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Affine inequalities for L_p-mixed mean zonoids

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Abstract
In this paper, we introduce the L_p-mixed mean zonoid of convex bodies K and L, and we prove some important properties for the L_p-mixed mean zonoid, such as monotonicity, GL(n) covariance, and so on. We also establish new affine isoperimetric inequalities for the L_p-mixed mean zonoid.

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1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, we write $\mathcal{K}^n_o$. $\mathcal{K}^n_s$ denotes the class of o-symmetric members of $\mathcal{K}^n_o$ (o denotes the origin in $\mathbb{R}^n$). Let S^{n-1} denote the unit sphere in Euclidean space $\mathbb{R}^n$ and let $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we write $\omega_n = V(B)$ for its volume.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty,\infty)$, is defined by (see [1, 2]) $h(K,x) = \max\{x \cdot y : y \in K\}, x \in \mathbb{R}^n$, where $x \cdot y$ denotes the standard inner product of x and y.

The zonoids are investigated by many authors (see [3–5]). The zonoid $\mathcal{Z}$ is a convex body with support function

$$h_{\mathcal{Z}}(u) = \frac{1}{2}\int_{S^{n-1}} \bigl|\langle u,v\rangle\bigr| \,\mathrm{d}\mu(v) \quad \text{for all } u \in S^{n-1},$$

where μ is some positive, even Borel measure on S^{n-1} and $\langle x,y\rangle$ denotes the standard inner product of vectors x and y in $\mathbb{R}^n$.

For $K \in \mathcal{K}^n$, the mean zonoid, $\bar{\mathcal{Z}}K$, was defined by Zhang [6]

$$h_{\bar{\mathcal{Z}}K}(u) = \frac{1}{V(K)^2}\int_K\int_K \bigl|\langle u,(x-y)\rangle\bigr| \,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } u \in S^{n-1}, \tag{1.1}$$

where $V(K)$ is the volume of the body K.

Further, Zhang [6] proved the affine isoperimetric inequality $V(\bar{\mathcal{Z}}K) \ge V(\bar{\mathcal{Z}}B_K)$, where B_K is the n-ball with the same volume as K.

For each convex subset in $\mathbb{R}^n$, it is well known that there is a unique ellipsoid with the following property: The moment of inertia of the ellipsoid and the moment of inertia of the convex set are the same about every 1-dimensional subspace of $\mathbb{R}^n$. This ellipsoid is called the ellipsoid of inertia $\Gamma_2 K$ (also called the Legendre ellipsoid) of the convex set. Namely, between the convex body K and the ellipsoid of inertia $\Gamma_2 K$ we have

$$\int_K |\langle x, y\rangle|^2 \,\mathrm{d}x = \int_{\Gamma_2 K} |\langle x, y\rangle|^2 \,\mathrm{d}x, \quad \forall y \in \mathbb{R}^n.$$

The Legendre ellipsoid and its polar (the Binet ellipsoid) are well-known concepts from classical mechanics. See [7–9] for historical references.

A non-negative finite Borel measure μ on the unit sphere S^{n-1} is said to be isotropic if it has the same moment of inertia about all lines through the origin or, equivalently, if, for all $x \in \mathbb{R}^n$,

$$|x|^2 = \int_{S^{n-1}} |\langle x, u\rangle|^2 \,\mathrm{d}\mu(u),$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$.

Based on the background of mechanics properties, the notion of L_p-zonoids was given by Schneider and Weil [10]. For $p \geq 1$, an L_p-zonoid was defined by

$$h_{\mathcal{Z}_p K}(u)^p = \int_{S^{n-1}} |\langle u, v\rangle|^p \,\mathrm{d}\mu(v) \quad \text{for all } u \in S^{n-1}, \tag{1.2}$$

where μ is some positive, even Borel measure on S^{n-1}. We also refer to [4, 11].

Xi, Guo and Leng [12] considered an extension for a class of bodies $\bar{\mathcal{Z}}_p K$ named L_p-mean zonoids as follows: For $K \in \mathcal{K}^n$ and $p \geq 1$, the L_p-mean zonoid, $\bar{\mathcal{Z}}_p K$, of K is defined by

$$h_{\bar{\mathcal{Z}}_p K}(z) = \left(\frac{1}{V(K)^2} \int_K \int_K \bigl|\langle z, (x-y)\rangle\bigr| \,\mathrm{d}x \,\mathrm{d}y\right)^{\frac{1}{p}} \quad \text{for all } z \in \mathbb{R}^n/\{o\}. \tag{1.3}$$

For $p = 1$, the body $\bar{\mathcal{Z}} K$ is the mean zonoid of K [6]. Xi *et al.* also showed that $\bar{\mathcal{Z}}_p K$ is an L_p-zonoid, and established the following affine isoperimetric inequality: For $K \in \mathcal{K}^n$ and $p \geq 1$,

$$V(\bar{\mathcal{Z}}_p K) \geq C_{n,p} V(K), \tag{1.4}$$

with equality if and only if K is an ellipsoid. Here $C_{n,p}$ is a constant depending on p and the dimension n.

The main purpose of this paper is to introduce the notion of L_p-mixed mean zonoids, which extends the L_p-mean zonoids by Xi, Guo and Leng [12].

Definition 1.1 For $K, L \in \mathcal{K}^n$ and $p \geq 1$, L_p-mixed mean zonoids, $\bar{\mathcal{Z}}_p(K, L)$, of K and L are defined by

$$h_{\bar{\mathcal{Z}}_p(K,L)}(z) = \left(\frac{1}{V(K)V(L)} \int_K \int_L \bigl|\langle z, (x-y)\rangle\bigr|^p \,\mathrm{d}x \,\mathrm{d}y\right)^{1/p} \quad \text{for all } z \in \mathbb{R}^n \backslash \{o\}. \tag{1.5}$$

Notice that when $K = L$, (1.5) is defined by Xi, Guo, and Leng in [12].

Let $\omega_p = \pi^{p/2}/\Gamma(1+p/2)$ and

$$C(n,p) = \left(\frac{2^{n+p}(2n+p+1)\omega_{2n+p}\omega_{2n+p+1}}{(n+1)\omega_2^2\omega_n^2\omega_{n+1}\omega_{p-1}\omega_{n+p-1}}\right)^{n/p}.$$

For the L_p-mixed mean zonoids, our main result is to establish the more general affine inequality as follows.

