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Sharp H_p-L_p type inequalities of weighted maximal operators of Vilenkin-Nörlund means and its applications

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Abstract

We prove and discuss some new H_p-L_p type inequalities of weighted maximal operators of Vilenkin-Nörlund means with monotone coefficients. It is also proved that these inequalities are the best possible in a special sense. We also apply these results to prove strong summability for such Vilenkin-Nörlund means. As applications, both some well-known and new results are pointed out.

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1 Introduction

The definitions and notations used in this introduction can be found in our next section. In the one-dimensional case the weak $(1,1)$-type inequality for maximal operator of Fejér means $\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|$ can be found in Schipp [1] for Walsh series and in Pál, Simon [2] for bounded Vilenkin series. Fujji [3] and Simon [4] verified that σ^* is bounded from H_1 to L_1. Weisz [5] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [6] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. In the case $p = 1/2$ a counterexample with respect to Walsh system was given by Goginava [7] and for the bounded Vilenkin system was proved by Tephnadze [8]. Weisz [9] proved that the maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

Weisz [10] proved that the maximal operator of Cesàro means $\sigma^{\alpha,*} f := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f|$ is bounded from the martingale space H_p to the space L_p for $p > 1/(1+\alpha)$. Goginava [11] gave a counterexample, which shows that boundedness does not hold for $0 < p \le 1/(1+\alpha)$. Simon and Weisz [12] showed that the maximal operator $\sigma^{\alpha,*}$ $(0 < \alpha < 1)$ of the (C, α) means is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space weak-$L_{1/(1+\alpha)}$. In [13] and [14] it was also proved that the maximal operator

$$\widetilde{\sigma}_p^{\alpha,*} := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f| / \big((n+1)^{1/p-\alpha-1} \log^{(1+\alpha)[p+\alpha(1+\alpha)]}(n+1)\big)$$

is bounded from the Hardy space H_p to the Lebesgue space L_p, where $0 < p \leq 1/(1 + \alpha)$. Moreover, the rate of the weights $\{(n + 1)^{1/p-\alpha-1} \log^{(1+\alpha)[p+\alpha(1+\alpha)]}(n + 1)\}_{n=1}^{\infty}$ in nth Cesàro mean is given exactly.

It is well known that Vilenkin systems do not form bases in the space $L_1(G_m)$. Moreover, there is a function in the Hardy space $H_1(G_m)$, such that the partial sums of f are not bounded in L_1-norm. Simon [15] (for unbounded Vilenkin systems in the case when $p = 1$ see [16] and for $0 < p < 1$ another proof was pointed out in [17]) proved that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[p]} n} \sum_{k=1}^{n} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p \quad (0 < p \leq 1) \tag{1}$$

for all $f \in H_p$ and $n \in \mathbb{N}_+$, where $[p]$ denotes the integer part of p. In [18] for Walsh system and in [19] with respect to bounded Vilenkin system it was proved that sequence $\{1/k^{2-p}\}_{k=1}^{\infty}$ $(0 < p < 1)$ in (1) cannot be improved.

In [20] it was proved that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p \quad (0 < p \leq 1/2, n = 2, 3, \dots). \tag{2}$$

An analogous result for (C, α) $(0 < \alpha < 1)$ means when $p = 1/(1 + \alpha)$ was generalized in [13] and when $0 < p < 1/(1 + \alpha)$ it was proved in [14]. In particular, the following inequality:

$$\frac{1}{\log^{[\alpha/(1+\alpha)+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k^{\alpha} f\|_p^p}{k^{2-(1+\alpha)p}} \leq c_p \|f\|_{H_p}^p \quad \left(0 < p \leq 1/(1 + \alpha), n = 2, 3, \dots\right)$$

holds.

Móricz and Siddiqi [21] investigated the approximation properties of some special Nörlund mean of the L_p function in norm. For more information on Nörlund means, see the paper of Blahota and Gát [22] and Nagy [23] (see also [24, 25], and [26]).

In [27] for $p = 1/(1 + \alpha)$ and in [28] for $0 < p < 1/(1 + \alpha)$ there was proved that for every $f \in H_p$ and for every Nörlund mean $t_n f$, generated by the non-increasing sequence $\{q_n : n \geq 0\}$, satisfying the conditions

$$n^{\alpha}/Q_n = O(1), \quad \text{as } n \to \infty, \tag{3}$$

and

$$(q_n - q_{n+1})/n^{\alpha-2} = O(1), \quad \text{as } n \to \infty, \tag{4}$$

there exists an absolute constant $c_{\alpha,p}$ such that

$$\frac{1}{\log^{[\alpha/(1+\alpha)+p]} n} \sum_{k=1}^{n} \frac{\|t_k f\|_p^p}{k^{2-(1+\alpha)p}} \leq c_{\alpha,p} \|f\|_{H_p}^p \quad (n = 2, 3, \dots) \tag{5}$$

and

$$\left\| \sup_{n \in \mathbb{N}} |t_n f| / \left((n+1)^{1/p-\alpha-1} \log^{(1+\alpha)[p+\alpha(1+\alpha)]}(n+1)\right) \right\|_{H_p} \le c_{\alpha,p} \|f\|_{H_p}. \tag{6}$$

In [29] it was proved that in the endpoint case $p = 1/(1+\alpha)$ both (3) and (4) conditions are sharp in a special sense.

In this paper we investigate the case when $0 < p < 1/(1+\alpha)$ and prove inequalities (5) and (6) for $f \in H_p$ and Vilenkin-Nörlund means with non-increasing coefficients, but with weaker conditions than (3) and (4), which give possibility to prove analogous results for the wider class of Vilenkin-Nörlund means when $0 < p < 1/(1+\alpha)$. As applications, both some well-known and new results are pointed out.

This paper is organized as follows: In order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results can be found in Section 3. For the proofs of the main results we need some lemmas, both well known, but also some new ones of independent interest. These results are presented in Section 4. The detailed proofs are given in Section 5. Some well-known and new consequences of our main results are presented in Section 6.

