Zhou et al. *Journal of Inequalities and Applications* (2016) 2016:249
DOI 10.1186/s13660-016-1191-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

An estimate on the thickness of boundary layer for nonlinear evolution equations with damping and diffusion

Wenshu Zhou[1], Xulong Qin[2] and Xiaodan Wei[3*]

*Correspondence: weixiaodancat@126.com
[3]School of Computer Science, Dalian Nationalities University, Dalian, 116600, China
Full list of author information is available at the end of the article

Abstract

The main purpose of this paper is to estimate the thickness of boundary layer for nonlinear evolution equations with damping and diffusion as the diffusion parameter β goes to zero. We prove that the thickness of layer is of the order $O(\beta^\gamma)$ with $0 < \gamma < 1$, thus improving the corresponding result in (Ruan and Zhu in Discrete Contin. Dyn. Syst. 32(1) 331-352, 2012) where $0 < \gamma < 1/2$ is obtained.

MSC: 35k50; 35B40; 76N20

Keywords: nonlinear evolution equation; zero diffusion limit; boundary layer thicknes

1 Introduction

In this paper, we consider the nonlinear evolution equations with damping and diffusion:

$$\begin{cases} \psi_t^\beta = -(\sigma-\alpha)\psi^\beta - \sigma\theta_x^\beta + \alpha\psi_{xx}^\beta, \\ \theta_t^\beta = -(1-\beta)\theta^\beta + \mu\beta\psi_x^\beta + 2\psi^\beta\theta_x^\beta + \beta\theta_{xx}^\beta, \quad 0 < x < 1, t > 0, \end{cases} \tag{1.1}$$

with the initial-boundary conditions

$$\begin{aligned} &(\psi^\beta, \theta^\beta)(x,0) = (\psi_0, \theta_0)(x), \quad 0 \le x \le 1, \\ &(\psi^\beta, \theta^\beta)(1,t) = (\psi^\beta, \theta^\beta)(0,t) = (0,0), \quad t \ge 0, \end{aligned} \tag{1.2}$$

where σ, α, β, and μ are positive constants with $\alpha < \sigma$ and $0 < \beta < 1$. The corresponding problem of zero diffusion limit as $\beta \to 0$ is the following:

$$\begin{cases} \psi_t^0 = -(\sigma-\alpha)\psi^0 - \sigma\theta_x^0 + \alpha\psi_{xx}^0, \\ \theta_t^0 = -\theta^0 + 2\psi^0\theta_x^0, \quad 0 < x < 1, t > 0, \end{cases} \tag{1.3}$$

with the initial-boundary conditions

$$\begin{aligned} &(\psi^0, \theta^0)(x,0) = (\psi_0, \theta_0)(x), \quad 0 \le x \le 1, \\ &\psi^0(1,t) = \psi^0(0,t) = 0, \quad t \ge 0. \end{aligned} \tag{1.4}$$

The system (1.1) was originally proposed by Hsieh in [2] to observe the nonlinear interaction between ellipticity and dissipation. In [3], Hsieh *et al.* established a link between this interaction and chaos. We also refer to [4, 5] for the physical background of (1.1). Some similar problems were studied in [6, 7] and the references therein.

Our main purpose is to estimate the thickness of boundary layer for problem (1.1)-(1.2) as $\beta \to 0$. Before stating the main result, we first recall the concept of BL-thickness in the sprit of [8].

Definition 1.1 A function $\delta(\beta)$ is called a BL-thickness for problem (1.1)-(1.2) with vanishing diffusion if $\delta(\beta) \downarrow 0$ as $\beta \downarrow 0$, and

$$\lim_{\beta\to 0}\|\psi^\beta - \psi^0\|_{L^\infty(0,T;L^\infty[0,1])} = 0,$$

$$\lim_{\beta\to 0}\|\theta^\beta - \theta^0\|_{L^\infty(0,T;L^\infty[\delta,1-\delta])} = 0,$$

$$\inf_{\beta\to 0}\lim\|\theta^\beta - \theta^0\|_{L^\infty(0,T;L^\infty[0,1])} > 0,$$

for any $T > 0$, where $(\psi^\beta, \theta^\beta)$ (resp. (ψ^0, θ^0)) is the solution for problem (1.1)-(1.2) (resp. problem (1.3)-(1.4)).