Theorem 1.2 *Let $K, L \in \mathcal{K}_o^n$ and $p \geq 1$. If $K \subseteq L$, then*

$$V\big(\bar{\mathcal{Z}}_p(K,L)\big) \geq C(n,p)V(K)^{\frac{n+p}{p}}V(L)^{-\frac{n}{p}}, \tag{1.6}$$

with equality if and only if $K = L$ is an ellipsoid.

If $L = K$, then the above inequality (1.6) reduces to the affine inequality (1.4).

An immediate consequence of Theorem 1.2 is the following.

Corollary 1.3 *Let $K, L \in \mathcal{K}_o^n$. If $K \subseteq L$, then*

$$\left(\frac{V(\bar{\mathcal{Z}}_1(K,L))}{V(K)}\right)^{\frac{1}{n}} \geq \big(C(n,1)\big)^{\frac{1}{n}}\left(\frac{\widetilde{V}_1(L,K)}{V(L)}\right)^{n}, \tag{1.7}$$

with equality if and only if $K = L$ is an ellipsoid.

2 Notation and preliminaries

We refer to the books Gardner [1] and Schneider [2] for some terminologies and notations as regards convex bodies.

The Hausdorff metric $\delta_H(K,L)$ between sets $K, L \in \mathcal{K}^n$ can be defined by

$$\delta_H(K,L) = \sup_{x\in S^{n-1}} \big|h(K,x) - h(L,x)\big|.$$

A set K is star-shaped (about $x_0 \in K$) if there exists $x_0 \in K$, such that the line segment from x_0 to any point $x \in K$ is contained in K. If K is a compact star-shaped (about the origin) set, then its radial function $\rho_K(x,z) : \mathbb{R}^n \backslash \{x\} \to [0,\infty)$ with respect to x is defined by

$$\rho_K(x,z) = \max\{c : x + cz \in Z\} \quad \text{for all } z \in \mathbb{R}^n \backslash \{x\}. \tag{2.1}$$

If ρ_K is positive and continuous, then K will be called a star body (about the origin), and $\mathcal{S}^n$ denotes the set of star bodies in $\mathbb{R}^n$. We will use $\mathcal{S}_o^n$ to denote the subset of star bodies in $\mathcal{S}^n$ containing the origin in their interiors. Two star bodies K and L are said to be dilates of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

For $K, L \in \mathcal{S}_o^n$, $p > 0$, and $\lambda, \mu \geq 0$ (not both zero), the L_p-radial combination, $\lambda \circ K \widetilde{+}_p \mu \circ L \in \mathcal{S}_o^n$, is defined by

$$\rho(\lambda \circ K \widetilde{+}_p \mu \circ L, \cdot)^p = \lambda\rho(K,\cdot)^p + \mu\rho(L,\cdot)^p.$$

The dual L_p-mixed volume $\widetilde{V}_p(K,L)$ of K,L was defined by

$$\widetilde{V}_p(K,L)=\frac{p}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\widetilde{+}_p\varepsilon\circ L)-V(K)}{\varepsilon}. \tag{2.2}$$

The integral representation of $\widetilde{V}_p(K,L)$ was proved by

$$\widetilde{V}_p(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-p}\rho_L(u)^p\,\mathrm{d}S(u).$$

The L_p-Minkowski inequality for the dual L_p-mixed volume is: If $K,L\in\mathcal{S}_o^n$ and $0<p<n$, then

$$\widetilde{V}_p(K,L)\le V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}, \tag{2.3}$$

with equality if and only if K and L are dilates.

The difference body $D(K,L)$ of K and L is defined by $D(K,L)=K-L=\{x-y:x\in K,y\in L\}$. Particularly, $DK=K-K=\{x-y:x\in K,y\in K\}$.

For a star-shaped K and $p\ge 1$, the L_p-centroid body of K, Γ_pK is the origin-symmetric convex body with the support function

$$h_{\Gamma_pK}(u)^p=\frac{1}{V(K)}\int_K\bigl|\langle u,x\rangle\bigr|^p\,\mathrm{d}x=\frac{1}{(n+p)V(K)}\int_{S^{n-1}}\bigl|\langle u,v\rangle\bigr|^p\rho_K(v)^{n+p}\,\mathrm{d}v, \tag{2.4}$$

for all $u\in S^{n-1}$.

For $K,L\in\mathcal{K}^n$, $p>-1$, and $K\subseteq L$, the generalized radial pth mean body, $R_p(K,L,\lambda_n)$, is defined by (see [13, 14])

$$\rho_{R_p(K,L,\lambda_n)}(u)=\left(\frac{1}{V(K)}\int_K\rho_L(x,u)^p\,\mathrm{d}x\right)^{1/p}, \tag{2.5}$$

for all $u\in S^{n-1}$, where λ_n is the n-dimensional Lebesgue measure in $\mathbb{R}^n$.

Lemma 2.1 ([13]) *For $K,L\in\mathcal{K}^n$ and $x\in\mathbb{R}^n$, the parallel section function on $\mathbb{R}^n$ is defined by $A_{K,L}(x):=V(K\cap(L+x))$. Then $g_{K,L}(x)=A_{K,L}(x)^{\frac{1}{n}}$ is concave on its support.*

If $K\subseteq L$ and $p>0$, then for all $u\in S^{n-1}$ (see [13, 14])

$$\int_K\rho_L(x,u)^p\,\mathrm{d}x=p\int_0^\infty A_{K,L}(ru)r^{p-1}\,\mathrm{d}r=p\int_0^{\rho_{DK}(u)}A_{K,L}(ru)r^{p-1}\,\mathrm{d}r. \tag{2.6}$$

For $p,q>0$, define the β-function by

$$\beta(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Lemma 2.2 *If $\lambda>0$ and $A_{K,L}(r,u):=V(K\cap(L+ru))$, then*

$$\int_0^\infty A_{K,L}(r,u)r^\lambda\,\mathrm{d}r\le n^{\lambda+1}\beta(\lambda+1,n)V(K)^{-\lambda}\left(\int_0^\infty A_{K,L}(r,u)\,\mathrm{d}r\right)^{\lambda+1}. \tag{2.7}$$

