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Moment convergence rates in the law of iterated logarithm for moving average process under dependence

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Abstract

We assume that $X_k = \sum_{i=-\infty}^{+\infty} a_i \xi_{i+k}$ is a moving average process and $\{\xi_i, -\infty < i < +\infty\}$ is a doubly infinite sequence of identically distributed and dependent random variables with zero mean and finite variance and $\{a_i, -\infty < i < +\infty\}$ is an absolutely summable sequence of real numbers. Under suitable conditions of dependence, we get the precise rates in the law of iterated logarithm for the first moment of the partial sums of the moving average process.

MSC: 60F15

Keywords: precise asymptotics; the law of iterated logarithm; moment convergence; moving average; φ -mixing

1 Introduction

Suppose that $\{\xi_i, -\infty < i < +\infty\}$ is a doubly infinite sequence of identically distributed random variables with zero mean and finite variance and $\{a_i, -\infty < i < +\infty\}$ is an absolutely summable sequence of real numbers and

$$X_k = \sum_{i=-\infty}^{+\infty} a_i \xi_{i+k}, \quad k \geq 1, \quad (1)$$

is a moving average process based on $\{\xi_i, -\infty < i < +\infty\}$. Set $S_n = \sum_{k=1}^n X_k, n \geq 1$.

Under the assumption that $\{\xi_i, -\infty < i < +\infty\}$ is a sequence of independent identically distributed (i.i.d.) random variables, many limiting results have been obtained for the moving average process $\{X_k, k \geq 1\}$. Burton and Dehling [1] got a large deviation principle; Yang [2] established the central limit theorem and the law of the iterated logarithm; Li *et al.* [3] and Zhang [4] obtained the complete convergence; and complete moment convergence was proved by Li [5] and Li and Zhang [6].

In this article we will discuss the case of φ -mixing. Suppose that $\{\xi_i, -\infty < i < +\infty\}$ is a sequence of identically distributed and φ -mixing random variables with

$$\varphi(m) := \sup_{k \geq 1} \left\{ \left| \mathbb{P}(B|A) - \mathbb{P}(B) \right|, A \in \mathcal{F}_{-\infty}^k, \mathbb{P}(A) \neq 0, B \in \mathcal{F}_{k+m}^\infty \right\} \rightarrow 0, \quad m \rightarrow \infty,$$

where $\mathcal{F}_a^b = \sigma(\xi_i, a \leq i \leq b)$. By definition, we know that $\{X_k, k \geq 1\}$ is φ -mixing if $\{\xi_i, -\infty < i < +\infty\}$ is a sequence of φ -mixing distributed random variables.

Since Hsu and Robbins [7] introduced complete convergence, there have been extensions in several directions. Some authors studied the precise asymptotics of complete convergence on moving average. For example, Li and Zhang [8] got precise asymptotics in the law of the iterated logarithm of moving average and Xiao and Yin [9] got moment convergence rates in the law of logarithm for moving average process under dependence, etc.

Xiao and Yin [10] studied precise asymptotics in the law of iterated logarithm for the first moment convergence of i.i.d. random variables. They got the following result.

Theorem 1.1 *Let $\{Y, Y_n, n \geq 1\}$ be a sequence of i.i.d. random variables and set $U_n = \sum_{i=1}^n Y_i$. \mathcal{N} is the standard normal random variable and $\mathbb{E}Y = 0, \mathbb{E}Y^2 = \sigma^2$, then, for $d > 0$ and $\beta > 0$,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\beta/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{\beta-1}}{n^{3/2} \log n} \mathbb{E}\{|U_n| - \varepsilon \sigma \sqrt{n}(\log \log n)^{d/2}\}_+ = \frac{d\sigma \mathbb{E}|\mathcal{N}|^{2\beta/d+1}}{\beta(2\beta + d)}. \tag{2}$$

Inspired by Xiao and Yin [10], we extend this result to moving average processes under φ -mixing random variable. Throughout this article, $\log x := \ln(x \vee e)$, and the symbol c denotes a positive constant which may be different in various places, and $[x]$ denotes the largest integer which is not greater than x , and \mathcal{N} is the standard normal random variable. Now, we state the main result of this article.

Theorem 1.2 *Suppose $\{X_i, i \geq 1\}$ is defined as in (1), where $\{a_i, -\infty < i < +\infty\}$ is a sequence of real numbers with $\sum_{i=-\infty}^{+\infty} |a_i| < \infty$, and $\{\xi_i, -\infty < i < +\infty\}$ is a sequence of identically distributed φ -mixing random variables with zero mean and finite variance and $0 < \sigma^2 := \mathbb{E}\xi_1^2 + 2 \sum_{k=2}^{\infty} \mathbb{E}\xi_1 \xi_k < \infty, \sum_{m=1}^{\infty} \varphi^{1/2}(2^m) < \infty, \tau := \sigma | \sum_{i=-\infty}^{+\infty} a_i|$. Then, for $d > 0$ and $\beta > 0$,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\beta/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{\beta-1}}{n^{3/2} \log n} \mathbb{E}\{|S_n| - \varepsilon \tau \sqrt{n}(\log \log n)^{d/2}\}_+ = \frac{d\tau \mathbb{E}|\mathcal{N}|^{2\beta/d+1}}{\beta(2\beta + d)}. \tag{3}$$

