\int_a^b |q(s)|\, ds > \frac{4^{\nu-1}\Gamma(\nu)\eta}{(b-a)^{\nu-1}f(\eta)},$$

where $\eta = \max_{t\in[a,b]} y(t)$.

But it is obvious that the Lyapunov inequality above is not controllable because a dummy variable η appears.

Motivated by [10–15] and [20–22], this paper aims to study the Lyapunov inequality and Lyapunov-type inequality for the following higher-order fractional BVP with a nonlinear term $q(t)f(y(t))$:

$$\begin{cases} ({}_a^C D^\nu y)(t) + q(t)f(y) = 0, & a < t < b, 2 < \nu \leq 3, \\ y(a) = y(b) = y''(a) = 0, \end{cases} \tag{1.3}$$

where ${}_a^C D^\nu$ is the Caputo derivative of order ν, $q : [a,b] \to \mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$ is a Lebesgue integrable function, $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

The paper is organized as follows. In Section 2, we present some preliminary conclusions. In Section 3, assuming that f is controllable, we obtain the Lyapunov inequality and Lyapunov-type inequality for (1.3). Applications of these inequalities to an eigenvalue problem and to a Mittag-Leffler function are given in Section 4.

2 Preliminaries

We first give the concepts of the Riemann-Liouville fractional integral and the Caputo fractional derivative.

Definition 2.1 *Let $\nu \geq 0$ and $\Gamma(\nu)$ be the Gamma function defined by $\Gamma(\nu) = \int_0^\infty t^{\nu-1}e^{-t}\,dt$, $\nu \geq 0$. The Riemann-Liouville fractional integral of order ν of $y(t)$ is defined by ${}_aI^0 y = y$ and*

$$\left({}_aI^\nu y\right)(t) := \frac{1}{\Gamma(\nu)} \int_a^t (t-s)^{\nu-1} y(s)\,ds, \quad t \in [a,b], \nu > 0.$$

Definition 2.2 *Let $\nu > 0$, $n = [\nu] + 1$, where $[\nu]$ denotes the integer part of number ν. The Caputo fractional derivative of order ν of $y(t)$ is defined by*

$$\left({}_a^C D^\nu y\right)(t) := \left({}_aI^{n-\nu} y^{(n)}\right)(t) = \frac{1}{\Gamma(n-\nu)} \int_a^t \frac{y^{(n)}(s)}{(t-s)^{\nu+1-n}}\,ds, \quad t \in [a,b].$$

Lemma 2.1 *$y \in C[a,b]$ is a solution to the problem* (1.3) *if and only if y satisfies the integral equation*

$$y(t) = \int_a^b G(t,s) q(s) f\left(y(s)\right) ds,$$

where $G(t,s)$ is the Green's function of the problem (1.3) *and*

$$G(t,s) = \begin{cases} \frac{(t-a)(b-s)^{\nu-1}}{\Gamma(\nu)(b-a)} - \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}, & a \leq s \leq t \leq b, \\ \frac{(t-a)(b-s)^{\nu-1}}{\Gamma(\nu)(b-a)}, & a \leq t \leq s \leq b. \end{cases} \tag{2.1}$$

Proof Suppose $y(t)$ is the solution of (1.3), then

$$\begin{aligned} y(t) &= \left({}_aI^{\nu\,C} D_a^\nu y\right)(t) + c_1(t-a)^2 + c_2(t-a) + c_3 \\ &= -\frac{1}{\Gamma(\nu)} \int_a^t (t-s)^{\nu-1} q(s) f\left(y(s)\right) ds + c_1(t-a)^2 + c_2(t-a) + c_3. \end{aligned} \tag{2.2}$$