2 Definitions and notations

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of the Z_{m_j}.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

In this paper we discuss bounded Vilenkin groups, *i.e.* the case when $\sup_n m_n < \infty$.

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots) \quad (x_j \in Z_{m_j}).$$

It is easy to give a base for the neighborhood of G_m:

$$I_0(x) := G_m, \qquad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\},$$

where $x \in G_m$, $n \in \mathbb{N}$.

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}_+$, and $\overline{I_n} := G_m \backslash I_n$.

$$\overline{I_N} = \left(\bigcup_{k=0}^{N-2} \bigcup_{s_k=1}^{m_k-1} \bigcup_{l=k+1}^{N-1} \bigcup_{s_l=1}^{m_l-1} I_{l+1}(s_k e_k + s_l e_l) \right) \cup \left(\bigcup_{k=1}^{N-1} \bigcup_{s_k=1}^{m_k-1} I_N(s_k e_k) \right). \tag{7}$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of the n_j differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad (i^2 = -1, x \in G_m, k \in \mathbb{N}).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system a Walsh-Paley system when $m \equiv 2$.

The norms (or quasi-norms) of the spaces $L_p(G_m)$ *and weak-*$L_p(G_m)$ $(0 < p < \infty)$ are, respectively, defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu, \qquad \|f\|_{\text{weak-}L_p}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda).$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [30]).

Next, we introduce analogs of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi}_n \, d\mu \quad (n \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \qquad D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}_+),$$

respectively.

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{8}$$

and

$$D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{k=m_j-n_j}^{m_j-1} r_j^k \right), \tag{9}$$

for $n = \sum_{i=0}^{\infty} n_i M_i$.

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n $(n \in \mathbb{N})$. Denote by $f = (f^{(n)}, n \in \mathbb{N})$ a martingale with respect to F_n $(n \in \mathbb{N})$ (for details see *e.g.* [31]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f = (f^{(n)}, n \in \mathbb{N})$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)} \overline{\psi}_i \, d\mu.$$

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The nth Nörlund means for a Fourier series of f is defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \tag{10}$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k.$$

A representation

$$t_n f(x) = \int_{G_m} f(t) F_n(x - t) \, d\mu(t)$$

plays a central role in the sequel, where

$$F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k$$

is the so-called Nörlund kernel.

We say that the Nörlund mean t_n is of (N, α) type if

$$\frac{n^\alpha}{Q_n} = O(1), \quad \text{as } n \to \infty \tag{11}$$

and for any $\varepsilon > 0$, we have

$$\frac{Q_n}{n^{\alpha+\varepsilon}} \to 0, \quad \text{as } n \to \infty. \tag{12}$$

For our further investigation it is much more convenient to replace condition (12) by its equivalent one:

$$Q_n \leq c_\alpha n^\alpha \varphi_n, \quad \text{where } \varlimsup_{j \to \infty} \frac{\varphi_j}{j^\varepsilon} = 0, \text{for every } \varepsilon > 0. \tag{13}$$

We always assume that $q_0 > 0$ and $\lim_{n \to \infty} Q_n = \infty$. In this case it is well known that the summability method generated by $\{q_k : k \geq 0\}$ is regular if and only if

$$\lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = 0.$$

Concerning this fact and related basic results, we refer to [32].

If $q_n \equiv 1$, then we get the nth Fejér mean and the Fejér kernel

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \qquad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k,$$

respectively.

Let $t, n \in \mathbb{N}$. It is well known that (see [33])

$$K_{M_n}(x) = \begin{cases} 0, & \text{if } x - x_t e_t \notin I_n, x \in I_t \backslash I_{t+1}, \\ \frac{M_t}{1 - r_t(x)}, & \text{if } x - x_t e_t \in I_n, x \in I_t \backslash I_{t+1}, \\ (M_n + 1)/2, & \text{if } x \in I_n. \end{cases} \tag{14}$$

The (C, α)-means of the Vilenkin-Fourier series are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha := 0, \qquad A_n^\alpha := \frac{(\alpha + 1) \cdots (\alpha + n)}{n!}, \qquad \alpha \neq -1, -2, \ldots.$$

For the martingale f we consider the following maximal operators:

$$\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|, \qquad \sigma^{\alpha,*} f := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f|, \qquad t^* f := \sup_{n \in \mathbb{N}} |t_n f|.$$

We also consider the following weighted maximal operators:

$$\widetilde{\sigma}_p^* := \sup_{n \in \mathbb{N}} |\sigma_n f| / \left((n+1)^{1/p-2} \log^{2[p+1/2]}(n+1) \right) \quad (0 < p \leq 1/2),$$

$$\widetilde{\sigma}_p^{\alpha,*} := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f| / \left((n+1)^{1/p-\alpha-1} \log^{(1+\alpha)[p+\alpha(1+\alpha)]}(n+1) \right) \quad \left(0 < p \leq 1/(1+\alpha) \right),$$

$$\widetilde{t}_p^* := \sup_{n \in \mathbb{N}} |t_n f| / \left((n+1)^{1/p-\alpha-1} \log^{(1+\alpha)[p+\alpha(1+\alpha)]}(n+1) \right) \quad \left(0 < p \leq 1/(1+\alpha) \right).$$

A bounded measurable function a is a p-atom, if there exists an interval I, such that

$$\int_I a \, d\mu = 0, \qquad \|a\|_\infty \leq \mu(I)^{-1/p}, \qquad \text{supp}(a) \subset I.$$

3 The main results

Our sharp H_p-L_p inequality reads as follows.