Proof If

$$F(\lambda) = \left(\frac{1}{\beta(\lambda+1,n)} \int_0^\infty \frac{A_{K,L}(r,u)}{A_{K,L}(0,u)} r^\lambda \,\mathrm{d}r \right)^{\frac{1}{\lambda+1}},$$

then $F(\lambda)$ is a decreasing function on $(-1,+\infty)$. Particularly, if $\lambda > 0$, then $F(\lambda) \le F(0)$ with equality if and only if

$$1 - \left(\frac{A_{K,L}(r,u)}{A_{K,L}(0,u)} \right)^{\frac{1}{n-1}} = \frac{r}{F(0)}.$$

Then

$$\int_0^\infty A_{K,L}(r,u) r^\lambda \,\mathrm{d}r \le n^{\lambda+1} \beta(\lambda+1,n) V(K)^{-\lambda} \left(\int_0^\infty A_{K,L}(r,u)\,\mathrm{d}r \right)^{\lambda+1}, \tag{2.8}$$

with equality if and only if $A_{K,L}(r,u) = V(K)(1 - \frac{rV(K)}{n\int_K \rho_L(x,u)\,\mathrm{d}x})^{n-1}$. □

3 L_p-Mixed mean zonoids

Suppose $K, L \in \mathcal{K}^n$ and $p \ge 1$. Define $\bar{\mathcal{Z}}_\infty(K,L)$ by

$$h_{\bar{\mathcal{Z}}_\infty(K,L)}(u) = \max_{x\in K, y\in L} \bigl|\langle u, (x-y)\rangle\bigr| \quad \text{for all } u \in S^{n-1}.$$

Since $\bar{\mathcal{Z}}_\infty(K,L) = D(K,L)$, it follows from Jensen's inequality that

$$\bar{\mathcal{Z}}_p(K,L) \subseteq \bar{\mathcal{Z}}_q(K,L) \subseteq D(K,L) \quad \text{for } 1 \le p \le q.$$

Property 3.1 Let $K, L \in \mathcal{K}^n$ with $K \subseteq L$. If $p \ge 1$, then

$$\bar{\mathcal{Z}}_p(K,L) = \left(\frac{V(R_{n+p}(K,L,\lambda_n))}{(n+p)V(L)} \right)^{1/p} \Gamma_p\bigl(R_{n+p}(K,L,\lambda_n)\bigr). \tag{3.1}$$

Proof From (1.5), (2.1), the Fubini theorem, (2.4), and (2.5), passing to spherical coordinates we have

$$\begin{aligned}
h_{\bar{\mathcal{Z}}_p(K,L)}(z) &= \left(\frac{1}{V(K)V(L)} \int_K \int_L \bigl|\langle z, (x-y)\rangle\bigr|^p \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p} \\
&= \left(\frac{1}{V(K)V(L)} \int_K \int_{S^{n-1}} \int_0^{\rho_L(y,v)} \bigl|\langle z, v\rangle\bigr|^p r^{n+p-1} \,\mathrm{d}r\,\mathrm{d}v\,\mathrm{d}y \right)^{1/p} \\
&= \left(\frac{1}{(n+p)V(K)V(L)} \int_{S^{n-1}} \bigl|\langle z, v\rangle\bigr|^p \int_K \rho_L(y,v)^{n+p} \,\mathrm{d}y\,\mathrm{d}v \right)^{1/p} \qquad (3.2) \\
&= \left(\frac{1}{(n+p)V(L)} \int_{S^{n-1}} \bigl|\langle z, v\rangle\bigr|^p \rho_{R_{n+p}(K,L,\lambda_n)}(v)^{n+p} \,\mathrm{d}v \right)^{1/p} \\
&= \left(\frac{V(R_{n+p}(K,L,\lambda_n))}{(n+p)V(L)} \right)^{1/p} h_{\Gamma_p(R_{n+p}(K,L,\lambda_n))}(z). \qquad (3.3)
\end{aligned}$$

Combining with (3.3), we have

$$\bar{\mathcal{Z}}_p(K,L) = \left(\frac{V(R_{n+p}(K,L,\lambda_n))}{(n+p)V(L)}\right)^{1/p}\Gamma_p\big(R_{n+p}(K,L,\lambda_n)\big). \qquad \square$$

Together (2.6) with (3.2), if $K \subseteq L$, then

$$h_{\bar{\mathcal{Z}}_p(K,L)}(z) = \left(\frac{1}{V(K)V(L)}\int_{S^{n-1}}\left|\langle z,u\rangle\right|^p\int_0^\infty A_{K,L}(ru)r^{n+p-1}\,\mathrm{d}r\,\mathrm{d}u\right)^{1/p}. \tag{3.4}$$

Let

$$C_{K,L}(n,p) = \left(\frac{n^{n+p}(n+p)\beta(n+p,n)V(R_1(K,L,\lambda_n))}{V(L)}\right)^{1/p}.$$

Property 3.2 Let $K,L \in \mathcal{K}^n$ and $p \geq 1$. If $K \subseteq L$,then

$$\bar{\mathcal{Z}}_p(K,L) \subseteq C_{K,L}(n,p)\Gamma_p\big(R_1(K,L,\lambda_n)\big).$$

Proof By (3.4), (2.7), (2.6), (2.5), and (2.4), we have

$$\begin{aligned}
h_{\bar{\mathcal{Z}}_p(K,L)}(u) &= \left(\frac{1}{V(K)V(L)}\int_{S^{n-1}}\left|\langle u,v\rangle\right|^p\int_0^\infty A_{K,L}(ru)r^{n+p-1}\,\mathrm{d}r\,\mathrm{d}v\right)^{1/p} \\
&\leq \left(\frac{n^{n+p}\beta(n+p,n)}{V(K)^{n+p}V(L)}\int_{S^{n-1}}\left|\langle u,v\rangle\right|^p\left(\int_0^\infty A_{K,L}(r,v)\,\mathrm{d}r\right)^{n+p}\right)^{1/p} \\
&= \left(\frac{n^{n+p}\beta(n+p,n)}{V(L)}\int_{S^{n-1}}\left|\langle u,v\rangle\right|^p\left(\frac{1}{V(K)}\int_K \rho_L(x,v)\,\mathrm{d}x\right)^{n+p}\right)^{1/p} \\
&= \left(\frac{n^{n+p}\beta(n+p,n)}{V(L)}\int_{S^{n-1}}\left|\langle u,v\rangle\right|^p\rho^{n+p}_{R_1(K,L,\lambda_n)}(v)\,\mathrm{d}v\right)^{1/p} \\
&= C_{K,L}(n,p)h_{\Gamma(R_1(K,L,\lambda_n))}(u).
\end{aligned}$$