Remark 1.3 Let $a_i = 1$ for $i = 0$ and $a_i = 0$ for $i \neq 0$, that is to say, $X_k = \xi_k$ with $\mathbb{E}\xi_1 = 0, \mathbb{E}\xi_1^2 < \infty$ and $0 < \sigma^2 := \mathbb{E}\xi_1^2 + 2 \sum_{k=2}^{\infty} \mathbb{E}\xi_1 \xi_k < \infty$, then, for $S_n = \sum_{k=1}^n \xi_k$, (3) still holds for $\tau = \sigma$ when $\{X_k; k \geq 1\}$ is a sequence of identically distributed φ -mixing random variables. Therefore, this result extends Theorem 1.1.

2 Lemmas

To prove our main result, we need the following lemmas.

Lemma 2.1 (Burton and Dehling [1]) *Let $\sum_{i=-\infty}^{+\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{+\infty} a_i$ and $k \geq 1$, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=-\infty}^{+\infty} \left| \sum_{j=i+1}^{i+n} a_j \right|^k = |a|^k.$$

Lemma 2.2 (Shao [11]) *Let $\{X_i, i \geq 1\}$ be a sequence of φ -mixing random variables with zero means and finite second moments. Set $S_n = \sum_{i=1}^n X_i$. Then*

$$\mathbb{E}S_n^2 \leq 8,000n \exp \left\{ 6 \sum_{i=1}^{\lfloor \log n \rfloor} \varphi^{1/2}(2^i) \right\} \max_{1 \leq i \leq n} \mathbb{E}X_i^2. \tag{4}$$

If there is some C_n such that $\max_{1 \leq i \leq n} \mathbb{E}S_i^2 \leq C_n$, then, for all $q \geq 2$, there exists a $C = C(q, \varphi(\cdot))$ such that

$$\mathbb{E} \max_{1 \leq i \leq n} |S_i|^q \leq C \left(C_n^{q/2} + \mathbb{E} \max_{1 \leq i \leq n} |X_i|^q \right). \tag{5}$$

Lemma 2.3 (Lin and Zhou [12]) *Let $\{X_i, i \geq 1\}$ be defined as in (1), and $\{\xi_i, -\infty < i < +\infty\}$ be a sequence of identically distributed φ -mixing random variables with zero means and finite variances and $0 < \sigma^2 = \mathbb{E}\xi_1^2 + 2 \sum_{k=2}^{\infty} \mathbb{E}\xi_1 \xi_k < \infty, \sum_{m=1}^{\infty} \varphi^{1/2}(2^m) < \infty$, then*

$$\frac{S_n}{\tau \sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1), \quad \tau = \sigma \left| \sum_{i=-\infty}^{+\infty} a_i \right|, \tag{6}$$

where \xrightarrow{D} denotes convergence in distribution.

3 Proof of main result

In this section, for $0 < \varepsilon < 1$ and $M > 1$, we set

$$b(\varepsilon) = \left[\exp \left\{ \exp \left\{ M \varepsilon^{-2/d} \right\} \right\} \right]. \tag{7}$$

Without loss of generality, we assume that $\tau = 1$.

Theorem 1.2 will be proved if we show the following propositions.

Proposition 3.1 *Under the conditions of Theorem 1.2, we have*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\beta/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{\beta-1}}{n \log n} \mathbb{E} \left\{ |\mathcal{N}| - \varepsilon (\log \log n)^{d/2} \right\}_+ = \frac{d \mathbb{E} |\mathcal{N}|^{2\beta/d+1}}{\beta(2\beta + d)}, \tag{8}$$

$$\lim_{M \rightarrow \infty} \varepsilon^{2\beta/d} \sum_{n > b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \mathbb{E} \left\{ |\mathcal{N}| - \varepsilon (\log \log n)^{d/2} \right\}_+ = 0, \tag{9}$$

where (9) uniformly holds true with respect to $0 < \varepsilon < 1$.

Proof See the proof of Proposition 1 and Proposition 3 in Xiao and Yin [10]. □

Proposition 3.2 *Under the conditions of Theorem 1.2, we have*

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} \varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \left| \mathbb{E} \left\{ |\mathcal{N}| - \varepsilon (\log \log n)^{d/2} \right\}_+ \right. \\ & \left. - \mathbb{E} \left\{ |S_n| / \sqrt{n} - \varepsilon (\log \log n)^{d/2} \right\}_+ \right| = 0. \end{aligned} \tag{10}$$