Theorem 1

(a) *Let $f \in H_p$, where $0 < p < 1/(1 + \alpha)$ for some $0 < \alpha \leq 1$, and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers satisfying conditions* (11) *and* (12). *Then the maximal operator*

$$\tilde{t}{}^{*}_{p,\alpha} := \sup_{n \in \mathbb{N}} \frac{|t_n f|}{(n+1)^{1/p-1-\alpha}}$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p, i.e. the following inequality holds:

$$\left\| \sup_{n \in \mathbb{N}} |t_n f| / \left((n+1)^{1/p-1-\alpha} \right) \right\|_p \leq c_{\alpha,p} \|f\|_{H_p}. \tag{15}$$

(b) *Let $0 < p < 1/(1 + \alpha)$ for some $0 < \alpha \leq 1$, and $\{\Phi_n : n \in \mathbb{N}_+\}$ be any non-decreasing sequence, satisfying the condition*

$$\varlimsup_{n \to \infty} \frac{(n+1)^{1/p-1-\alpha}}{\Phi_n} = \infty. \tag{16}$$

Then the inequality (15) *is sharp in the sense that there exist a Nörlund mean with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying the conditions* (11) *and* (12) *and a martingale $f \in H_P$ such that*

$$\sup_{k \in \mathbb{N}} \frac{\left\| \frac{t_{M_{2n_k}+1} f_k}{\Phi_{M_{2n_k}+1}} \right\|_{\text{weak-}L_p}}{\|f_k\|_{H_p}} = \infty.$$

Our new result concerning strong summability of Nörlund means with non-increasing sequences reads as follows.

Theorem 2 *Let $f \in H_p$, where $0 < \alpha < 1$, $0 < p < 1/(1 + \alpha)$, and let $\{q_n : n \geq 0\}$ be a sequence of non-increasing numbers, satisfying conditions* (11) *and* (12). *Then there exists an absolute constant $c_{\alpha,p}$, depending only on α and p, such that the inequality*

$$\sum_{k=1}^{\infty} \frac{\|t_k f\|_p^p}{k^{2-(1+\alpha)p}} \leq c_{\alpha,p} \|f\|_{H_p}^p$$

holds.

4 Lemmas

We need the following well-known lemma of Weisz [34].

Lemma 1 *Suppose that an operator T is σ-linear and for some $0 < p \leq 1$ and*

$$\int_{\bar{I}} |Ta|^p \, d\mu \leq c_p < \infty,$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_p \leq c_p \|f\|_{H_p}.$$

The next results are due to Blahota, Persson, and Tephnadze [27].

Lemma 2 *Let $s_n M_n < r \leq (s_n + 1)M_n$, where $1 \leq s_n \leq m_n - 1$. Then for every Nörlund mean, without any restriction on the generative sequence $\{q_k : k \in \mathbb{N}\}$ we have the following equality*:

$$Q_r F_r = Q_r D_{s_n M_n} - \psi_{s_n M_n - 1} \sum_{l=1}^{s_n M_n - 2} (q_{r - s_n M_n + l} - q_{r - s_n M_n + l + 1}) l \overline{K_l}$$
$$- \psi_{s_n M_n - 1}(s_n M_n - 1)q_{r-1} \overline{K_{s_n M_n - 1}} + \psi_{s_n M_n} Q_{r - s_n M_n} F_{r - s_n M_n}.$$

We also need the following new lemmas of independent interest.

Lemma 3 *Let $0 < \alpha \leq 1$ and $\{q_n : n \geq 0\}$ be a sequence of non-increasing numbers satisfying conditions* (11) *and* (12). *Then*

$$|Q_n F_n| \leq c_\alpha \left\{ \sum_{j=0}^{|n|} M_j^\alpha \varphi_j |K_{M_j}| \right\},$$

where

$$\varlimsup_{j \to \infty} \frac{\varphi_j}{j^\varepsilon} = 0, \quad \text{for every } \varepsilon > 0. \tag{17}$$

Lemma 4 *Let $0 < \alpha \leq 1$ and $\{q_n : n \geq 0\}$ be a sequence of non-increasing numbers, satisfying conditions* (11) *and* (12). *If $r \geq M_N$, then*

$$\int_{I_N} |F_r(x - t)| \, d\mu(t) \leq \frac{c_\alpha M_l^\alpha \varphi_l M_k}{r^\alpha M_N} \leq \frac{c_\alpha M_l^\alpha \varphi_l M_k}{M_N^{1+\alpha}}, \quad x \in I_{l+1}(s_k e_k + s_l e_l),$$

where

$$1 \leq s_k \leq m_k - 1, \qquad 1 \leq s_l \leq m_l - 1 \quad (k = 0, \ldots, N - 2, l = k + 1, \ldots, N - 1)$$

and

$$\int_{I_N} |F_r(x - t)| \, d\mu(t) \leq \frac{c_\alpha M_k}{M_N}, \quad x \in I_N(s_k e_k),$$

where

$$1 \leq s_k \leq m_k - 1 \quad (k = 0, \ldots, N - 1).$$

5 Proofs

Proof of Lemma 3 Let $0 < \alpha \leq 1$ and $\{q_k : k \geq 0\}$ satisfy the conditions (11) and (12). Since

$$n^\alpha \varphi_n \geq Q_n \geq nq_{n-1}$$

we obtain

$$q_{n-1} \leq n^{\alpha-1} \varphi_n, \tag{18}$$

where φ_n satisfies condition (17).

By using an Abel transformation we get

$$Q_n = \sum_{j=1}^{n-2} (q_j - q_{j+1})j + q_{n-1}(n-1) + q_0 \tag{19}$$

and

$$\sum_{j=1}^{n-2} |q_j - q_{j+1}|j \leq Q_n \leq n^\alpha \varphi_n. \tag{20}$$

Suppose that

$$|q_j - q_{j+1}| \geq j^{\alpha-2} \phi_j \delta_j,$$

for all $j \in \mathbb{N}$, where δ_j is any function, such that

$$\varlimsup_{j \to \infty} \delta_j = \infty. \tag{21}$$

Under condition (19) there exists an increasing sequence $\{\alpha_k : k \geq 0\}$, such that $\alpha_{k+1} \geq 2\alpha_k$ and

$$\delta_{\alpha_k} \uparrow \infty.$$

Hence,

$$\sum_{j=\alpha_k}^{\alpha_{k+1}+1} |q_j - q_{j+1}|j \geq c\phi_{\alpha_k}\delta_{\alpha_k} \sum_{j=\alpha_k}^{\alpha_{k+1}+1} j^{\alpha-1}$$

$$\geq c\phi_{\alpha_k}\delta_{\alpha_k} \int_{\alpha_k}^{\alpha_{k+1}+2} x^{\alpha-1}\,dx$$

$$\geq \frac{c\phi_{\alpha_k}\delta_{\alpha_k}}{\alpha} x^\alpha \Big|_{\alpha_k}^{\alpha_{k+1}} \geq c\phi_{\alpha_k}\delta_{\alpha_k}\alpha_k^\alpha. \tag{22}$$