The theory of boundary layers is one of the most fundamental and important issues in fluid dynamics (*cf.* [9, 10]) since the seminal work by Prandtl in 1904. There are a number of papers dedicated to the questions of boundary layers for the Navier-Stokes equations; see for instance [8, 11–17] and the references therein. Moreover, the boundary layer problem also arises in the theory of hyperbolic systems when parabolic equations with small viscosity are applied as perturbations; see for instance [18–23] and the references therein.

Recently, Ruan and Zhu [1], Theorem 1.3, discussed the existence and zero diffusion limit for problem (1.1)-(1.2), and proved that the thickness of boundary layer is of the order $O(\beta^\gamma)$ with $0 < \gamma < 1/2$ if $\frac{(\sigma+\mu\beta)^2}{4(1-\beta)} < \alpha < \sigma$ and if the initial data satisfy $\psi_0 \in H^2([0,1]), \theta_0 \in H^3([0,1]), (\psi_0,\theta_0)(1) = (\psi_0,\theta_0)(0) = (0,0)$, and $\|(\psi_0,\theta_0)\|_2$ is sufficiently small. Here $H^l([0,1])$ denotes the usual lth order Sobolev space with the norm $\|f\|_l = (\sum_{i=0}^{l}\int_0^1 |\partial_x^i f|^2\,dx)^{1/2}$. In the present paper, we improve the result by extending the range of γ to $(0,1)$. Our main result can be stated as follows.

Theorem 1.2 *Let* $0 < \beta < 1$ *and* $\frac{(\sigma+\mu\beta)^2}{4(1-\beta)} < \alpha < \sigma$. *Assume that the initial data satisfy* $\psi_0 \in H^2([0,1]), \theta_0 \in H^3([0,1]), (\psi_0,\theta_0)(1) = (\psi_0,\theta_0)(0) = (0,0)$, *and* $\|(\psi_0,\theta_0)\|_2$ *is sufficiently small. Then any function* $\delta(\beta)$ *satisfying* $\delta(\beta) \downarrow 0$ *and* $\frac{\beta}{\delta(\beta)} \to 0$ *as* $\beta \downarrow 0$ *is a BL-thickness such that*

$$\|\theta^\beta - \theta^0\|_{L^\infty(0,T;L^\infty[\delta,1-\delta])} \le C\sqrt{\frac{\beta}{\delta}}, \quad \forall \delta \in (0,1/2), \tag{1.5}$$

where $T > 0$, *and* C *is a positive constant independent of* β *and* δ.

The proof of Theorem 1.2 will be given in the next section.

2 Proof of Theorem 1.2

To prove Theorem 1.2, we need the following result, which can be found in [1], Lemmas 2.2, 2.4, 2.5 and 3.1.

Lemma 2.1 *Let the assumptions of Theorem* 1.2 *hold. Then there exists a positive constant independent of* β *such that*

$$\int_0^1 \big[(\psi_x^\beta)^2 + (\theta_x^\beta)^2 + (\psi_{xx}^\beta)^2 + \beta(\theta_{xx}^\beta)^2\big]\,dx \le C, \tag{2.1}$$

$$\int_0^1 \big[(\psi_x^0)^2 + (\theta_x^0)^2 + (\theta_{xx}^0)^2 + (\psi_{xx}^0)^2\big]\,dx \le C \tag{2.2}$$

and

$$\int_0^1 \big[(\psi^\beta - \psi^0)^2 + (\theta^\beta - \theta^0)^2\big]\,dx + \int_0^T \int_0^1 (\psi^\beta - \psi^0)_x^2\,dx\,dt \le C\beta. \tag{2.3}$$