Proof Let $\Delta_n = \sup_{x \in \mathbb{R}} |\mathbb{P}(|\mathcal{N}| \geq x) - \mathbb{P}(|S_n|/\sqrt{n} \geq x)|$. By Lemma 2.3, then we have $\Delta_n \rightarrow 0$ as $n \rightarrow \infty$. We have

$$\begin{aligned}
 & \varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \left| \mathbb{E}\{|\mathcal{N}| - \varepsilon(\log \log n)^{d/2}\}_+ - \mathbb{E}\{|S_n|/\sqrt{n} - \varepsilon(\log \log n)^{d/2}\}_+ \right| \\
 &= \varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \left| \int_0^\infty \mathbb{P}(|\mathcal{N}| \geq x + \varepsilon(\log \log n)^{d/2}) \, dx \right. \\
 &\quad \left. - \int_0^\infty \mathbb{P}(|S_n|/\sqrt{n} \geq x + \varepsilon(\log \log n)^{d/2}) \, dx \right| \\
 &\leq \varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_0^\infty \left| \mathbb{P}(|\mathcal{N}| \geq (x + \varepsilon)(\log \log n)^{d/2}) \right. \\
 &\quad \left. - \mathbb{P}(|S_n|/\sqrt{n} \geq (x + \varepsilon)(\log \log n)^{d/2}) \right| \, dx \\
 &\leq \varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_0^{l(n)} \left| \mathbb{P}(|\mathcal{N}| \geq (x + \varepsilon)(\log \log n)^{d/2}) \right. \\
 &\quad \left. - \mathbb{P}(|S_n|/\sqrt{n} \geq (x + \varepsilon)(\log \log n)^{d/2}) \right| \, dx \\
 &\quad + \varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_{l(n)}^\infty \mathbb{P}(|\mathcal{N}| \geq (x + \varepsilon)(\log \log n)^{d/2}) \, dx \\
 &\quad + \varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_{l(n)}^\infty \mathbb{P}(|S_n|/\sqrt{n} \geq (x + \varepsilon)(\log \log n)^{d/2}) \, dx \\
 &:= I_1 + I_2 + I_3, \tag{11}
 \end{aligned}$$

where $l(n) = (\log \log n)^{-d/2} \Delta_n^{-1/2}$. Hence, for I_1 , by applying Lemma 2.3, we have

$$\begin{aligned}
 I_1 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_0^{l(n)} \Delta_n \, dx \leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \Delta_n l(n) \\
 &= c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \Delta_n^{1/2} \leq \frac{cM^\beta}{(\log \log b(\varepsilon))^\beta} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \Delta_n^{1/2}. \tag{12}
 \end{aligned}$$

So, by the Toeplitz lemma, we have

$$\lim_{\varepsilon \searrow 0} I_1 = 0. \tag{13}$$

As for I_2 , by the Markov inequality, (12), and (13), we have

$$\begin{aligned}
 I_2 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_{l(n)}^\infty \frac{1}{(x + \varepsilon)^2 (\log \log n)^d} \, dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1-d/2}}{n \log n} l(n)^{-1} \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \Delta_n^{1/2} \rightarrow 0, \quad \text{as } \varepsilon \searrow 0. \tag{14}
 \end{aligned}$$

For I_3 , note that $S_n = \sum_{k=1}^n X_k = \sum_{i=-\infty}^{+\infty} \sum_{k=1}^n a_i \xi_{i+k} = \sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} \xi_j$. By Lemma 2.1, without loss of generality, we assume $\sum_{i=-\infty}^{+\infty} |a_i| \leq 1$, and set $S'_n = \sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} \xi'_j$, where $\xi'_j = \xi_j I(|\xi_j| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2})$. By $\mathbb{E}\xi_1 = 0$, then, for all $x > 0$, we have $\mathbb{E}\xi_1 I(|\xi_1| \leq x) = -\mathbb{E}\xi_1 I(|\xi_1| > x)$. So, we have

$$\begin{aligned} |\mathbb{E}S'_n| &= \left| \sum_{i=-\infty}^{+\infty} a_i \mathbb{E} \sum_{j=i+1}^{i+n} \xi_j I(|\xi_j| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) \right| \\ &\leq \sum_{i=-\infty}^{+\infty} |a_i| \left| \mathbb{E} \sum_{j=i+1}^{i+n} \xi_j I(|\xi_j| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) \right| \\ &= \sum_{i=-\infty}^{+\infty} |a_i| \left| \mathbb{E} \sum_{j=i+1}^{i+n} \xi_j I(|\xi_j| > (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) \right| \\ &\leq n \mathbb{E}|\xi_1| I(|\xi_1| > (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) \\ &\leq \frac{\sqrt{n} \mathbb{E}\xi_1^2}{(x + \varepsilon)(\log \log n)^{d/2}}, \end{aligned} \tag{15}$$

thus for $x \geq l(n)$ and n large enough, we have

$$\frac{|\mathbb{E}S'_n|}{(x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}} \leq \frac{c \mathbb{E}\xi_1^2}{(x + \varepsilon)^2(\log \log n)^d} \leq \frac{c}{l(n)^2(\log \log n)^d} \leq c \Delta_n < \frac{1}{4}, \tag{16}$$