This implies $h_{\bar{\mathcal{Z}}_p(K,L)}(u) \leq C_{K,L}(n,p)h_{\Gamma_p(R_1(K,L,\lambda_n))}(u)$. $\square$

The following property will be used to prove that $\bar{\mathcal{Z}}_p : \mathcal{K}^n \times \mathcal{K}^n \to \mathcal{K}^n$ is continuous.

Property 3.3 If $p \geq 1, K_i, L_i \in \mathcal{K}^n$ and $K_i \to K \in \mathcal{K}^n, L_i \to L \in \mathcal{K}^n$, then

$$\bar{\mathcal{Z}}_p(K_i,L_i) \to \bar{\mathcal{Z}}_p(K,L).$$

Proof Since $K_i \to K$, $\{K_i\}$ are uniformly bounded. Thus there is $R_K > 0$, such that $K_i \subseteq R_K B^n$. Similarly, $L_i \subseteq R_L B^n$ with $R_L > 0$. Taking (1.5) together with Minkowski's inequality, it follows that for $u_0 \in S^{n-1}$

$$\begin{aligned}
&\left|h_{\bar{\mathcal{Z}}_p(K_i,L_i)}(u_0) - h_{\bar{\mathcal{Z}}_p(K,L)}(u_0)\right| \\
&\quad = \left|\left(\frac{1}{V(K_i)V(L_i)}\int_{R_K B^n}\int_{R_L B^n}\mathbf{1}_{K_i}(x)\mathbf{1}_{L_i}(y)\left|\langle u_0,(x-y)\rangle\right|^p\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}\right.
\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{1}{V(K)V(L)} \int_{R_K B^n} \int_{R_L B^n} \mathbf{1}_K(x)\mathbf{1}_L(y) \left|\langle u_0, (x-y)\rangle\right|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \Bigg| \\
&\leq \left(\frac{1}{V(K_i)V(L_i)} \int_{R_K B^n} \int_{R_L B^n} \left|\mathbf{1}_{K_i}(x)\mathbf{1}_{L_i}(y) - \mathbf{1}_K(x)\mathbf{1}_L(y)\right| \left|\langle u_0, (x-y)\rangle\right|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \\
&+ \left| \left(\left(\frac{1}{V(K_i)V(L_i)} - \frac{1}{V(K)V(L)}\right) \int_{R_K B^n} \int_{R_L B^n} \mathbf{1}_K(x)\mathbf{1}_L(y) \left|\langle u_0, (x-y)\rangle\right|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \right|.
\end{aligned}
$$

This means $h_{\bar{\mathcal{Z}}_p(K_i,L_i)}(u_0) \to h_{\bar{\mathcal{Z}}_p(K,L)}(u_0)$, which is the desired result. □

The following property will prove that $\bar{\mathcal{Z}}_p : \mathcal{K}^n \times \mathcal{K}^n \to \mathcal{K}^n$ is $\mathrm{GL}(n)$ covariant.

Property 3.4 If $p \geq 1$, $K \in \mathcal{K}^n$ and $T \in \mathrm{GL}(n)$, then

$$
\bar{\mathcal{Z}}_p(TK, TL) = T\big(\bar{\mathcal{Z}}_p(K,L)\big).
$$

Proof Combining (1.5) with the substitution $x = Tx_1, y = Ty_1$, we obtain

$$
\begin{aligned}
h_{\bar{\mathcal{Z}}_p(TK,TL)}(z) &= \left(\frac{1}{V(TK)V(TL)} \int_{TK} \int_{TL} \left|\langle z, (x-y)\rangle\right|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \\
&= \left(\frac{1}{V(TK)V(TL)} |T|^2 \int_K \int_L \left|\langle z, (Tx - Ty)\rangle\right|^p \,\mathrm{d}x_1\,\mathrm{d}y_1\right)^{1/p} \\
&= \left(\frac{1}{V(K)V(L)} \int_K \int_L \left|\langle T^t z, (x_1 - y_1)\rangle\right|^p \,\mathrm{d}x_1\,\mathrm{d}y_1\right)^{1/p} \\
&= h_{\bar{\mathcal{Z}}_p(K,L)}\big(T^t z\big) \\
&= h_{T\bar{\mathcal{Z}}_p(K,L)}(z).
\end{aligned}
$$

Namely, $\bar{\mathcal{Z}}_p(TK, TL) = T(\bar{\mathcal{Z}}_p(K,L))$. □

4 Proof of main result

If $u \in S^{n-1}$, then we denote by $u^\perp$ the $(n-1)$-dimensional subspace orthogonal to u, by l_u the line through o parallel to u, and by $l_u(x)$ the line through the point x parallel to u. We denote by K_u the image of the orthogonal projection of K onto $u^\perp$ for a convex body K. Let $\bar{l}_u(K; y') : K_u \to \mathbb{R}$ and $\underline{l}_u(K; y') : K_u \to \mathbb{R}$ for the overgraph and undergraph functions of K in the direction u; namely,

$$
K = \big\{y' + tu : -\underline{l}_u\big(K; y'\big) \leq t \leq \bar{l}_u\big(K; y'\big) \text{ for } y' \in K_u\big\}.
$$

Thus, the overgraph and undergraph functions of the Steiner symmetrical S_u of $K \in \mathcal{K}^n$ in direction u are defined by

$$
\bar{l}_u\big(S_u K; y'\big) = \underline{l}_u\big(S_u K; y'\big) = \frac{1}{2}\big(\bar{l}_u\big(K; y'\big) + \underline{l}_u\big(K; y'\big)\big).
$$

For $y' \in K_u$, $m_{y'} = m_{y'}(u)$ denotes $m_{y'}(u) = \frac{1}{2}(\bar{l}_u(K; y') - \underline{l}_u(K; y'))$. Let the midpoint of the chord $K \cap l_u(y')$ be $y' + m_{y'}(u)u$, note that $l_u(y')$ is the line through y' parallel to u, and let the length $|K \cap l_u(y')|$ of this chord be $\sigma_{y'} = \sigma_{y'}(u)$. For $x = (x', s) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we write $h_K(x', s)$ throughout this section.