Combining $2 < \nu \leq 3$ and $y(a) = y(b) = y''(a) = 0$, we get $c_2 = \frac{1}{(b-a)\Gamma(\nu)} \int_a^b (b-s)^{\nu-1} q(s) \times f(y(s))\,ds$ and $c_1 = c_3 = 0$. Substituting c_1, c_2, and c_3 into (2.2) yields

$$\begin{aligned} y(t) &= \frac{1}{(b-a)\Gamma(\nu)} \int_a^b (t-a)(b-s)^{\nu-1} q(s) f\bigl(y(s)\bigr)\,ds - \frac{1}{\Gamma(\nu)} \int_a^t (t-s)^{\nu-1} q(s) f\bigl(y(s)\bigr)\,ds \\ &= \int_a^b G(t,s) q(s) f\bigl(y(s)\bigr)\,ds. \end{aligned}$$

□

Lemma 2.2 *The Green's function $G(t,s)$ defined in* (2.1) *has the following properties.*

(i) *$G(t,s) \geq 0$, for all $a \leq t, s \leq b$.*

(ii) *For any $s \in [a,b]$,*

$$\max_{t\in[a,b]} G(t,s) = G(t_0,s) = \frac{(s-a)(b-s)^{\nu-1}}{(b-a)\Gamma(\nu)} + \frac{(\nu-2)(b-s)^{\frac{(\nu-1)^2}{\nu-2}}}{(\nu-1)^{\frac{\nu-1}{\nu-2}}(b-a)^{\frac{\nu-1}{\nu-2}}\Gamma(\nu)},$$

where $t_0 = s + (\frac{(b-s)^{\nu-1}}{(b-a)(\nu-1)})^{\frac{1}{\nu-2}} \in [s,b]$.

(iii) *$G(t,s) \geq \frac{(t-a)(b-t)}{(b-a)^2} G(t_0,s)$ for all $a \leq t, s \leq b$.*

(iv) $\max_{s\in[a,b]} G(t_0,s) \leq \frac{(b-a)^{\nu-1}}{\Gamma(\nu)}$.

Proof (i) Considering $\nu > 2$, we have $\frac{t-a}{b-a} \geq (\frac{t-a}{b-a})^{\nu-1}$. We also get $(\frac{(t-a)(b-s)}{b-a})^{\nu-1} \geq (t-s)^{\nu-1}$ because $\frac{(t-a)(b-s)}{b-a} \geq (t-s)$. So when $s \leq t$, we have

$$\begin{aligned} \Gamma(\nu) G(t,s) &= \left(\frac{t-a}{b-a}\right)(b-s)^{\nu-1} - (t-s)^{\nu-1} \\ &\geq \left(\frac{t-a}{b-a}\right)^{\nu-1} (b-s)^{\nu-1} - (t-s)^{\nu-1} \\ &= \left(\frac{(t-a)(b-s)}{b-a}\right)^{\nu-1} - (t-s)^{\nu-1} \geq 0, \end{aligned}$$

which means $G(t,s) \geq 0$. When $s \geq t$, the conclusion holds obviously.

(ii) For any $s \in [a,b]$, when $t \geq s$, by (2.1),

$$\Gamma(\nu) G'_t(t,s) = \frac{(b-s)^{\nu-1}}{b-a} - (\nu-1)(t-s)^{\nu-2} \begin{cases} \geq 0, & t \leq t_0, \\ \leq 0, & t \geq t_0, \end{cases}$$

where $t_0 = s + (\frac{(b-s)^{\nu-1}}{(b-a)(\nu-1)})^{\frac{1}{\nu-2}} \in [s,b]$. So

$$\max_{t\in[s,b]} G(t,s) = G(t_0,s) = \frac{(s-a)(b-s)^{\nu-1}}{(b-a)\Gamma(\nu)} + \frac{(\nu-2)(b-s)^{\frac{(\nu-1)^2}{\nu-2}}}{(\nu-1)^{\frac{\nu-1}{\nu-2}}(b-a)^{\frac{\nu-1}{\nu-2}}\Gamma(\nu)}. \tag{2.3}$$

When $t \leq s$, we easily get by (2.1)

$$\max_{t\in[a,s]} G(t,s) \leq G(s,s) \leq G(t_0,s). \tag{2.4}$$

Equations (2.3) and (2.4) show that (ii) is correct.