so we obtain

$$\begin{aligned} &\int_{l(n)}^{\infty} \mathbb{P}(|S_n|/\sqrt{n} \geq (x + \varepsilon)(\log \log n)^{d/2}) \, dx \\ &\leq c \int_{l(n)}^{\infty} \mathbb{P} \left(\left| \sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} \xi_j I(|\xi_j| > (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) \right| \right. \\ &\quad \left. > \frac{(x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}}{2} \right) \, dx \\ &\quad + c \int_{l(n)}^{\infty} \mathbb{P} \left(\left| \sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} (\xi'_j - \mathbb{E}\xi'_j) \right| > \frac{(x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}}{4} \right) \, dx \\ &:= I_{31} + I_{32}. \end{aligned} \tag{17}$$

By (11) and (17), we have

$$I_3 \leq c \varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} (I_{31} + I_{32}).$$

On the one hand, by (12) and (13), we obtain

$$\begin{aligned} &\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} I_{31} \\ &\leq c \varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_{l(n)}^{\infty} \frac{n \mathbb{E}|\xi_1| I(|\xi_1| > (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2})}{\sqrt{n}(x + \varepsilon)(\log \log n)^{d/2}} \, dx \end{aligned}$$

$$\begin{aligned}
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_{l(n)}^\infty \frac{\mathbb{E}\xi_1^2 I(|\xi_1| > (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2})}{(x + \varepsilon)^2 (\log \log n)^d} dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1-d/2}}{n \log n} \int_{l(n)}^\infty (x + \varepsilon)^{-2} dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1-d/2}}{n \log n} l(n)^{-1} \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \Delta_n^{1/2} \rightarrow 0, \quad \text{as } \varepsilon \searrow 0.
 \end{aligned} \tag{18}$$

On the other hand, by the Markov inequality, the Hölder inequality, and Lemma 2.2, and noting that $\sum_{m=1}^\infty \varphi^{1/2}(2^m) < \infty$, we have

$$\begin{aligned}
 &\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \text{I}_{32} \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2-dq/2}}{n^{1+q/2} \log n} \int_{l(n)}^\infty (x + \varepsilon)^{-q} \mathbb{E} \left| \sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} (\xi'_j - \mathbb{E}\xi'_j) \right|^q dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2-dq/2}}{n^{1+q/2} \log n} \\
 &\quad \times \int_{l(n)}^\infty (x + \varepsilon)^{-q} \mathbb{E} \left[\sum_{i=-\infty}^{+\infty} (|a_i|^{1-\frac{1}{q}}) \left(|a_i|^{\frac{1}{q}} \left| \sum_{j=i+1}^{i+n} (\xi'_j - \mathbb{E}\xi'_j) \right| \right) \right]^q dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2-dq/2}}{n^{1+q/2} \log n} \\
 &\quad \times \int_{l(n)}^\infty (x + \varepsilon)^{-q} \mathbb{E} \left[\left(\sum_{i=-\infty}^{+\infty} |a_i|^{1-\frac{1}{q}, \frac{1}{1-\frac{1}{q}}} \right)^{1-\frac{1}{q}} \left(\sum_{i=-\infty}^{+\infty} |a_i|^{\frac{1}{q}, q} \left| \sum_{j=i+1}^{i+n} (\xi'_j - \mathbb{E}\xi'_j) \right|^q \right)^{\frac{1}{q}} \right]^q dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2-dq/2}}{n^{1+q/2} \log n} \\
 &\quad \times \int_{l(n)}^\infty (x + \varepsilon)^{-q} \mathbb{E} \left(\sum_{i=-\infty}^{+\infty} |a_i| \right)^{(1-\frac{1}{q})q} \sum_{i=-\infty}^{+\infty} |a_i| \left| \sum_{j=i+1}^{i+n} (\xi'_j - \mathbb{E}\xi'_j) \right|^q dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2-dq/2}}{n^{1+q/2} \log n} \\
 &\quad \times \int_{l(n)}^\infty (x + \varepsilon)^{-q} \sum_{i=-\infty}^{+\infty} |a_i| \mathbb{E} \left| \sum_{j=i+1}^{i+n} (\xi'_j - \mathbb{E}\xi'_j) \right|^q dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2-dq/2}}{n^{1+q/2} \log n} \int_{l(n)}^\infty (x + \varepsilon)^{-q} \left((n\mathbb{E}\xi_1'^2)^{\frac{q}{2}} + n\mathbb{E}|\xi_1'|^q \right) dx \\
 &:= \text{I}_{321} + \text{I}_{322}.
 \end{aligned}$$

For I_{321} , $q \geq 2$. By (12) and (13), we have

$$\begin{aligned}
 I_{321} &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2-dq/2}}{n \log n} \int_{l(n)}^{\infty} (x + \varepsilon)^{-q} dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2-dq/2}}{n \log n} l(n)^{1-q} \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \Delta_n^{\frac{q-1}{2}} \rightarrow 0, \quad \text{as } \varepsilon \searrow 0.
 \end{aligned}
 \tag{19}$$