Proof of Theorem 1.2 It suffices to prove (1.5). Set

$$u^\beta = \psi^\beta - \psi^0, \qquad v^\beta = \theta^\beta - \theta^0.$$

Then it follows from the equation of θ^β that

$$v_t^\beta = -(1-\beta)v^\beta + 2\psi^\beta v_x^\beta + 2u^\beta\theta_x^0 + \beta v_{xx}^\beta + \beta\big(\mu\psi_x^\beta + \theta^0 + \theta_{xx}^0\big).$$

Differentiating the equation, we see that $z := v_x^\beta$ satisfies

$$z_t = -(1-\beta)z + 2\big(\psi^\beta z\big)_x + 2\big(u^\beta\theta_x^0\big)_x + \beta z_{xx} + \beta\big(\mu\psi_{xx}^\beta + \theta_x^0 + \theta_{xxx}^0\big). \tag{2.4}$$

Denote φ_ε for $\varepsilon \in (0,1)$ and ξ_δ for $\delta \in (0,1/2)$ by

$$\varphi_\varepsilon(s) = \sqrt{s^2 + \varepsilon^2}, \qquad \xi_\delta(x) = \begin{cases} x, & 0 \le x \le \delta, \\ \delta, & \delta \le x \le 1-\delta, \\ 1-x, & 1-\delta \le x \le 1. \end{cases}$$

It is easy to check that φ_ε satisfies

$$\begin{cases} |s| \le |\varphi_\varepsilon(s)| \le |s| + 1, \\ |\varphi_\varepsilon'(s)| \le 1, \quad 0 \le s\varphi_\varepsilon'(s) \le \varphi_\varepsilon(s), \\ \varphi_\varepsilon''(s) \ge 0, \quad s^2\varphi_\varepsilon''(s) \ge 0, \end{cases}$$

and ξ_δ satisfies

$$0 \le \xi_\delta \le \delta, \qquad \xi_\delta(1) = \xi_\delta(0) = 0.$$

Multiplying (2.4) by $\varphi_\varepsilon'(z)\xi_\delta$ and integrating it over $(0,1) \times (0,t)$, we have

$$\begin{aligned} &\int_0^1 \varphi_\varepsilon(z)\xi_\delta\,dx - \varepsilon\int_0^1 \xi_\delta\,dx \\ &\quad = -(1-\beta)\int_0^t\int_0^1 z\varphi_\varepsilon'(z)\xi_\delta\,dx\,d\tau + 2\int_0^t\int_0^1 \big(\psi^\beta z\big)_x\varphi_\varepsilon'(z)\xi_\delta\,dx\,d\tau \end{aligned}$$

$$+ 2\int_0^t\int_0^1 \big(u^\beta\theta_x^0\big)_x\varphi_\varepsilon'(z)\xi_\delta\,dx\,d\tau + \beta\int_0^t\int_0^1 z_{xx}\varphi_\varepsilon'(z)\xi_\delta\,dx\,d\tau$$

$$+ \beta\int_0^t\int_0^1 \varphi_\varepsilon'(z)\xi_\delta\big(\mu\psi_{xx}^\beta + \theta_x^0 + \theta_{xxx}^0\big)\,dx\,d\tau =: \sum_{i=1}^5 E_j. \tag{2.5}$$

Next we estimate $E_i (i = 1, 2, 3, 4, 5)$. From $0 \le s\varphi_\varepsilon'(s) \le \varphi_\varepsilon(s)$, we have

$$E_1 \le \int_0^t\int_0^1 \varphi_\varepsilon(z)\xi_\delta\,dx\,d\tau. \tag{2.6}$$

To estimate E_2, we note, using integration by parts,

$$\begin{aligned}
E_2 &= 2\int_0^t\int_0^1 \psi_x^\beta z\varphi_\varepsilon'(z)\xi_\delta\,dx\,d\tau + 2\int_0^t\int_0^1 \psi^\beta z_x\varphi_\varepsilon'(z)\xi_\delta\,dx\,d\tau \\
&= 2\int_0^t\int_0^1 \psi_x^\beta z\varphi_\varepsilon'(z)\xi_\delta\,dx\,d\tau - 2\int_0^t\int_0^1 \varphi_\varepsilon(z)\psi_x^\beta\xi_\delta\,dx\,d\tau \\
&\quad - 2\int_0^t\int_0^1 \varphi_\varepsilon(z)\psi^\beta\xi_\delta'\,dx\,d\tau \\
&=: E_2^1 + E_2^2 + E_2^3.
\end{aligned} \tag{2.7}$$