By combining (18) and (22) we get

$$\frac{Q_{\alpha_{k+1}+3}}{(\alpha_{k+1}+3)^\alpha(\phi_{\alpha_{k+1}}+3)} \geq \frac{\sum_{j=1}^{\alpha_{k+1}+1} |q_j - q_{j+1}|j}{(\alpha_{k+1}+3)^\alpha(\phi_{\alpha_{k+1}}+3)} \geq c\delta_{\alpha_k} \to \infty, \quad \text{as } k \to \infty.$$

This is a contradiction with condition (13), that is,

$$\frac{Q_n}{n^\alpha \phi_n} = O(1), \quad \text{as } n \to \infty. \tag{23}$$

It follows that

$$|q_j - q_{j+1}| \le j^{\alpha-2}\phi_j. \tag{24}$$

It is easy to see that

$$Q_k|D_{sM_n}| \le cM_n^\alpha \phi_n|D_{sM_n}| \tag{25}$$

and

$$(sM_n - 1)q_{k-1}|K_{sM_n-1}| \le c\phi_k k^{\alpha-1}M_n|K_{sM_n-1}| \le cM_n^\alpha \phi_n|K_{sM_n-1}|. \tag{26}$$

Let

$$n = s_{n_1}M_{n_1} + s_{n_2}M_{n_2} + \cdots + s_{n_r}M_{n_r}, \quad n_1 > n_2 > \cdots > n_r,$$

and

$$n^{(k)} = s_{n_{k+1}}M_{n_{k+1}} + \cdots + s_{n_r}M_{n_r}, \quad 1 \le s_{n_l} \le m_l - 1, l = 1,\dots,r.$$

By combining (24)-(26) and Lemma 2 we have

$$|Q_n F_n|$$

$$\le c_\alpha M_{n_1}^\alpha \phi_{n_1}|D_{s_{n_1}M_{n_1}}| + c_\alpha \sum_{l=1}^{s_{n_1}M_{n_1}-1} \left|\left(n^{(1)} + l\right)^{\alpha-2}\right|\phi_{n^{(1)}+l}|lK_l|$$

$$+ c_\alpha M_{n_1}^\alpha \phi_{n_1}|K_{s_{n_1}M_{n_1}-1}| + c_\alpha|Q_{n^{(1)}}F_{n^{(1)}}|.$$

By repeating this process r times we get

$$|Q_n F_n|$$

$$\le c_\alpha \sum_{k=1}^r \left(M_{n_k}^\alpha \phi_{n_k}|D_{s_{n_k}M_{n_k}}| + \sum_{l=1}^{s_{n_k}M_{n_k}-1} \left(n^{(k)} + l\right)^{\alpha-2}\phi_{n^{(k)}+l}|lK_l| + M_{n_k}^\alpha \phi_{n_k}|K_{s_{n_k}M_{n_k}-1}|\right)$$

$$:= I + II + III.$$

By combining (8), (9), and (14) we find that

$$I \le c_\alpha \sum_{k=1}^{|n|} M_k^\alpha \phi_k|D_{M_k}| \le c_\alpha \sum_{k=1}^{|n|} M_k^\alpha \phi_k|K_{M_k}|$$

and

$$III \le c_\alpha \sum_{k=1}^r M_{n_k}^{\alpha-1}\phi_{n_k}|M_{n_k}K_{s_{n_k}M_{n_k}} - D_{s_{n_k}M_{n_k}}| \le c_\alpha \sum_{k=1}^r M_k^\alpha \phi_{n_k}|K_{M_k}|.$$

Moreover,

$$
\begin{aligned}
II &= c_\alpha \sum_{k=1}^{r} \sum_{A=1}^{n_k} \sum_{l=s_{A-1}M_{A-1}}^{s_A M_A - 1} \left(n^{(k)} + l\right)^{\alpha-2} \phi_{n^{(k)}+l} |lK_l| \\
&= c_\alpha \sum_{k=1}^{r} \sum_{A=1}^{n_{k+1}} \sum_{l=s_{A-1}M_{A-1}}^{s_A M_A - 1} \left(n^{(k)} + l\right)^{\alpha-2} \phi_{n^{(k)}+l} |lK_l| \\
&\quad + c_\alpha \sum_{k=1}^{r} \sum_{A=n_{k+1}+1}^{n_k} \sum_{l=s_{A-1}M_{A-1}}^{s_A M_A - 1} \left(n^{(k)} + l\right)^{\alpha-2} \phi_{n^{(k)}+l} |lK_l| \\
&\leq c_\alpha \sum_{k=1}^{r} M_{n_{k+1}}^{\alpha-2} \phi_{n_{k+1}} \sum_{A=1}^{n_{k+1}} \sum_{l=s_{A-1}M_{A-1}}^{s_A M_A - 1} |lK_l| \\
&\quad + c_\alpha \sum_{k=1}^{r} \sum_{A=n_{k+1}+1}^{n_k} M_A^{\alpha-2} \phi_A \sum_{l=s_{A-1}M_{A-1}}^{s_A M_A - 1} |lK_l| := II_1 + II_2 .
\end{aligned}
$$