For I_{322} , by (12) and (13), we have

$$\begin{aligned}
 I_{322} &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2-dq/2}}{n^{q/2} \log n} \\
 &\quad \times \int_{l(n)}^{\infty} (x + \varepsilon)^{-q} \mathbb{E}|\xi_1|^q I(|\xi_1| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1-d/2}}{n \log n} \\
 &\quad \times \int_{l(n)}^{\infty} (x + \varepsilon)^{-2} \mathbb{E}\xi_1^2 I(|\xi_1| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1-d/2}}{n \log n} l(n)^{-1} \\
 &= c\varepsilon^{2\beta/d} \sum_{n \leq b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \Delta_n^{1/2} \rightarrow 0, \quad \text{as } \varepsilon \searrow 0.
 \end{aligned}
 \tag{20}$$

Thus, combining with (13), (14), and (18)-(20), we complete the proof of this proposition. □

Proposition 3.3 *Under the conditions of Theorem 1.2,*

$$\lim_{M \rightarrow \infty} \varepsilon^{2\beta/d} \sum_{n > b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \mathbb{E}\{ |S_n|/\sqrt{n} - \varepsilon(\log \log n)^{d/2} \}_+ = 0
 \tag{21}$$

uniformly holds true with respect to $0 < \varepsilon < 1$.

Proof For $x \geq 0$, M large enough, we have

$$\frac{|\mathbb{E}S'_n|}{\sqrt{n}(x + \varepsilon)(\log \log n)^{d/2}} \leq \frac{c}{(x + \varepsilon)^2(\log \log n)^d} \leq \frac{c}{M^d} < \frac{1}{4}.$$

Then

$$\begin{aligned}
 &\mathbb{P}(|S_n| \geq \sqrt{n}(\log \log n)^{d/2}(x + \varepsilon)) \\
 &\leq \mathbb{P}\left(\left| \sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} \xi_j I(|\xi_j| > (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) \right| > \frac{\sqrt{n}(\log \log n)^{d/2}(x + \varepsilon)}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \mathbb{P}\left(\left|S'_n\right| > \frac{\sqrt{n}(\log \log n)^{d/2}(x+\varepsilon)}{2}\right) \\
 \leq & \mathbb{P}\left(\left|\sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} \xi_j I(|\xi_j| > (x+\varepsilon)\sqrt{n}(\log \log n)^{d/2})\right| > \frac{\sqrt{n}(\log \log n)^{d/2}(x+\varepsilon)}{2}\right) \\
 & + \mathbb{P}\left(\left|S'_n - \mathbb{E}S'_n\right| \geq \left|S'_n\right| - \left|\mathbb{E}S'_n\right|\right. \\
 & \left. > \frac{\sqrt{n}(\log \log n)^{d/2}(x+\varepsilon)}{2} - \frac{\sqrt{n}(\log \log n)^{d/2}(x+\varepsilon)}{4}\right) \\
 = & \mathbb{P}\left(\left|\sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} \xi_j I(|\xi_j| > (x+\varepsilon)\sqrt{n}(\log \log n)^{d/2})\right| > \frac{\sqrt{n}(\log \log n)^{d/2}(x+\varepsilon)}{2}\right) \\
 & + \mathbb{P}\left(\left|\sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} (\xi'_j - \mathbb{E}\xi'_j)\right| > \frac{\sqrt{n}(\log \log n)^{d/2}(x+\varepsilon)}{4}\right). \tag{22}
 \end{aligned}$$

Therefore, by (22), we obtain

$$\begin{aligned}
 & \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \mathbb{E}\left\{\left|S_n\right|/\sqrt{n} - \varepsilon(\log \log n)^{\frac{d}{2}}\right\}_+ \\
 = & \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{n \log n} \int_0^\infty \mathbb{P}\left(\left|S_n\right|/\sqrt{n} \geq x + \varepsilon(\log \log n)^{d/2}\right) dx \\
 = & \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}}}{n \log n} \int_0^\infty \mathbb{P}\left(\left|S_n\right| \geq \sqrt{n}(\log \log n)^{\frac{d}{2}}(x+\varepsilon)\right) dx \quad (x = y(\log \log n)^{d/2}) \\
 \leq & \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}}}{n \log n} \int_0^\infty \mathbb{P}\left(\left|\sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} \xi_j I(|\xi_j| \right. \right. \\
 & \left. \left. > (x+\varepsilon)\sqrt{n}(\log \log n)^{\frac{d}{2}}\right) > \frac{(x+\varepsilon)\sqrt{n}(\log \log n)^{\frac{d}{2}}}{2}\right) dx \\
 & + \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}}}{n \log n} \int_0^\infty \mathbb{P}\left(\left|\sum_{i=-\infty}^{+\infty} a_i \sum_{j=i+1}^{i+n} (\xi'_j - \mathbb{E}\xi'_j)\right| \right. \\
 & \left. > \frac{(x+\varepsilon)\sqrt{n}(\log \log n)^{\frac{d}{2}}}{4}\right) dx \\
 & := J_1 + J_2.
 \end{aligned}$$