(iii) For any $s \in [a,b]$, when $t \le s$, by (2.1),

$$\begin{aligned} G(t,s) &= \frac{(t-a)(b-s)^{\nu-1}}{\Gamma(\nu)(b-a)} \\ &\ge \frac{(t-a)}{(b-a)} \cdot \frac{(t_0-a)(b-s)^{\nu-1}}{\Gamma(\nu)(b-a)} \\ &\ge \frac{(t-a)}{(b-a)} \left[\frac{(t_0-a)(b-s)^{\nu-1}}{\Gamma(\nu)(b-a)} - \frac{(t_0-s)^{\nu-1}}{\Gamma(\nu)} \right] \\ &= \frac{(t-a)}{(b-a)} G(t_0,s) \ge \frac{(t-a)(b-t)}{(b-a)^2} G(t_0,s). \end{aligned} \tag{2.5}$$

When $t > s$, by (2.1),

$$\Gamma(\nu) G''_{tt}(t,s) = -(\nu-1)(\nu-2)(t-s)^{\nu-3} \le 0,$$

which means that $G(t,s)$ is concave about t on $[s,b]$. Thus, for any $t \in [s,t_0]$, by the concavity of $G(t,s)$, we have

$$\begin{aligned} G(t,s) &\ge \frac{G(t_0,s) - G(s,s)}{t_0 - s}(t-s) + G(s,s) \\ &= \frac{G(t_0,s)}{t_0-s}(t-s) + \frac{t_0-t}{t_0-s} G(s,s) \\ &\ge \frac{G(t_0,s)}{t_0-s}(t-s) + \frac{(t_0-t)}{(t_0-s)} \frac{(s-a)}{(b-a)} G(t_0,s) \\ &\ge \frac{(t-a)}{(b-a)} G(t_0,s) \ge \frac{(t-a)(b-t)}{(b-a)^2} G(t_0,s). \end{aligned} \tag{2.6}$$

For any $t \in [t_0,1]$, by the concavity of $G(t,s)$, we have

$$\begin{aligned} G(t,s) &\ge \frac{G(t_0,s) - G(b,s)}{t_0 - b}(t-b) + G(b,s) \\ &= \frac{G(t_0,s)}{b-t_0}(b-t) \\ &\ge \frac{(b-t)}{(b-a)} G(t_0,s) \ge \frac{(t-a)(b-t)}{(b-a)^2} G(t_0,s). \end{aligned} \tag{2.7}$$

(2.5), (2.6), and (2.7) tell us (iii) is correct.

In the end, we prove (iv). It is easy to see that $(\frac{(\nu-2)^{\nu-2}(b-s)}{(\nu-1)^{\nu-1}(b-a)})^{\frac{1}{\nu-2}} \le 1$, so

$$\begin{aligned} \max_{s\in[a,b]} G(t_0,s) &= \max_{s\in[a,b]} \left\{ \frac{(s-a)(b-s)^{\nu-1}}{(b-a)\Gamma(\nu)} + \frac{(\nu-2)(b-s)^{\frac{(\nu-1)^2}{\nu-2}}}{(\nu-1)^{\frac{\nu-1}{\nu-2}}(b-a)^{\frac{\nu-1}{\nu-2}}\Gamma(\nu)} \right\} \\ &= \frac{1}{(b-a)\Gamma(\nu)} \max_{s\in[a,b]} \left[(s-a)(b-s)^{\nu-1} + \left(\frac{(\nu-2)^{\nu-2}(b-s)}{(\nu-1)^{\nu-1}(b-a)} \right)^{\frac{1}{\nu-2}} (b-s)^{\nu} \right] \\ &\le \frac{1}{\Gamma(\nu)} \max_{s\in[a,b]} (b-s)^{\nu-1} = \frac{(b-a)^{\nu-1}}{\Gamma(\nu)}. \end{aligned}$$

□

Remark 2.1 In Lemma 2.2, we give an accurate bound for Green's function $G(t,s)$, not only an upper bound but also a lower bound, and coincidentally, the only difference between the two bounds is a function factor whose value is in $[0,1]$, which leads to Lyapunov-type inverse inequalities; see Theorem 3.1 and Corollary 3.1. In [10–15] and [20–22], only an upper bound of corresponding Green's function was given and thus only a one-side Lyapunov inequality could be obtained.