By (2.1) and the embedding $W^{1,1}[0,1] \hookrightarrow L^\infty[0,1]$, we have

$$\big|\psi_x^\beta(x,t)\big| \le \int_0^1 \big|\psi_x^\beta\big|\,dx + \int_0^1 \big|\psi_{xx}^\beta\big|\,dx \le C, \tag{2.8}$$

where C denotes the generic positive constant independent of β, δ, and ε, so

$$\begin{aligned}
&\big|\psi^\beta(x,t)\big| \le \int_0^x \big|\psi_y^\beta(y,t)\big|\,dy \le Cx \le C\xi_\delta(x), \quad \forall x \in [0,\delta], \\
&\big|\psi^\beta(x,t)\big| \le \int_x^1 \big|\psi_y^\beta(y,t)\big|\,dy \le C(1-x) \le C\xi_\delta(x), \quad \forall x \in [1-\delta,1].
\end{aligned} \tag{2.9}$$

By $0 \le s\varphi_\varepsilon'(s) \le \varphi_\varepsilon(s)$ and (2.8), we obtain

$$E_2^1 + E_2^2 \le C\int_0^t\int_0^1 \varphi_\varepsilon(z)\xi_\delta\,dx\,d\tau. \tag{2.10}$$

By the definition of ξ_δ and (2.9), we have

$$\begin{aligned}
E_2^3 &= -2\int_0^t\int_0^\delta \varphi_\varepsilon(z)\psi^\beta\,dx\,d\tau + 2\int_0^t\int_{1-\delta}^1 \varphi_\varepsilon(z)\psi^\beta\,dx\,d\tau \\
&\le C\int_0^t\int_0^\delta \varphi_\varepsilon(z)\xi_\delta\,dx\,d\tau + C\int_0^t\int_{1-\delta}^1 \varphi_\varepsilon(z)\xi_\delta\,dx\,d\tau \\
&\le C\int_0^t\int_0^1 \varphi_\varepsilon(z)\xi_\delta\,dx\,d\tau.
\end{aligned} \tag{2.11}$$

Thus

$$E_2 \le C\int_0^t\int_0^1 \varphi_\varepsilon(z)\xi_\delta\,dx\,d\tau. \tag{2.12}$$

Using integration by parts and noticing $\varphi_\varepsilon'' \geq 0$ and $|\varphi_\varepsilon'| \leq 1$, we have

$$\begin{aligned} E_4 &= -\beta \int_0^t \int_0^1 z_x^2 \varphi_\varepsilon''(z)\xi_\delta \, dx \, d\tau - \beta \int_0^t \int_0^1 z_x \varphi_\varepsilon'(z)\xi_\delta' \, dx \, d\tau \\ &\leq -\beta \int_0^t \int_0^1 z_x \varphi_\varepsilon'(z)\xi_\delta' \, dx \, d\tau \\ &= -\beta \int_0^t \int_0^\delta z_x \varphi_\varepsilon'(z) \, dx \, d\tau + \beta \int_0^t \int_{1-\delta}^1 z_x \varphi_\varepsilon'(z) \, dx \, d\tau \\ &\leq \beta \left(\int_0^t \int_0^\delta |z_x| \, dx \, d\tau + \int_0^t \int_{1-\delta}^1 |z_x| \, dx \, d\tau \right), \end{aligned}$$

and, by Hölder's inequality and (2.1), we obtain

$$\begin{aligned} E_4 &\leq C\beta\delta^{1/2} \left[\left(\int_0^t \int_0^\delta |z_x|^2 \, dx \, d\tau \right)^{1/2} + \left(\int_0^t \int_{1-\delta}^1 |z_x|^2 \, dx \, d\tau \right)^{1/2} \right] \\ &\leq C\beta^{1/2}\delta^{1/2}. \end{aligned} \tag{2.13}$$