By applying (14) for II_1 we get

$$
\begin{aligned}
II_1 &\leq c_\alpha \sum_{k=1}^{r} M_{n_{k+1}}^{\alpha-2} \phi_{n_{k+1}} \sum_{A=1}^{n_{k+1}} \sum_{l=s_{A-1}M_{A-1}}^{s_A M_A - 1} \sum_{j=0}^{A} M_j |K_{M_j}| \\
&\leq c_\alpha \sum_{k=1}^{n_1} M_k^{\alpha-2} \phi_k \sum_{A=1}^{k} M_A \sum_{j=0}^{A} M_j |K_{M_j}| \\
&\leq c_\alpha \sum_{k=0}^{n_1} M_k^{\alpha-1} \phi_k \sum_{j=0}^{k} M_j |K_{M_j}| \\
&= c_\alpha \sum_{j=0}^{n_1} M_j |K_{M_j}| \sum_{k=j}^{n_1} \phi_k M_k^{\alpha-1} \leq c_\alpha \sum_{j=0}^{n_1} \phi_j M_j^{\alpha} |K_{M_j}| .
\end{aligned}
$$

By using (14) for II_2 we have similarly

$$
\begin{aligned}
II_2 &\leq c_\alpha \sum_{k=1}^{r} \sum_{A=n_{k+1}+1}^{n_k} M_A^{\alpha-1} \phi_A \sum_{j=0}^{A} M_j |K_{M_j}| \\
&\leq c_\alpha \sum_{A=1}^{n_1} M_A^{\alpha-1} \phi_A \sum_{j=0}^{A} M_j |K_{M_j}| \leq c_\alpha \sum_{j=0}^{n_1} \phi_j M_j^{\alpha} |K_{M_j}| .
\end{aligned}
$$

The proof is complete by combining the estimates above. $\qquad\square$

Proof of Lemma 4 Let $x \in I_{l+1}(s_k e_k + s_l e_l), 1 \leq s_k \leq m_k - 1, 1 \leq s_l \leq m_l - 1$. Then, by applying (14), we have

$$
K_{M_n}(x) = 0, \quad \text{when } n > l > k.
$$

Suppose that $k < n \le l$. Moreover, by using (14) we get

$$\left|K_{M_n}(x)\right| \le cM_k.$$

Let $n \le k < l$. Then

$$\left|K_{M_n}(x)\right| = (M_n + 1)/2 \le cM_k.$$

If we now apply Lemma 3 and (14) we can conclude that

$$Q_r\left|F_r(x)\right| \le c_\alpha \sum_{A=0}^{l} M_A^\alpha \varphi_A \left|K_{M_A}(x)\right| \le c_\alpha \sum_{A=0}^{l} M_A^\alpha \varphi_A M_k \le c_\alpha M_l^\alpha \varphi_l M_k. \tag{27}$$

Let $x \in I_{l+1}(s_k e_k + s_l e_l)$, for some $0 \le k < l \le N - 1$. Since $x - t \in I_{l+1}(s_k e_k + s_l e_l)$, for $t \in I_N$ and $r \ge M_N$ from (27) we obtain

$$\int_{I_N} \left|F_r(x - t)\right| d\mu(t) \le \frac{c_\alpha M_l^\alpha \varphi_l M_k}{r^\alpha M_N}. \tag{28}$$

Let $x \in I_N(s_k e_k)$, $k = 0, \ldots, N - 1$. Then, by applying (8) and (9) we have

$$\int_{I_N} \left|F_r(x - t)\right| d\mu(t) \le \frac{1}{Q_n} \sum_{m=1}^{n} q_{n-m} \int_{I_N} \left|D_m(x - t)\right| d\mu(t)$$

$$\le \frac{1}{Q_n} \sum_{m=1}^{n} q_{n-m} \frac{cM_k}{M_N} \le \frac{cM_k}{M_N}. \tag{29}$$

By combining (28) and (29) we complete the proof of Lemma 4. □

Proof of Theorem 1 According to Lemma 1 the proof of the first part of Theorem 1 will be complete, if we show that

$$\int_{\overline{I_N}} \left|\widetilde{t}^{*,p} a(x)\right|^p d\mu(x) < \infty,$$

for every $1/(1 + \alpha - \varepsilon)$-atom a. We may assume that a is an arbitrary p-atom with support I, $\mu(I) = M_N^{-1}$ and $I = I_N$. It is easy to see that $t_n(a) = 0$, when $n \le M_N$. Therefore, we can suppose that $n > M_N$.

By using Lemma 3 we easily see that $\widetilde{t}^{*,p}$ is bounded from L_∞ to L_∞. Let $x \in I_N$. Since $\|a\|_\infty \le M_N^{1/p}$ we obtain

$$\left|t_n a(x)\right| \le \int_{I_N} \left|a(t)\right| \left|F_n(x - t)\right| d\mu(t)$$

$$\le \|a\|_\infty \int_{I_N} \left|F_n(x - t)\right| d\mu(t) \le M_N^{1/p} \int_{I_N} \left|F_n(x - t)\right| d\mu(t).$$

Let $x \in I_{l+1}(s_k e_k + s_l e_l)$, $0 \le k < l < N$. From Lemma 4 we get

$$\left|t_n a(x)\right| \le \frac{c_\alpha M_l^\alpha \varphi_l M_k M_N^{1/p}}{n^\alpha M_N} \le \frac{c_\alpha \varphi_l M_l^\alpha M_k}{n^\alpha M_N^{1-1/p}} \le \frac{c_\alpha \varphi_l M_l^\alpha M_k}{M_N^{1+\alpha-1/p}}. \tag{30}$$

Let $x \in I_N(s_k e_k)$, $0 \le k < N$. According to Lemma 4 we have

$$\left| t_n a(x) \right| \le \frac{c_\alpha M_N^{1/p} M_k}{M_N} \le c_\alpha M_N^{1/p-1} M_k. \tag{31}$$