Lemma 4.1 ([15]) *If $K \in \mathcal{K}_o^n$, $u \in S^{n-1}$ and $y' \in \operatorname{relint} K_u$, then*

$$\bar{l}_u(K;y') = \min_{x' \in u^{\perp}} \{h_K(x',1) - \langle x',y'\rangle\}, \tag{4.1}$$

$$\underline{l}_u(K;y') = \min_{x' \in u^{\perp}} \{h_K(x',-1) - \langle x',y'\rangle\}. \tag{4.2}$$

Lemma 4.2 *If $K \in \mathcal{K}^n, p \geq 1$, $u \in S^{n-1}$, and $z_1', z_2' \in u^{\perp}$, then*

$$h_{\bar{\mathcal{Z}}_p(S_uK,S_uL)}\left(\frac{z_1'+z_2'}{2},1\right) \leq \frac{1}{2}h_{\bar{\mathcal{Z}}_p(K,L)}(z_1',1) + \frac{1}{2}h_{\bar{\mathcal{Z}}_p(K,L)}(z_2',-1), \tag{4.3}$$

$$h_{\bar{\mathcal{Z}}_p(S_uK,S_uL)}\left(\frac{z_1'+z_2'}{2},-1\right) \leq \frac{1}{2}h_{\bar{\mathcal{Z}}_p(K,L)}(z_1',1) + \frac{1}{2}h_{\bar{\mathcal{Z}}_p(K,L)}(z_2',-1). \tag{4.4}$$

Equality in (4.3) *or* (4.4) *implies that all of the chords of K and L parallel to u have midpoints that lie in a hyperplane, respectively.*

Proof We only prove (4.3). Inequality (4.4) can be established in the same way. It follows from the definition of the L_p-mixed mean zonoid that

$$\begin{aligned}
&h_{\bar{\mathcal{Z}}_p(K,L)}(z_1',1) \\
&\quad= \left(\frac{1}{V(K)V(L)}\int_K\int_L \left|\langle (z_1',1),(x-y)\rangle\right|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \\
&\quad= \Bigg(\frac{1}{V(K)V(L)} \\
&\qquad\times \int_{K_u}\int_{m_{y'}-\frac{\sigma_{y'}}{2}}^{m_{y'}+\frac{\sigma_{y'}}{2}}\int_{L_u}\int_{m_{x'}-\frac{\sigma_{x'}}{2}}^{m_{x'}+\frac{\sigma_{x'}}{2}} \left|\langle (z_1',1),((x',s_1)-(y',s_2))\rangle\right|^p \,\mathrm{d}s_1\,\mathrm{d}x'\,\mathrm{d}s_2\,\mathrm{d}y'\Bigg)^{1/p} \\
&\quad= \left(\frac{1}{V(K)V(L)}\int_{K_u}\int_{m_{y'}-\frac{\sigma_{y'}}{2}}^{m_{y'}+\frac{\sigma_{y'}}{2}}\int_{L_u}\int_{m_{x'}-\frac{\sigma_{x'}}{2}}^{m_{x'}+\frac{\sigma_{x'}}{2}} \left|\langle z_1',(x'-y')\rangle + s_1 - s_2\right|^p \,\mathrm{d}s_1\,\mathrm{d}x'\,\mathrm{d}s_2\,\mathrm{d}y'\right)^{1/p} \\
&\quad= \Bigg(\frac{1}{V(K)V(L)} \\
&\qquad\times \int_{K_u}\int_{-\frac{\sigma_{y'}}{2}}^{\frac{\sigma_{y'}}{2}}\int_{L_u}\int_{-\frac{\sigma_{x'}}{2}}^{\frac{\sigma_{x'}}{2}} \left|\langle z_1',(x'-y')\rangle + t_1 - t_2 + m_{x'} - m_{y'}\right|^p \,\mathrm{d}t_1\,\mathrm{d}x'\,\mathrm{d}t_2\,\mathrm{d}y'\Bigg)^{1/p} \\
&\quad= \left(\frac{1}{V(S_uK)V(S_uL)}\int_{S_uK}\int_{S_uL} \left|\langle z_1',(x'-y')\rangle + t_1 - t_2 + m_{x'} - m_{y'}\right|^p \,\mathrm{d}t_1\,\mathrm{d}x'\,\mathrm{d}t_2\,\mathrm{d}y'\right)^{1/p},
\end{aligned}$$

by $t_1 = -m_{x'} + s_1, t_2 = -m_{y'} + s_2$.