Let $E = C[a,b]$ be endowed with the norm $\|y\| = \max_{t\in[a,b]} |y(t)|$. Let $E^+ = \{y \in E, y(t) \geq 0$ for any $t \in [a,b]$ and $\|y\| \neq 0\}$.

Lemma 2.3 *Assume that $y \in E^+$ is a solution for* (1.3), *then*

$$\frac{(t-a)(b-t)}{(b-a)^2}\|y\| \leq y(t) \leq \|y\|, \quad \forall t \in [a,b].$$

Proof Suppose y is a positive solution of (1.3), then from Lemma 2.1 we know

$$y(t) = \int_a^b G(t,s)q(s)f\bigl(y(s)\bigr)\,ds.$$

Then by (iii) of Lemma 2.2,

$$\begin{aligned} y(t) &\geq \int_a^b \frac{(t-a)(b-t)}{(b-a)^2} G(t_0,s)q(s)f\bigl(y(s)\bigr)\,ds \\ &= \frac{(t-a)(b-t)}{(b-a)^2}\int_a^b \max_{t\in[a,b]} G(t,s)q(s)f\bigl(y(s)\bigr)\,ds \\ &\geq \frac{(t-a)(b-t)}{(b-a)^2}\max_{t\in[a,b]}\int_a^b G(t,s)q(s)f\bigl(y(s)\bigr)\,ds = \frac{(t-a)(b-t)}{(b-a)^2}\|y\|. \end{aligned}$$

□

3 Main results

A Lyapunov inequality and two Lyapunov-type inequalities are given in this section.

Theorem 3.1 *Assume that f is controlled by two lines, i.e., there exist two positive constants M and N satisfying $Ny \leq f(y) \leq My$ for any $y \in \mathbb{R}^+$. If* (1.3) *has a solution in E^+, then the following Lyapunov inequality* (3.1), *Lyapunov-type inequalities* (3.2) *and* (3.3) *hold*:

$$\int_a^b q(s)\,ds > \frac{\Gamma(\nu)}{M(b-a)^{\nu-1}}, \tag{3.1}$$

$$\int_a^b (s-a)^2(b-s)^\nu q(s)\,ds \leq \frac{4\Gamma(\nu)(b-a)^3}{N}, \tag{3.2}$$

$$\int_a^b (s-a)(b-s)^{\frac{\nu^2-\nu-1}{\nu-2}} q(s)\,ds \leq \frac{4\Gamma(\nu)(b-a)^{\frac{3\nu-5}{\nu-2}}(\nu-1)^{\frac{\nu-1}{\nu-2}}}{(\nu-2)N}. \tag{3.3}$$

Proof Suppose $y \in E^+$ is a solution for (1.3), then $\|y\| \neq 0$. From Lemma 2.1 and Lemma 2.3, we know that

$$y(t) = \int_a^b G(t,s)q(s)f\bigl(y(s)\bigr)\,ds, \quad t \in [a,b], \tag{3.4}$$

$$\frac{(s-a)(b-s)}{(b-a)^2}\|y\| \le y(s), \quad \forall s \in [a,b], \tag{3.5}$$

respectively. For any $t \in [a,b]$, by (3.4) and (iv) of Lemma 2.2, we have

$$\begin{aligned} 0 \le y(t) &\le \int_a^b G(t_0,s)q(s)My(s)\,ds \\ &< \int_a^b \max_{s\in[a,b]} G(t_0,s)q(s)\,dsM\|y\| \\ &\le \int_a^b \frac{(b-a)^{\nu-1}}{\Gamma(\nu)}q(s)\,dsM\|y\|, \end{aligned}$$

which yields

$$\int_a^b q(s)\,ds > \frac{\Gamma(\nu)}{M(b-a)^{\nu-1}}. \tag{3.6}$$

On the other hand, by (3.5), we get

$$f\bigl(y(s)\bigr) \ge Ny(s) \ge N\|y\|\frac{(s-a)(b-s)}{(b-a)^2}, \quad \forall s \in [a,b]. \tag{3.7}$$