By combining (7) and (30)-(31) we obtain

$$\int_{\overline{I_N}} \left| t^{\widetilde{*,p}} a \right|^p d\mu$$

$$= \sum_{k=0}^{N-2} \sum_{s_k=1}^{m_k-1} \sum_{l=k+1}^{N-1} \sum_{s_l=1}^{m_l-1} \int_{I_{l+1}(s_k e_k + s_l e_l)} \sup_{n > M_N} \left| \frac{t_n a}{n^{1/p-1-\alpha}} \right|^p d\mu$$

$$+ \sum_{k=0}^{N-1} \sum_{s_k=1}^{m_k-1} \int_{I_N(s_k e_k)} \sup_{n > M_N} \left| \frac{t_n a}{n^{1/p-1-\alpha}} \right|^p d\mu$$

$$\le c_\alpha \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(m_k-1)(m_l-1)}{M_{l+1}} \left(\varphi_l M_l^\alpha M_k \right)^p + c_\alpha \sum_{k=0}^{N-1} \frac{(m_k-1)}{M_N} M_N^{\alpha p} M_k^p$$

$$\le c_\alpha \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(\varphi_l M_l^\alpha M_k)^p}{M_{l+1}} + c_\alpha \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^{(1-\alpha p)}}$$

$$\le c_\alpha \sum_{k=0}^{N-2} M_k^p \sum_{l=k+1}^{N-1} \frac{\varphi_l^p}{M_l^{1-\alpha p}} + c_\alpha \le \sum_{k=0}^{N-2} M_k^p \frac{\varphi_k^p}{M_k^{1-\alpha p}} + c_\alpha$$

$$\le c_\alpha \sum_{k=0}^{N-2} \frac{\varphi_k^p}{M_k^{1-p(\alpha+1)}} + c_\alpha \le c_\alpha \sum_{k=0}^{N-2} \frac{\varphi_k^p}{2^{k(1-p(1+\alpha))}} + c_\alpha \le c_\alpha < \infty.$$

The proof of part (a) is complete.

Under condition (16) there exist positive integers n_k such that

$$\lim_{k \to \infty} \frac{(M_{2n_k}+1)^{1/p-1-\alpha}}{\Phi_{M_{2n_k}+1}} = \infty, \quad 0 < p < 1/(1+\alpha).$$

Let t_n be Nörlund mean with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying (11) and condition (12), but in the restricted form

$$\frac{c_\alpha (M_{2n_k}+1)^{1/p-1-\alpha}}{\varphi_{M_{2n_k}+1} \Phi_{M_{2n_k}+1}} \to \infty, \quad \text{as } k \to \infty.$$

Set

$$f_k := D_{M_{2n_k+1}} - D_{M_{2n_k}}.$$

Then

$$\widehat{f_k}(i) = \begin{cases} 1, & i = M_{2n_k}, \ldots, M_{2n_k+1} - 1, \\ 0, & \text{otherwise,} \end{cases} \tag{32}$$

and

$$S_i f_k = \begin{cases} D_i - D_{M_{2n_k}}, & i = M_{2n_k}+1, \ldots, M_{2n_k+1}-1, \\ f_k, & i \geq M_{2n_k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

Moreover,

$$\|f_k\|_{H_p} \leq \lambda M_{2n_k}^{1-1/p}, \tag{34}$$

where $\lambda = \sup_n m_n$.

By using (33) we get

$$\frac{|t_{M_{2n_k}+1}f_k|}{\Phi_{M_{2n_k}+1}} = \frac{q_0|S_{M_{2n_k}+1}|}{Q_{M_{2n_k}+1}\Phi_{M_{2n_k}+1}}$$

$$= \frac{q_0|D_{M_{2n_k}+1} - D_{M_{2n_k}}|}{Q_{M_{2n_k}+1}\Phi_{M_{2n_k}+1}} = \frac{q_0|\psi_{M_{2n_k}}|}{Q_{M_{2n_k}+1}\Phi_{M_{2n_k}+1}} \geq \frac{c_\alpha}{M_{2n_k}^\alpha \Phi_{M_{2n_k}+1}\varphi_{M_{2n_k}+1}}.$$

Hence,

$$\mu\left\{ x \in G_m : \frac{|t_{M_{2n_k}+1}f_k(x)|}{\Phi_{M_{2n_k}+1}} \geq \frac{c_\alpha}{M_{2n_k}^\alpha \Phi_{M_{2n_k}+1}\varphi_{M_{2n_k}+1}} \right\} = 1. \tag{35}$$

By applying (35) we have

$$\frac{\frac{c_\alpha}{M_{2n_k}^\alpha \Phi_{M_{2n_k}+1}\varphi_{M_{2n_k}+1}}\left(\mu\left\{ x \in G_m : \frac{|t_{M_{2n_k}+1}f_k(x)|}{\Phi_{M_{2n_k}+1}} \geq \frac{q_0}{M_{2n_k}^\alpha \Phi_{M_{2n_k}+1}\varphi_{M_{2n_k}+1}} \right\}\right)^{1/p}}{\|f_k\|_{H_p}}$$

$$\geq \frac{c_\alpha M_{2n_k}^{1/p-1-\alpha}}{\Phi_{M_{2n_k}+1}\varphi_{M_{2n_k}+1}} \geq \frac{c_\alpha (M_{2n_k}+1)^{1/p-1-\alpha}}{\Phi_{M_{2n_k}+1}\varphi_{M_{2n_k}+1}} \to \infty, \quad \text{as } k \to \infty.$$

The proof is complete. □

Proof of Theorem 2 According to Lemma 1 the proof of Theorem 2 will be complete, if we show that

$$\sum_{m=1}^\infty \frac{\|t_m a\|_p^p}{m^{2-(1+\alpha)p}} \leq c_\alpha < \infty,$$

for every p-atom a. Analogously to the first part of Theorem 1 we can assume that $n > M_N$ and a be an arbitrary p-atom, with support I, $\mu(I) = M_N$, and $I = I_N$.