By $|\varphi_\varepsilon'| \leq 1$, $0 \leq \xi_\delta \leq \delta$, Hölder's inequality, (2.2), and (2.3), we have

$$\begin{aligned} E_3 &= 2\int_0^t \int_0^1 u_x^\beta \theta_x^0 \varphi_\varepsilon'(z)\xi_\delta \, dx \, d\tau + 2\int_0^t \int_0^1 u^\beta \theta_{xx}^0 \varphi_\varepsilon'(z)\xi_\delta \, dx \, d\tau \\ &\leq C\delta \left(\int_0^t \int_0^1 \left(u_x^\beta\right)^2 dx \, d\tau \right)^{1/2} \left(\int_0^t \int_0^1 \left(\theta_x^0\right)^2 dx \, d\tau \right)^{1/2} \\ &\quad + C\delta \left(\int_0^t \int_0^1 \left(u^\beta\right)^2 dx \, d\tau \right)^{1/2} \left(\int_0^t \int_0^1 \left(\theta_{xx}^0\right)^2 dx \, d\tau \right)^{1/2} \\ &\leq C\delta\beta^{1/2}. \end{aligned} \tag{2.14}$$

Finally, we estimate E_5. By $|\varphi_\varepsilon'| \leq 1$, $0 \leq \xi_\delta \leq \delta$, and Lemma 2.1, we have

$$\begin{aligned} E_5 &\leq C\beta\delta \int_0^t \int_0^1 \left(\left|\psi_{xx}^\beta\right| + \left|\theta_x^0\right| + \left|\theta_{xxx}^0\right| \right) dx \, d\tau \\ &\leq C\beta\delta. \end{aligned} \tag{2.15}$$

Combining (2.6), (2.12)-(2.15) with (2.5) and noticing

$$\varepsilon \int_0^1 \xi_\delta \, dx \leq \varepsilon\delta,$$

we obtain

$$\int_0^1 \varphi_\varepsilon(z)\xi_\delta \, dx \leq C \int_0^t \int_0^1 \varphi_\varepsilon(z)\xi_\delta \, dx \, d\tau + \varepsilon\delta + C\beta^{1/2}\delta^{1/2},$$

so an application of Gronwall's inequality leads to

$$\int_0^1 \varphi_\varepsilon(z)\xi_\delta \, dx \leq C\left(\varepsilon\delta + \beta^{1/2}\delta^{1/2}\right).$$

From this and the definition of ξ_δ and $|z| \le \varphi_\varepsilon(z)$, we obtain

$$\int_\delta^{1-\delta} |z|\,dx \le C\left(\varepsilon + \sqrt{\frac{\beta}{\delta}}\right).$$

Letting $\varepsilon \to 0$ yields

$$\int_\delta^{1-\delta} |z|\,dx \le C\sqrt{\frac{\beta}{\delta}}. \tag{2.16}$$

From (2.3), (2.16), and the embedding $W^{1,1}([\delta, 1-\delta]) \hookrightarrow L^\infty([\delta, 1-\delta])$ it follows that

$$\begin{aligned}\left\|\left(\theta^\beta - \theta^0\right)(\cdot, t)\right\|_{L^\infty[\delta,1-\delta]} &\le \int_0^1 \left|\theta^\beta - \theta^0\right|dx + \int_\delta^{1-\delta} |z|\,dx \\ &\le C\sqrt{\frac{\beta}{\delta}}.\end{aligned}$$

Thus (1.5) is proved, and the proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally.

Author details
[1]Department of Mathematics, Dalian Nationalities University, Dalian, 116600, China. [2]Department of Mathematics, Sun-Yat University, Guangzhou, 510275, China. [3]School of Computer Science, Dalian Nationalities University, Dalian, 116600, China.

Acknowledgements
We would like to thank the referees for their important comments which have improved our paper. The research was supported in part by the NSFC (grants 11571062, 11571380), the Program for Liaoning Excellent Talents in University (grant LJQ2013124) and the Fundamental Research Fund for the Central Universities (grant DC201502050202).

Received: 1 June 2016 Accepted: 30 September 2016 Published online: 12 October 2016

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