$$\begin{aligned}
&h_{\bar{\mathcal{Z}}_p(K,L)}(z_2',-1) \\
&\quad= \left(\frac{1}{V(K)V(L)}\int_K\int_L \left|\langle (z_2',-1),(x-y)\rangle\right|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \\
&\quad= \Bigg(\frac{1}{V(K)V(L)}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{K_u}\int_{m_{y'}-\frac{\sigma_{y'}}{2}}^{m_{y'}+\frac{\sigma_{y'}}{2}}\int_{L_u}\int_{m_{x'}-\frac{\sigma_{x'}}{2}}^{m_{x'}+\frac{\sigma_{x'}}{2}} \left|\left\langle (z_2',-1),\left((x',s_1)-(y',s_2)\right)\right\rangle\right|^p \,\mathrm{d}s_1\,\mathrm{d}x'\,\mathrm{d}s_2\,\mathrm{d}y'\Bigg)^{1/p}\\
&=\Bigg(\frac{1}{V(K)V(L)}\int_{K_u}\int_{m_{y'}-\frac{\sigma_{y'}}{2}}^{m_{y'}+\frac{\sigma_{y'}}{2}}\int_{L_u}\int_{m_{x'}-\frac{\sigma_{x'}}{2}}^{m_{x'}+\frac{\sigma_{x'}}{2}} \left|\left\langle z_2',(x'-y')\right\rangle - s_1+s_2\right|^p \,\mathrm{d}s_1\,\mathrm{d}x'\,\mathrm{d}s_2\,\mathrm{d}y'\Bigg)^{1/p}\\
&=\Bigg(\frac{1}{V(K)V(L)}\\
&\times \int_{K_u}\int_{-\frac{\sigma_{y'}}{2}}^{\frac{\sigma_{y'}}{2}}\int_{L_u}\int_{-\frac{\sigma_{x'}}{2}}^{\frac{\sigma_{x'}}{2}} \left|\left\langle z_2',(x'-y')\right\rangle + t_1-t_2-m_{x'}+m_{y'}\right|^p \,\mathrm{d}t_1\,\mathrm{d}x'\,\mathrm{d}t_2\,\mathrm{d}y'\Bigg)^{1/p}\\
&=\Bigg(\frac{1}{V(S_uK)V(S_uL)}\int_{S_uK}\int_{S_uL} \left|\left\langle z_2',(x'-y')\right\rangle + t_1-t_2-m_{x'}+m_{y'}\right|^p \,\mathrm{d}t_1\,\mathrm{d}x'\,\mathrm{d}t_2\,\mathrm{d}y'\Bigg)^{1/p}.
\end{aligned}
$$

Let $t_1 = m_{x'} - s_1, t_2 = m_{y'} - s_2$. Thus, combining with Minkowski's inequality we have

$$
\begin{aligned}
&2h_{\bar{\mathcal{Z}}_p(S_uK,S_uL)}\left(\frac{z_1'+z_2'}{2},1\right)\\
&=2\Bigg(\frac{1}{V(S_uK)V(S_uL)}\int_{S_uK}\int_{S_uL}\left|\left\langle\left(\frac{z_1'+z_2'}{2},1\right),(x-y)\right\rangle\right|^p \,\mathrm{d}x\,\mathrm{d}y\Bigg)^{1/p}\\
&=\Bigg(\frac{1}{V(S_uK)V(S_uL)}\int_{S_uK}\int_{S_uL}\left|\left\langle (z_1'+z_2'),(x'-y')\right\rangle + 2t_1-2t_2\right|^p \,\mathrm{d}t_1\,\mathrm{d}x'\,\mathrm{d}t_2\,\mathrm{d}y'\Bigg)^{1/p}\\
&\le\Bigg(\frac{1}{V(S_uK)V(S_uL)}\int_{S_uK}\int_{S_uL}\left|\left\langle z_1',(x'-y')\right\rangle + t_1-t_2+m_{x'}-m_{y'}\right|^p \,\mathrm{d}t_1\,\mathrm{d}x'\,\mathrm{d}t_2\,\mathrm{d}y'\Bigg)^{1/p}\\
&\quad+\Bigg(\frac{1}{V(S_uK)V(S_uL)}\\
&\quad\times\int_{S_uK}\int_{S_uL}\left|\left\langle z_2',(x'-y')\right\rangle + t_1-t_2-m_{x'}+m_{y'}\right|^p \,\mathrm{d}t_1\,\mathrm{d}x'\,\mathrm{d}t_2\,\mathrm{d}y'\Bigg)^{1/p}\\
&=h_{\bar{\mathcal{Z}}_p(K,L)}(z_1',1)+h_{\bar{\mathcal{Z}}_p(K,L)}(z_2',-1).
\end{aligned}
$$

From the condition of inequality in Minkowski's inequality, we know that equality in (4.3) or (4.4) holds if and only if for $\lambda \ge 0$, we have

$$\left\langle z_1',(x'-y')\right\rangle + t_1-t_2+m_{x'}-m_{y'} = \lambda\left(\left\langle z_2',(x'-y')\right\rangle + t_1-t_2-m_{x'}+m_{y'}\right),$$

for all $(x_1', t_1) \in K, (y_1', t_2) \in L$. This is equivalent to

$$\left\langle (z_1'-\lambda z_2'),(x'-y')\right\rangle + (1+\lambda)(m_{x'}-m_{y'}) = (\lambda-1)(t_1-t_2), \tag{4.5}$$

for all $(x_1', t_1) \in K, (y_1', t_2) \in L$.

We fix x', y'. If change t_1, t_2 in (4.5) with $(x_1', t_1) \in K, (y_1', t_2) \in L$, then the left of (4.5) will not change; thus we obtain $\lambda = 1$. Namely, equality in (4.3) or (4.4) implies all of the chords of K and L parallel to u have midpoints that lie in a hyperplane, respectively. □

Lemma 4.3 *If $K, L \in \mathcal{K}^n, p \ge 1$ and $u \in S^{n-1}$, then*

$$\bar{\mathcal{Z}}_p(S_uK, S_uL) \subseteq S_u\left(\bar{\mathcal{Z}}_p(K,L)\right). \tag{4.6}$$

If the inclusion is an identity, then all of the chords of K and L parallel to u have midpoints that lie in a hyperplane, respectively.

Proof Let $y' \in \operatorname{relint}(\bar{Z}_p(K,L))_u$. Lemma 4.1 means that there exist $z_1' = z_1'(y')$ and $z_2' = z_2'(y')$ in $u^\perp$ with

$$\bar{l}_u\big(\bar{Z}_p(K,L);y'\big) = h_{\bar{Z}_p(K,L)}\big(z_1',1\big) - \langle z_1',y'\rangle,$$
$$\underline{l}_u\big(\bar{Z}_p(K,L);y'\big) = h_{\bar{Z}_p(K,L)}\big(z_2',-1\big) - \langle z_2',y'\rangle.$$