Let $x \in I_N$. Since $\|a\|_\infty \leq cM_N^{1/p}$ if we apply Lemma 3 we obtain

$$\int_{I_N} |t_m a|^p \, d\mu \leq \int_{I_N} \|F_m\|_1^p \|a\|_\infty^p \, d\mu \leq c_{\alpha,p}\varphi_m^p \int_{I_N} \|a\|_\infty^p \, d\mu \leq c_{\alpha,p}\varphi_m^p.$$

Hence

$$\sum_{m=M_N+1}^\infty \frac{\int_{I_N} |t_m a|^p \, d\mu}{m^{2-(1+\alpha)p}} \leq c_{\alpha,p} \sum_{m=M_N+1}^\infty \frac{\varphi_m^p}{m^{2-(1+\alpha)p}} \leq \frac{c_{\alpha,p}\varphi_N^p}{2^{N(1-(1+\alpha)p)}} \leq c_{\alpha,p} < \infty.$$

By combining (7), (30), and (31) analogously to first part of Theorem 1 we can write

$$\sum_{m=M_N+1}^{\infty} \frac{\int_{\overline{I_N}} |t_m a|^p \, d\mu}{m^{2-(1+\alpha)p}}$$

$$= \sum_{m=M_N+1}^{\infty} \left(\sum_{k=0}^{N-2} \sum_{s_k=1}^{m_k-1} \sum_{l=k+1}^{N-1} \sum_{s_l=1}^{m_l-1} \frac{\int_{I_{l+1}(s_k e_k + s_l e_l)} |t_m a|^p \, d\mu}{m^{2-(1+\alpha)p}} + \sum_{k=0}^{N-1} \sum_{s_k=0}^{m_k-1} \frac{\int_{I_N(s_k e_k)} |t_m a|^p \, d\mu}{m^{2-(1+\alpha)p}} \right)$$

$$\leq \sum_{m=M_N+1}^{\infty} \left(\frac{c_{\alpha,p} M_N^{1-p}}{m^{2-p}} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_l^{p\alpha} \varphi_l^p M_k^p}{M_l} + \frac{c_{\alpha,p} M_N^{1-p}}{m^{2-(1+\alpha)p}} \sum_{k=0}^{N-1} \frac{M_k^p}{M_N} \right)$$

$$< c_{\alpha,p} M_N^{1-p} \sum_{m=M_N+1}^{\infty} \frac{1}{m^{2-p}} + c_{\alpha,p} \sum_{m=M_N+1}^{\infty} \frac{1}{m^{2-(1+\alpha)p}} \leq c_{\alpha,p} < \infty,$$

which completes the proof. □

6 Applications and final remarks

Remark 1 We note that under the conditions (3) and (4) we see that the conditions (11) and (12) are also fulfilled.

Proof Let $0 < \alpha \leq 1$. We can write

$$|Q_n| = \left| \sum_{j=1}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n-1) + q_0 \right|.$$

First suppose that $\alpha = 1$. Then

$$q_{n-1} \leq q_0 \leq c$$

and

$$|Q_n| \leq \left| \sum_{j=1}^{n-2} j^0 + c(n-1) + q_0 \right| \leq cn.$$

Moreover, condition (12) automatically holds,

$$\frac{Q_n}{n^{1+\varepsilon}} \leq \frac{q_0 n}{n^{1+\varepsilon}} \to 0, \quad \text{as } n \to \infty,$$

for any $\varepsilon > 0$.

Since the case $q_0 n / Q_n = O(1)$, as $n \to \infty$, has already been considered, we can exclude it. Hence, we may assume that $\{q_k : k \geq 0\}$ satisfies conditions (3) and (4) and, in addition, satisfies the following condition:

$$\frac{Q_n}{q_0 n} = o(1), \quad \text{as } n \to \infty.$$

It follows that

$$q_n = q_0 \frac{q_n n}{q_0 n} \leq q_0 \frac{Q_n}{q_0 n} = o(1), \quad \text{as } n \to \infty,$$

$$q_n = \sum_{l=n}^{\infty}(q_l - q_{l+1}) \le \sum_{l=n}^{\infty}\frac{1}{l^{2-\alpha}} \le \frac{c}{n^{1-\alpha}},$$

and

$$|Q_n| \le \left|\sum_{j=1}^{n-2}j^{\alpha-1} + c(n-1)^{\alpha} + q_0\right| \le cn^{\alpha}.$$

$\square$

From Remark 1 we immediately see that the following is true.

Corollary 1 *Conditions* (3) *and* (4) *provide a wider class of Nörlund means with non-increasing coefficients than conditions* (11) *and* (12).

From the proof of Remark 1 for $\alpha = 1$ we immediately have the following.

Remark 2 Let $\alpha = 1$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. Then condition (12) automatically holds,

$$\frac{Q_n}{n^{1+\varepsilon}} \to 0, \quad \text{as } n \to \infty,$$

for any $\varepsilon > 0$.

By applying Remark 2 and Theorem 1 we get the following.

Theorem 3

(a) *Let $f \in H_p$, where $0 < p < 1/2$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers satisfying condition* (11) *for $\alpha = 1$. Then the maximal operator*

$$\widetilde{t}^{*}_{p,1} := \sup_{n\in\mathbb{N}} \frac{|t_n f|}{(n+1)^{1/p-2}}$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p, i.e. the inequality

$$\left\|\sup_{n\in\mathbb{N}}|t_n f|/\left((n+1)^{1/p-2}\right)\right\|_p \le c_{\alpha,p}\|f\|_{H_p} \tag{36}$$

holds.

(b) *Let $\{\Phi_n : n \in \mathbb{N}_+\}$ be any non-decreasing sequence, satisfying the condition*

$$\overline{\lim_{n\to\infty}}\frac{(n+1)^{1/p-2}}{\Phi_n} = \infty.$$

Then the inequality (36) *is sharp in the sense that there exists a Nörlund mean with non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying the condition* (11) *such that*

$$\sup_{k\in\mathbb{N}}\frac{\left\|\frac{t_{M_{2n_k}+1}f_k}{\Phi_{M_{2n_k}+1}}\right\|_{\text{weak-}L_p}}{\|f_k\|_{H_p}} = \infty.$$

By applying Remark 2 and Theorem 2 we get the following.