From (3.4) and (3.7), together with (iii) of Lemma 2.2, we get

$$\begin{aligned} y\left(\frac{a+b}{2}\right) &= \int_a^b G\left(\frac{a+b}{2},s\right)q(s)f\bigl(y(s)\bigr)\,ds \\ &\ge \frac{1}{4}\int_a^b G(t_0,s)q(s)f\bigl(y(s)\bigr)\,ds \\ &\ge \frac{1}{4}\int_a^b G(t_0,s)\frac{(s-a)(b-s)}{(b-a)^2}q(s)\,dsN\|y\| \\ &= \frac{1}{4}\int_a^b \frac{(s-a)^2(b-s)^\nu}{(b-a)^3\Gamma(\nu)}q(s)\,dsN\|y\| \\ &\quad + \frac{1}{4}\int_a^b \frac{(\nu-2)(s-a)(b-s)^{\frac{(\nu^2-\nu-1)}{\nu-2}}}{(\nu-1)^{\frac{\nu-1}{\nu-2}}(b-a)^{\frac{3\nu-5}{\nu-2}}\Gamma(\nu)}q(s)\,dsN\|y\|, \end{aligned}$$

which yields

$$\|y\| \ge \frac{1}{4}\int_a^b \frac{(s-a)^2(b-s)^\nu}{(b-a)^3\Gamma(\nu)}q(s)\,dsN\|y\| \tag{3.8}$$

and

$$\|y\| \ge \frac{1}{4}\int_a^b \frac{(\nu-2)(s-a)(b-s)^{\frac{(\nu^2-\nu-1)}{\nu-2}}}{(\nu-1)^{\frac{\nu-1}{\nu-2}}(b-a)^{\frac{3\nu-5}{\nu-2}}\Gamma(\nu)}q(s)\,dsN\|y\|. \tag{3.9}$$

(3.2) and (3.3) follow easily from (3.8) and (3.9), respectively.

In fact, the condition that f is controlled by two lines in Theorem 3.1 is very common. For example, for the piecewise continuous function

$$f(y) = \begin{cases} 2y, & 0 \le y \le 1, \\ 3 - y, & 1 \le y \le 2, \\ \frac{1}{2}y, & 2 \le y \le +\infty, \end{cases}$$

we may choose $M = 2$ and $N = \frac{1}{2}$ in Theorem 3.1.

Let $f(y) = y$ in (1.3), we get a fractional BVP with a linear term as follows:

$$\begin{cases} ({}_a^C D^\nu y)(t) + q(t)y(t) = 0, & a < t < b, 2 < \nu \le 3, \\ y(a) = y(b) = y''(a) = 0. \end{cases} \tag{3.10}$$

By choosing $M = N = 1$ in Theorem 3.1, we obtain the following result. □

Corollary 3.1 *If* (3.10) *has a solution* $y \in E^+$, *then*

$$\int_a^b q(s)\,ds > \frac{\Gamma(\nu)}{(b-a)^{\nu-1}}, \tag{3.11}$$

$$\int_a^b (s-a)^2(b-s)^\nu q(s)\,ds \le 4\Gamma(\nu)(b-a)^3, \tag{3.12}$$

$$\int_a^b (s-a)(b-s)^{\frac{\nu^2-\nu-1}{\nu-2}} q(s)\,ds \le \frac{4\Gamma(\nu)(b-a)^{\frac{3\nu-5}{\nu-2}}(\nu-1)^{\frac{\nu-1}{\nu-2}}}{(\nu-2)}. \tag{3.13}$$

4 Applications

4.1 Eigenvalue problems

Let $a = 0$ and $b = 1$ in (3.10), we now discuss the eigenvalue problem (4.1).

$$\begin{cases} ({}_a^C D^\nu)(t) + \lambda y(t) = 0, & 0 < t < 1, 2 < \nu \le 3, \\ y(0) = y(1) = y''(0) = 0. \end{cases} \tag{4.1}$$

By Corollary 3.1, we obtain the following result.