For J_1 , when $2\beta < d$, we have

$$\begin{aligned}
 \varepsilon^{2\beta/d} J_1 & \leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_0^\infty \frac{n\mathbb{E}|\xi_1|I(|\xi_1| > (x+\varepsilon)\sqrt{n}(\log \log n)^{d/2})}{(x+\varepsilon)\sqrt{n}(\log \log n)^{d/2}} dx \\
 & \leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1-d/2}}{n \log n} \int_0^\infty \frac{\mathbb{E}\xi_1^2 I(|\xi_1| > (x+\varepsilon)\sqrt{n}(\log \log n)^{d/2})}{(x+\varepsilon)^2} dx \\
 & \leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1-d/2}}{n \log n} \int_0^\infty (x+\varepsilon)^{-2} dx
 \end{aligned}$$

$$\begin{aligned} &\leq c\varepsilon^{2\beta/d-1} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1-d/2}}{n \log n} \\ &\leq cM^{\beta-d/2}, \end{aligned}$$

thus

$$\lim_{M \rightarrow \infty} \varepsilon^{2\beta/d} J_1 = 0 \tag{23}$$

uniformly holds true with respect to $0 < \varepsilon < 1$.

When $2\beta \geq d$, we have

$$\begin{aligned} J_1 &\leq c \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+d/2}}{n \log n} \int_0^\infty \frac{n\mathbb{E}|\xi_1| I(|\xi_1| > (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2})}{(x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}} dx \\ &= c \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{\sqrt{n} \log n} \int_0^\infty \frac{\mathbb{E}|\xi_1| I(|\xi_1| > (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2})}{x + \varepsilon} dx \\ &= c \int_0^\infty \frac{1}{x + \varepsilon} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{\sqrt{n} \log n} \sum_{j=n}^\infty \mathbb{E}|\xi_1| I\left(\sqrt{j}(\log \log j)^{d/2} < \frac{|\xi_1|}{x + \varepsilon}\right) \\ &\quad \leq \sqrt{j+1}(\log \log(j+1))^{d/2} dx \\ &= c \int_0^\infty \frac{1}{x + \varepsilon} \sum_{j>b(\varepsilon)} \mathbb{E}|\xi_1| I\left(\sqrt{j}(\log \log j)^{\frac{d}{2}} < \frac{|\xi_1|}{x + \varepsilon} \leq \sqrt{j+1}(\log \log(j+1))^{\frac{d}{2}}\right) \\ &\quad \times \sum_{n=b(\varepsilon)+1}^j \frac{(\log \log n)^{\beta-1}}{\sqrt{n} \log n} dx. \end{aligned}$$

Since we have $(\log \log n)^{\beta-1}(\log n)^{-1} \rightarrow 0$, as $n > b(\varepsilon) \rightarrow \infty$, then we obtain

$$\begin{aligned} J_1 &\leq c \int_0^\infty \frac{1}{x + \varepsilon} \sum_{j>b(\varepsilon)} \sqrt{j}\mathbb{E}|\xi_1| I\left(\sqrt{j}(\log \log j)^{d/2} < \frac{|\xi_1|}{x + \varepsilon} \leq \sqrt{j+1}(\log \log(j+1))^{d/2}\right) dx \\ &\leq c \int_0^\infty \frac{1}{(x + \varepsilon)^2} \mathbb{E}\xi_1^2 I(|\xi_1| > (x + \varepsilon)\sqrt{b(\varepsilon)}(\log \log b(\varepsilon))^{d/2}) dx \\ &\leq c \int_0^\infty \frac{1}{(x + \varepsilon)^2} \mathbb{E}\xi_1^2 I(|\xi_1| > \varepsilon(\log \log b(\varepsilon))^{d/2}) dx \\ &= c\varepsilon^{-1}\mathbb{E}\xi_1^2 I(|\xi_1| > \varepsilon(\log \log b(\varepsilon))^{d/2}) = M^{d/2}. \end{aligned}$$

Hence, when $2\beta \geq d$,

$$\lim_{M \rightarrow \infty} \varepsilon^{2\beta/d} J_1 \leq \lim_{M \rightarrow \infty} \mathbb{E}\xi_1^2 I(|\xi_1| > M^{d/2}) = 0 \tag{24}$$

uniformly holds true with respect to $0 < \varepsilon < 1$.