Theorem 4 *Let $f \in H_p$, where $0 < p < 1/2$ and $\{q_n : n \geq 0\}$ be a sequence of non-increasing numbers, satisfying condition* (11). *Then there exists an absolute constant $c_{\alpha,p}$, depending only on α and p, such that*

$$\sum_{k=1}^{\infty} \frac{\|t_k f\|_{H_p}^p}{k^{2-(1+\alpha)p}} \leq c_{\alpha,p} \|f\|_{H_p}^p.$$

From Theorem 3 we get the following result by Tephnadze [35].

Corollary 2

(a) *Let $f \in H_p$, where $0 < p < 1/2$. Then the maximal operator*

$$\overset{\sim}{\sigma}{}_p^* := \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{(n+1)^{1/p-2}}$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p.

(b) *Let $\{\Phi_n : n \in \mathbb{N}_+\}$ be any non-decreasing sequence, satisfying the condition*

$$\varlimsup_{n \to \infty} \frac{(n+1)^{1/p-2}}{\Phi_n} = \infty.$$

Then

$$\sup_{k \in \mathbb{N}} \frac{\left\| \frac{\sigma_{M_{2n_k}+1} f_k}{\Phi_{M_{2n_k}+1}} \right\|_{\text{weak-}L_p}}{\|f_k\|_{H_p}} = \infty.$$

Moreover, Theorem 4 implies the following result by Tephnadze [18].

Corollary 3 *Let $f \in H_p$, where $0 < p < 1/2$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\sum_{k=1}^{\infty} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p.$$

Next we note that Theorem 1 and Remark 1 imply the following results of Blahota, Tephnadze [13] for $0 < \alpha < 1$.

Corollary 4

(a) *Let $f \in H_p$, where $0 < p < 1/(1+\alpha)$ for some $0 < \alpha < 1$. Then the maximal operator*

$$\overset{\sim}{\sigma}{}_p^{*,\alpha} := \sup_{n \in \mathbb{N}} \frac{|\sigma_n^\alpha f|}{(n+1)^{1/p-1-\alpha}}$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p.

(b) *Let $\{\Phi_n : n \in \mathbb{N}_+\}$ be any non-decreasing sequence, satisfying the condition*

$$\varlimsup_{n \to \infty} \frac{(n+1)^{1/p-1-\alpha}}{\Phi_n} = \infty.$$

Then

$$\sup_{k \in \mathbb{N}} \frac{\left\| \frac{\sigma^{\alpha}_{M_{2n_k}+1} f_k}{\Phi_{M_{2n_k}+1}} \right\|_{\text{weak-}L_p}}{\|f_k\|_{H_p}} = \infty.$$

Similarly, Theorem 2 and Remark 1 immediately imply the following result of Blahota, Tephnadze [13] for $0 < \alpha < 1$.

Corollary 5 *Let $f \in H_p$, where $0 < p < 1/(1+\alpha)$, for some $0 < \alpha < 1$. Then there exists an absolute constant $c_{\alpha,p}$, depending only on α and p, such that*

$$\sum_{k=1}^{\infty} \frac{\|\sigma_k^{\alpha} f\|_p^p}{k^{2-(1+\alpha)p}} \leq c_{\alpha,p} \|f\|_{H_p}^p.$$

Let $0 < \alpha \leq 1$, $\beta > 0$, and $\theta_n^{\alpha,\beta}$ denote the Nörlund mean, where

$$\{q_0 = 0, q_k = k^{\alpha-1} \log^{\beta} k : k \geq 1\},$$

that is,

$$\theta_n^{\alpha,\beta} f = \frac{1}{Q_n} \sum_{k=1}^{n} (n-k)^{\alpha-1} S_k f.$$

Remark 3 $0 < \alpha \leq 1$ and $\beta = 0$. Then $\theta_n^{\alpha,\beta}$ satisfy conditions (3) and (4) and also conditions (11) and (12).

Remark 4 $0 < \alpha \leq 1$ and $\beta > 0$. Then $\theta_n^{\alpha,\beta}$ satisfies conditions (11) and (12), but does not satisfy (3) and (4).

Finally, we also point out some new consequences of our results.

First we note that Theorem 1 and Remark 1 immediately imply the following new result.

Corollary 6

(a) *Let $f \in H_p$, where $0 < p < 1/(1+\alpha)$ for some $0 < \alpha \leq 1$. Then for every $\beta > 0$ the maximal operator*

$$\overset{\sim *,\alpha,\beta}{\theta}_p := \sup_{n \in \mathbb{N}} \frac{|\theta_n^{\alpha,\beta} f|}{(n+1)^{1/p-1-\alpha}}$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p.

(b) *Let $\{\Phi_n : n \in \mathbb{N}_+\}$ be any non-decreasing sequence, satisfying the condition*

$$\varlimsup_{n \to \infty} \frac{(n+1)^{1/p-1-\alpha}}{\Phi_n} = \infty.$$

Then

$$\sup_{k \in \mathbb{N}} \frac{\left\| \frac{\theta^{\alpha,\beta}_{M_{2n_k}+1} f_k}{\Phi_{M_{2n_k}+1}} \right\|_{\text{weak-}L_p}}{\|f_k\|_{H_p}} = \infty.$$

In a similar way we see that Theorem 2 and Remark 1 immediately generates the following new result.

Corollary 7 *Let $f \in H_p$, where $0 < p < 1/(1+\alpha)$, for some $0 < \alpha \leq 1$. Then for every $\beta > 0$ there exists an absolute constant $c_{\alpha,\beta,p}$, depending only on α, β, and p, such that*

$$\sum_{k=1}^{\infty} \frac{\|\theta_k^{\alpha,\beta} f\|_p^p}{k^{2-(1+\alpha)p}} \leq c_{\alpha,\beta,p} \|f\|_{H_p}^p.$$

Authors' contributions
All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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