Corollary 4.1 *For any* $\lambda \in [0, \Gamma(\nu)] \cup (\frac{4\Gamma(\nu)}{B(3,\nu+1)}, +\infty)$, *where*

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1}\,ds, \quad x > 0, y > 0,$$

eigenvalue problem (4.1) *has no corresponding eigenfunction* $y \in E^+$.

Proof Assume that $y_0 \in E^+$ is an eigenfunction of (4.1) corresponding to an eigenvalue $\lambda_0 \in [0, \Gamma(\nu)] \cup (\frac{4\Gamma(\nu)}{B(3,\nu+1)}, +\infty)$. By (3.11), (3.12), and (3.13) in Corollary 3.1, we have

$$\lambda_0 > \Gamma(\nu), \qquad \lambda_0 \int_0^1 s^2(1-s)^\nu\,ds \le 4\Gamma(\nu),$$

$$\lambda_0 \int_0^1 s(1-s)^{\frac{\nu^2-\nu-1}{\nu-2}}\,ds \le \frac{4\Gamma(\nu)(\nu-1)^{\frac{\nu-1}{\nu-2}}}{(\nu-2)},$$

and thus

$$\Gamma(\nu) < \lambda_0 \le \min\left\{\frac{4\Gamma(\nu)}{B(3,\nu+1)}, \frac{4\Gamma(\nu)(\nu-1)^{\frac{\nu-1}{\nu-2}}}{(\nu-2)B(2,\frac{\nu^2-3}{\nu-2})}\right\} = \frac{4\Gamma(\nu)}{B(3,\nu+1)}, \tag{4.2}$$

which is a contradiction. □

Remark 4.1 We supplement the proof for the minimum value in (4.2). $2 < \nu \le 3$ leads to $\frac{\nu-1}{\nu-2} \ge 2$. Considering $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we get

$$\frac{1}{B(3,\nu+1)} = \frac{\Gamma(4+\nu)}{\Gamma(3)\Gamma(\nu+1)} = \frac{(3+\nu)(2+\nu)(1+\nu)}{2},$$
$$\frac{(\nu-1)^{\frac{\nu-1}{\nu-2}}}{(\nu-2)B(2,\frac{\nu^2-3}{\nu-2})} \ge \frac{(\nu-1)^2\Gamma(2+\frac{\nu^2-3}{\nu-2})}{(\nu-2)\Gamma(2)\Gamma(\frac{\nu^2-3}{\nu-2})} = \frac{(\nu-1)^2(\nu^2+\nu-5)(\nu^2-3)}{(\nu-2)^3}.$$

Obviously,

$$\frac{(\nu-1)^2(\nu^2+\nu-5)(\nu^2-3)}{(\nu-2)^3} > \frac{(3+\nu)(2+\nu)(1+\nu)}{2},$$

which means $\min\{\frac{4\Gamma(\nu)}{B(3,\nu+1)}, \frac{4\Gamma(\nu)(\nu-1)^{\frac{\nu-1}{\nu-2}}}{(\nu-2)B(2,\frac{\nu^2-3}{\nu-2})}\} = \frac{4\Gamma(\nu)}{B(3,\nu+1)}$.