Next we consider J_2 , similar to the proof of I_{32} , we have

$$\begin{aligned} \varepsilon^{2\beta/d} J_2 &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n^{1+\frac{q}{2}} \log n} \int_0^\infty (x + \varepsilon)^{-q} ((n\mathbb{E}\xi_1'^2)^{q/2} + n\mathbb{E}|\xi_1'|^q) dx \\ &:= J_{21} + J_{22}. \end{aligned}$$

For J_{21} , haven taken $q > \max\{2\beta/d + 1, 2\}$, then we have

$$\begin{aligned} J_{21} &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n \log n} \int_0^\infty (x + \varepsilon)^{-q} dx \\ &\leq c\varepsilon^{2\beta/d-q+1} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n \log n} \\ &\leq c\varepsilon^{2\beta/d-q+1} (\log \log b(\varepsilon))^{\beta+\frac{d}{2}-\frac{dq}{2}} \\ &= M^{\beta+\frac{d}{2}-\frac{dq}{2}}. \end{aligned}$$

Thus

$$\lim_{M \rightarrow \infty} J_{21} = 0 \tag{25}$$

uniformly holds true with respect to $0 < \varepsilon < 1$.

For J_{22} , if $2\beta < d$, we have

$$\begin{aligned} J_{22} &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n^{\frac{q}{2}} \log n} \\ &\quad \times \int_0^\infty (x + \varepsilon)^{-q} \mathbb{E}|\xi_1|^q I(|\xi_1| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) dx \\ &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1-\frac{d}{2}}}{n \log n} \\ &\quad \times \int_0^\infty (x + \varepsilon)^{-2} \mathbb{E}\xi_1^2 I(|\xi_1| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) dx \\ &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1-\frac{d}{2}}}{n \log n} \int_0^\infty (x + \varepsilon)^{-2} dx \\ &\leq c\varepsilon^{2\beta/d-1} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1-\frac{d}{2}}}{n \log n} \\ &= cM^{\beta-d/2}. \end{aligned}$$

Thus

$$\lim_{M \rightarrow \infty} J_{22} = 0 \tag{26}$$

uniformly holds true with respect to $0 < \varepsilon < 1$.

If $2\beta \geq d$, we divide J_{22} into two parts:

$$\begin{aligned}
 J_{22} &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n^{\frac{q}{2}} \log n} \\
 &\quad \times \int_0^\infty (x + \varepsilon)^{-q} \mathbb{E}|\xi_1|^q I(|\xi_1| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2}) \, dx \\
 &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n^{\frac{q}{2}} \log n} \int_0^\infty (x + \varepsilon)^{-q} \mathbb{E}|\xi_1|^q [I(|\xi_1| \leq \sqrt{n}(\log \log n)^{d/2}\varepsilon) \\
 &\quad + I(\sqrt{n}(\log \log n)^{d/2}\varepsilon < |\xi_1| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{d/2})] \, dx \\
 &:= J_{221} + J_{222}.
 \end{aligned}$$

For J_{221} , since $q > 2$, so we have

$$\begin{aligned}
 J_{221} &\leq c\varepsilon^{2\beta/d-q+1} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n^{\frac{q}{2}} \log n} \mathbb{E}|\xi_1|^q I(|\xi_1| \leq \sqrt{b(\varepsilon)}(\log \log b(\varepsilon))^{\frac{d}{2}}\varepsilon) \\
 &\quad + c\varepsilon^{2\beta/d-q+1} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n^{\frac{q}{2}} \log n} \\
 &\quad \times \sum_{j=b(\varepsilon)+1}^n \mathbb{E}|\xi_1|^q I(\sqrt{j-1}(\log \log(j-1))^{\frac{d}{2}}\varepsilon < |\xi_1| \leq \sqrt{j}(\log \log j)^{\frac{d}{2}}\varepsilon) \\
 &\leq c\varepsilon^{2\beta/d-1} \frac{(\log \log b(\varepsilon))^{\beta-1-\frac{d}{2}}}{\log b(\varepsilon)} \mathbb{E}\xi_1^2 I(|\xi_1| \leq \sqrt{b(\varepsilon)}(\log \log b(\varepsilon))^{\frac{d}{2}}\varepsilon) \\
 &\quad + c\varepsilon^{2\beta/d-q+1} \sum_{j=b(\varepsilon)+1}^\infty \mathbb{E}|\xi_1|^q I(\sqrt{j-1}(\log \log(j-1))^{\frac{d}{2}}\varepsilon < |\xi_1| \leq \sqrt{j}(\log \log j)^{\frac{d}{2}}\varepsilon) \\
 &\quad \times \sum_{n=j}^\infty \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n^{\frac{q}{2}} \log n} \\
 &\leq c\varepsilon^{2\beta/d-1} \frac{1}{\log \log b(\varepsilon)} \\
 &\quad + c\varepsilon^{2\beta/d-q+1} \sum_{j=b(\varepsilon)+1}^\infty \mathbb{E}|\xi_1|^q I(\sqrt{j-1}(\log \log(j-1))^{\frac{d}{2}}\varepsilon < |\xi_1| \leq \sqrt{j}(\log \log j)^{\frac{d}{2}}\varepsilon) j^{1-\frac{q}{2}} \\
 &\quad \times \frac{(\log \log j)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{\log j} \\
 &\leq c\varepsilon^{2\beta/d-1+d/2} M^{-1} + c\varepsilon^{2\beta/d-1} \sum_{j=b(\varepsilon)+1}^\infty \frac{(\log \log j)^{\beta-1-\frac{d}{2}}}{\log j} \\
 &\quad \times \mathbb{E}\xi_1^2 I(\sqrt{j-1}(\log \log(j-1))^{\frac{d}{2}}\varepsilon < |\xi_1| \leq \sqrt{j}(\log \log j)^{\frac{d}{2}}\varepsilon) \\
 &\leq cM^{-1} + c\mathbb{E}\xi_1^2 I(|\xi_1| > \sqrt{b(\varepsilon)}(\log \log b(\varepsilon))^{\frac{d}{2}}\varepsilon > M^{\frac{d}{2}}).
 \end{aligned}$$