Combining (4.1), (4.2), (4.3), and (4.4), it follows that

$$\begin{aligned}
\bar{l}_u\big(S_u\big(\bar{Z}_p(K,L)\big);y'\big) &= \frac{1}{2}\bar{l}_u\big(\bar{Z}_p(K,L);y'\big) + \frac{1}{2}\underline{l}_u\big(\bar{Z}_p(K,L);y'\big) \\
&= \frac{1}{2}\big(h_{\bar{Z}_p(K,L)}\big(z_1',1\big) - \langle z_1',y'\rangle\big) + \frac{1}{2}\big(h_{\bar{Z}_p(K,L)}\big(z_2',-1\big) - \langle z_2',y'\rangle\big) \\
&= \frac{1}{2}h_{\bar{Z}_p(K,L)}\big(z_1',1\big) + \frac{1}{2}h_{\bar{Z}_p(K,L)}\big(z_2',-1\big) - \left\langle \left(\frac{1}{2}z_1' + \frac{1}{2}z_2'\right), y'\right\rangle \\
&\geq h_{\bar{Z}_p(S_u(K,L))}\left(\frac{z_1'+z_2'}{2},1\right) - \left\langle \left(\frac{1}{2}z_1' + \frac{1}{2}z_2'\right), y'\right\rangle \\
&\geq \min_{x'\in u^\perp}\big\{h_{\bar{Z}_p(S_uK,S_uL)}\big(x',1\big) - \langle x',y'\rangle\big\} \\
&= \bar{l}_u\big(\bar{Z}_p(S_uK,S_uL);y'\big)
\end{aligned}$$

and

$$\begin{aligned}
\underline{l}_u\big(S_u\big(\bar{Z}_p(K,L)\big);y'\big) &= \frac{1}{2}\bar{l}_u\big(\bar{Z}_p(K,L);y'\big) + \frac{1}{2}\underline{l}_u\big(\bar{Z}_p(K,L);y'\big) \\
&= \frac{1}{2}\big(h_{\bar{Z}_p(K,L)}\big(z_1',1\big) - \langle z_1',y'\rangle\big) + \frac{1}{2}\big(h_{\bar{Z}_p(K,L)}\big(z_2',-1\big) - \langle z_2',y'\rangle\big) \\
&= \frac{1}{2}h_{\bar{Z}_p(K,L)}\big(z_1',1\big) + \frac{1}{2}h_{\bar{Z}_p(K,L)}\big(z_2',-1\big) - \left\langle \left(\frac{1}{2}z_1' + \frac{1}{2}z_2'\right), y'\right\rangle \\
&\geq h_{\bar{Z}_p(S_u(K,L))}\left(\frac{z_1'+z_2'}{2},-1\right) - \left\langle \left(\frac{1}{2}z_1' + \frac{1}{2}z_2'\right), y'\right\rangle \\
&\geq \min_{x'\in u^\perp}\big\{h_{\bar{Z}_p(S_uK,S_uL)}\big(x',-1\big) - \langle x',y'\rangle\big\} \\
&= \underline{l}_u\big(\bar{Z}_p(S_uK,S_uL);y'\big).
\end{aligned}$$

Let the inclusion be an identity. Then equality in both (4.3) and (4.4) holds; thus all of the chords of K and L parallel to u have midpoints that lie in a hyperplane, respectively. □

Now, we show that $\bar{Z}_p(K,L)$ contains the origin in its interior.

Lemma 4.4 *Suppose that* $K,L \in \mathcal{K}^n, p \geq 1$. *Then there exists* $c_0 > 0$ *such that*

$$\int_K\int_L \big|\langle u,(x-y)\rangle\big|^p \,\mathrm{d}x\,\mathrm{d}y \geq c_0,$$

for all $u \in S^{n-1}$.

Proof Given $u_0 \in S^{n-1}$. Taking the Euclidean n-balls $B_1 \subseteq K$ and $B_2 \subseteq L$ such that $x - y$ is not orthogonal to u_0 for all $(x,y) \in B_1 \times B_2$, then it follows from the continuity that the above result holds. □

Proof of Theorem 1.2 It follows from the standard Steiner symmetrization argument that there exists a sequence of directions $\{u_i\}$ with the sequences of convex bodies $\{K_i\}$ and $\{L_i\}$, defined by

$$K_{i+1} = S_{u_i}K_i, \qquad K_0 = K,$$

and

$$L_{i+1} = S_{u_i}L_i, \qquad L_0 = L,$$

converge to B_K and B_L, respectively. Note that B_K (B_L) is the n-ball, where $V(K) = V(B_K)(V(L) = V(B_L))$.

By Property 3.3 and Lemma 4.3, we have

$$\begin{aligned} V\bigl(\bar{\mathcal{Z}}_p(K_i,L_i)\bigr) &= V\bigl(\bar{\mathcal{Z}}_p(S_{u_{i-1}}K_{i-1},S_{u_{i-1}}L_{i-1})\bigr) \\ &\le V\bigl(\bar{\mathcal{Z}}_p(K_{i-1},L_{i-1})\bigr) \le \cdots \\ &\le V\bigl(\bar{\mathcal{Z}}_p(K,L)\bigr). \end{aligned} \tag{4.7}$$

From Lemma 4.4, Lemma 4.3, (4.7), and the definitions of B_K and B_L, we get

$$V\bigl(\bar{\mathcal{Z}}_p(K,L)\bigr) \ge V\bigl(\bar{\mathcal{Z}}_p(B_K,B_L)\bigr).$$

Since K and L are the ellipsoids, it follows from Property 3.4 that

$$V\bigl(\bar{\mathcal{Z}}_p(K,L)\bigr) = V\bigl(\bar{\mathcal{Z}}_p(B_K,B_L)\bigr).$$

Conversely, let $V(\bar{\mathcal{Z}}_p(K,L)) = V(\bar{\mathcal{Z}}_p(B_K,B_L))$. Clearly, for all $u \in S^{n-1}$ the inclusion in (4.6) is the identity. Thus we see that all of the chords of K and L parallel to u have midpoints that lie in a hyperplane, respectively, for all $u \in S^{n-1}$, namely, K and L are ellipsoids.

This implies

$$V\bigl(\bar{\mathcal{Z}}_p(K,L)\bigr) \ge V\bigl(\bar{\mathcal{Z}}_p(B_K,B_L)\bigr), \tag{4.8}$$

where $V(B_K) = V(K)$ and $V(B_L) = V(L)$, and equality holds if and only if K and L are dilated ellipsoids having the same midpoints.