4.2 Real zeros for Mittag-Leffler function

We consider the two-parameter Mittag-Leffler function

$$E_{\nu,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\nu+\beta)}, \quad \nu > 0, \beta > 0, z \in \mathbb{R}. \tag{4.3}$$

Obviously, $E_{\nu,\beta}(z) > 0$ for all $z \ge 0$. Hence, the real zeros of $E_{\nu,\beta}(z)$, if they exist, must be negative real numbers. In the following, we will use Corollary 4.1 to obtain an interval in which the Mittag-Leffler function (4.3) with $\beta = 2, 2 < \nu \le 3$ has no real zeros.

Corollary 4.2 *Let* $2 < \nu \le 3$. *Then the Mittag-Leffler function* $E_{\nu,2}(z)$ *has no real zeros for* $z \in (-\infty, -\frac{4\Gamma(\nu)}{B(3,\nu+1)}) \cup [-\Gamma(\nu), +\infty)$.

Proof We first of all recall some elementary knowledge as regards the eigenvalue problem (4.1). By Theorem 1 in [23], the general solution $y(t)$ satisfying the fractional differential equation in (4.1) is

$$y(t) = AE_{\nu,1}(-\lambda t^\nu) + BtE_{\nu,2}(-\lambda t^\nu) + Ct^2E_{\nu,3}(-\lambda t^\nu).$$

Using the boundary conditions in (4.1), we get

$$A = 0, \qquad BE_{\nu,2}(-\lambda) = 0, \qquad C = 0.$$

Thus, if there exists a positive real number λ satisfying $E_{\nu,2}(-\lambda) = 0$, then λ must be a positive eigenvalue of (4.1) and the corresponding eigenfunction is given by

$$y(t) = tE_{\nu,2}(-\lambda t^\nu).$$

Now, suppose $\lambda_0 \in (-\infty, -\frac{4\Gamma(\nu)}{B(3,\nu+1)}) \cup [-\Gamma(\nu), +\infty)$ is a real zero of $E_{\nu,2}(z)$, then $\lambda_0 < 0$, and thus $-\lambda_0$ must be a positive eigenvalue of (4.1). From Corollary 4.1, we know

$$\Gamma(\nu) < -\lambda_0 \leq \frac{4\Gamma(\nu)}{B(3,\nu+1)},$$

and thus

$$-\Gamma(\nu) > \lambda_0 \geq -\frac{4\Gamma(\nu)}{B(3,\nu+1)},$$

which is a contradiction.

Therefore, the Mittag-Leffler function $E_{\nu,2}(z)$ has no real zeros on $(-\infty, -\frac{4\Gamma(\nu)}{B(3,\nu+1)}) \cup [-\Gamma(\nu), +\infty)$. □

5 Conclusions

We obtain a Lyapunov inequality (3.1), two Lyapunov-type inequalities (3.2) and (3.3) for the following higher-order fractional boundary value problem with a nonlinear term:

$$\begin{cases} ({}_a^C D^\nu y)(t) + q(t)f(y) = 0, & a < t < b, 2 < \nu \leq 3, \\ y(a) = y(b) = y''(a) = 0. \end{cases}$$

We get these inequalities when the nonlinear term f is controllable. Accurate properties of the Green's function in Lemma 2.2 are important.

As applications, on the one hand, we conclude that, for any $\lambda \in [0, \Gamma(\nu)] \cup (\frac{4\Gamma(\nu)}{B(3,\nu+1)}, +\infty)$, the following eigenvalue problem:

$$\begin{cases} ({}_a^C D^\nu)(t) + \lambda y(t) = 0, & 0 < t < 1, 2 < \nu \leq 3, \\ y(0) = y(1) = y''(0) = 0 \end{cases} \tag{5.1}$$

has no corresponding eigenfunction $y \in E^+$; On the other hand, we prove that the Mittag-Leffler function $E_{\nu,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\nu+2)}$ has no real zeros on $(-\infty, -\frac{4\Gamma(\nu)}{B(3,\nu+1)}) \cup [-\Gamma(\nu), +\infty)$ for $\nu \in (2, 3]$.

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author contributed independently in drafting this manuscript. The author read and approved the final manuscript.

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