Thus

$$\lim_{M \rightarrow \infty} J_{221} \leq \lim_{M \rightarrow \infty} M^{-1} + \lim_{M \rightarrow \infty} \mathbb{E} \xi_1^2 I(|\xi_1| > M^{\frac{d}{2}}) = 0 \tag{27}$$

uniformly holds true with respect to $0 < \varepsilon < 1$.

Finally we consider J_{222} , since $2\beta \geq d$ and $q > 2$, we have

$$\begin{aligned} J_{222} &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n^{\frac{q}{2}} \log n} \mathbb{E} |\xi_1|^q I(|\xi_1| > \sqrt{n}(\log \log n)^{\frac{d}{2}} \varepsilon) \\ &\quad \times \int_0^\infty (x + \varepsilon)^{-q} I(|\xi_1| \leq (x + \varepsilon)\sqrt{n}(\log \log n)^{\frac{d}{2}}) dx \\ &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1+\frac{d}{2}-\frac{dq}{2}}}{n^{\frac{q}{2}} \log n} \mathbb{E} |\xi_1|^q I(|\xi_1| > \sqrt{n}(\log \log n)^{\frac{d}{2}} \varepsilon) \\ &\quad \times \int_{\frac{|\xi_1|}{\sqrt{n}(\log \log n)^{\frac{d}{2}}-\varepsilon}}^\infty (x + \varepsilon)^{-q} dx \\ &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{\sqrt{n} \log n} \mathbb{E} |\xi_1| I(|\xi_1| > \sqrt{n}(\log \log n)^{\frac{d}{2}} \varepsilon) \\ &\leq c\varepsilon^{2\beta/d} \sum_{n>b(\varepsilon)} \frac{(\log \log n)^{\beta-1}}{\sqrt{n} \log n} \\ &\quad \times \sum_{j=n}^\infty \mathbb{E} |\xi_1| I(\sqrt{j}(\log \log j)^{\frac{d}{2}} \varepsilon < |\xi_1| \leq \sqrt{j+1}(\log \log(j+1))^{\frac{d}{2}} \varepsilon) \\ &\leq c\varepsilon^{2\beta/d} \sum_{j=b(\varepsilon)}^\infty \mathbb{E} |\xi_1| I(\sqrt{j}(\log \log j)^{\frac{d}{2}} \varepsilon < |\xi_1| \leq \sqrt{j+1}(\log \log(j+1))^{\frac{d}{2}} \varepsilon) \\ &\quad \times \sum_{n=b(\varepsilon)}^j \frac{(\log \log n)^{\beta-1}}{\sqrt{n} \log n} \\ &\leq c\varepsilon^{2\beta/d} \sum_{j=b(\varepsilon)}^\infty \sqrt{j} \mathbb{E} |\xi_1| I(\sqrt{j}(\log \log j)^{\frac{d}{2}} \varepsilon < |\xi_1| \leq \sqrt{j+1}(\log \log(j+1))^{\frac{d}{2}} \varepsilon) \\ &\leq c\varepsilon^{2\beta/d-1} \sum_{j=b(\varepsilon)}^\infty \mathbb{E} \xi_1^2 I(\sqrt{j}(\log \log j)^{\frac{d}{2}} \varepsilon < |\xi_1| \leq \sqrt{j+1}(\log \log(j+1))^{\frac{d}{2}} \varepsilon) \\ &\leq c\varepsilon^{2\beta/d-1} \mathbb{E} \xi_1^2 I(|\xi_1| \geq \varepsilon \sqrt{b(\varepsilon)} (\log \log b(\varepsilon))^{\frac{d}{2}} > M^{\frac{d}{2}}). \end{aligned}$$

Hence,

$$\lim_{M \rightarrow \infty} J_{222} \leq \lim_{M \rightarrow \infty} \varepsilon^{2\beta/d-1} \mathbb{E} \xi_1^2 I(|\xi_1| > M^{d/2}) \leq \lim_{M \rightarrow \infty} \mathbb{E} \xi_1^2 I(|\xi_1| > M^{d/2}) = 0 \tag{28}$$

uniformly holds true with respect to $0 < \varepsilon < 1$.

Thus, combining with (23)-(28), we complete the proof of this proposition. □

Combining with Proposition 3.1, Proposition 3.2, and Proposition 3.3 we complete the proof of Theorem 1.2.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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