Furthermore, we know that $h(\bar{\mathcal{Z}}_p(B_K,B_L),u)$ is a constant independent of u. Thus $\bar{\mathcal{Z}}_p(B_K,B_L)$ is an n-ball. Thus one has

$$\left(\frac{V(\bar{\mathcal{Z}}_p(B_K,B_L))}{\omega_n}\right)^{1/n} = \left(\frac{1}{V(B_K)V(B_L)}\int_{B_K}\int_{B_L}\bigl|\langle u,(x-y)\rangle\bigr|^p\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}. \tag{4.9}$$

The following formula is well known:

$$\int_{S^{n-1}} \bigl|\langle u,(x-y)\rangle\bigr|^p \,\mathrm{d}u = \frac{(n+p)\omega_{n+p}}{\omega_2\omega_{p-1}}|x-y|^p. \tag{4.10}$$

Together (4.9) with (4.10), it follows that

$$\left(\frac{V(\bar{\mathcal{Z}}_p(B_K,B_L))}{\omega_n}\right)^{1/n} = \left(\frac{(n+p)\omega_{n+p}}{n\omega_2\omega_{p-1}\omega_n V(B_K)V(B_L)}\int_{B_K}\int_{B_L}|x-y|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}. \tag{4.11}$$

Suppose that r_{B_K} and r_{B_L} denote radii of the balls B_K and B_L, respectively. Without loss of generality, let $B_K \subseteq B_L$. For all $u \in S^{n-1}$, it is obvious that $\rho_{DB_K}(u) = 2r_{B_K}$ and $\rho_{DB_L}(u) = 2r_{B_L}$. It follows from the spherical polar coordinates, (2.1), the Fubini theorem, and (2.6) that

$$\begin{aligned}\int_{B_K}\int_{B_L}|x-y|^p \,\mathrm{d}x\,\mathrm{d}y &= \int_{B_K}\int_{S^{n-1}}\int_0^{\rho_{B_L}(y,u)} r^{n+p-1}\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}y \\ &= \frac{1}{n+p}\int_{S^{n-1}}\int_{B_K}\rho_{B_L}(y,u)^{n+p}\,\mathrm{d}y\,\mathrm{d}u \\ &= \int_{S^{n-1}}\int_0^{\rho_{DB_K}(u)} V\bigl(B_K\cap(B_L+tu)\bigr)t^{n+p-1}\,\mathrm{d}t\,\mathrm{d}u \\ &= \int_{S^{n-1}}\int_0^{2r_{B_K}} V\bigl(B_K\cap(B_L+tu)\bigr)t^{n+p-1}\,\mathrm{d}t\,\mathrm{d}u. \end{aligned} \tag{4.12}$$

Since $g_{B_K,B_L}(tu)^{1/n} = V(B_K\cap(B_L+tu))^{1/n}$ is concave on DB_K, it follows from Lemma 2.1 that

$$V\bigl(B_K\cap(B_L+tu)\bigr) \ge V(B_K)\left(1-\frac{t}{2r_{B_K}}\right)^n \quad \text{for } 0\le t\le 2r_{B_K}, \tag{4.13}$$

with equality if and only if $B_K = B_L$.

Taking (4.12) together with (4.13), it follows that

$$\begin{aligned}\int_{B_K}\int_{B_L}|x-y|^p \,\mathrm{d}x\,\mathrm{d}y &\ge n\omega_n(2r_{B_K})^{n+p}V(K)\int_0^1 y^{n+p-1}(1-y)^n\,\mathrm{d}y \\ &= n\omega_n(2r_{B_K})^{n+p}\beta(n+p,n+1)V(K) \\ &= 2^{n+p}n\omega_n^{-\frac{p}{n}}\beta(n+p,n+1)V(K)^{\frac{2n+p}{n}}. \end{aligned} \tag{4.14}$$

Combining (4.11) with (4.14), we have

$$\left(\frac{V(\bar{\mathcal{Z}}_p(B_K,B_L))}{\omega_n}\right)^{1/n} \ge \left(\frac{2^{n+p}(n+p)\omega_{n+p}\beta(n+p,n+1)}{\omega_2\omega_{p-1}\omega_n^{\frac{n+p}{n}}}\right)^{1/p}\left(\frac{V(K)^{\frac{n+p}{n}}}{V(L)}\right)^{1/p}. \tag{4.15}$$

From (4.8) and (4.15), this yields

$$V\bigl(\bar{\mathcal{Z}}_p(K,L)\bigr) \ge C(n,p)V(K)^{\frac{n+p}{p}}V(L)^{-\frac{n}{p}}, \tag{4.16}$$

with equality if and only if $K = L$ is an ellipsoid. □

Proof of Corollary 1.3 Let $p = 1$. From the L_p-Minkowski inequality (2.3), we have

$$V(K)^{n-1}V(L) \geq \widetilde{V}_1(K,L)^n.$$

Exchange the order of K and L, then

$$V(L)^{n-1}V(K) \geq \widetilde{V}_1(L,K)^n,$$

namely,

$$\frac{V(K)}{V(L)} \geq \left(\frac{\widetilde{V}_1(L,K)}{V(L)}\right)^n, \tag{4.17}$$

with equality if and only if K and L are dilates.

On the other hand, from inequality (1.6), we have

$$\left(\frac{V(\bar{\mathcal{Z}}_1(K,L))}{V(K)}\right)^{\frac{1}{n}} \geq \left(C(n,1)\right)^{\frac{1}{n}}\frac{V(K)}{V(L)}, \tag{4.18}$$

with equality if and only if $K = L$ is an ellipsoid.

Taking (4.17) together with (4.18), it follows that

$$\left(\frac{V(\bar{\mathcal{Z}}_1(K,L))}{V(K)}\right)^{\frac{1}{n}} \geq \left(C(n,1)\right)^{\frac{1}{n}}\left(\frac{\widetilde{V}_1(L,K)}{V(L)}\right)^n.$$

Together with the equality conditions of inequalities (4.17) and (4.18), we see with equality in (1.7) if and only if $K = L$ is an ellipsoid. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper, and read and approved the final manuscript.

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