Song et al. *Journal of Inequalities and Applications* (2016) 2016:220
DOI 10.1186/s13660-016-1162-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# On the stability of finding approximate fixed points by simplicial methods

Qi-Qing Song[1,2], Hui Yang[1*], Guang-Hui Yang[1] and Chong-Yi Zhong[1]

*Correspondence:
hui-yang@163.com
[1]College of Science, Guizhou University, Guiyang, 550025, China
Full list of author information is available at the end of the article

**Abstract**
This paper reports some new results in relation to simplicial algorithms considering continuities of approximate fixed point sets. The upper semi-continuity of a set-valued mapping of approximate fixed points using vector-valued simplicial methods is proved, and thus one obtains the existence of finite essential connected components in approximate fixed point sets by vector-valued labels; examples are given to show that this is very different from the property for integer-valued labeling simplicial methods. The existence of essential sets is also proved focusing on both perturbations of domains and functions.

**Keywords:** stability; approximate fixed point; simplex

## 1 Introduction

Fixed point theorems have important effects in mathematical and economic sciences. The famous Brouwer fixed point theorem [1] plays a key role in many existence problems and also has prompted a wave of finding many kinds of equilibria and other applications, such as the Nash equilibrium[2], the general equilibrium [3], network problems [4–6], approximation theory [7], computer science [8], *etc.*

Naturally, designing algorithms to compute a Brouwer fixed point is also important field. It is well known that Sperner's lemma became a simple tool for the proof of the existence of Brouwer fixed points. Based on Sperner's lemma, simplicial algorithms continue to spring up after the excellent work by Scarf [9], such as Kuhn's algorithm [10, 11], the restart algorithms [12–14], variable dimension algorithms [15], and homotopy algorithms [16, 17]. For simplicial algorithms, one frequently finds a complete labeled sub-simplex (full labeled sub-simplex) in a simplex to approximate a fixed point. Two common labels are integer-valued and vector-valued. Given a fixed grid size, it is well known that the approximate degree of a complete vector-valued sub-simplex to fixed points is better than the other. Is there any difference between the stability of these algorithms? Is a complete labeled sub-simplex able to resist the perturbation of functions or simplices? This paper will focus on these problems.

The stability of fixed points has attracted much attention. After the seminal work for essential fixed points of continuous functions (Brouwer type fixed points) in [18], essential components and essential sets of fixed points were introduced [19, 20]. From the view point of stability, as the analogs of singletons, minimal essential sets seem to be good choices [21]. Essential stabilities (which are related to lower semi-continuity) were used to analyze many problems, such as coincidence points [22, 23], fixed points[24], KKM points [25, 26], game equilibrium points [27–31], maximal elements [32], and variational relation problems [33–35], *etc.*

In fact, when a simplicial algorithm is in order, we must face approximate fixed points as the grid size shrinks. By employing the essential stabilities, this concerns the stability of approximate fixed point sets using simplicial methods under the perturbation of the corresponding functions and domains. We show that there is a significant difference between vector-valued simplicial methods and integer-valued methods. The upper semi-continuity of a set-valued mapping for approximate fixed points using vector-valued labeling is proved. The existence of finite essential connected components of approximate fixed point sets is also proved for vector-labeled simplicial methods. These results are new.

## 2 Preliminaries and motivations

Let $S$ be an $n$-simplex in $R^{n+1}$ with vertices $v^1, v^2, \dots, v^{n+1}$, $C(S)$ the space of continuous functions $f$ on $S$ with uniform metric and $I_k = \{1, 2, \dots, k\}$. The $i$th unit vector of $R^{n+1}$ is denoted by $e(i)$, $i = 1, 2, \dots, n$, and the $(n+1)$-vector $(1, 1, \dots, 1)^T$ is represented by $e$.

We recall some definitions involving simplicial fixed point algorithms. Given the grid size $\frac{1}{q}$, the standard triangulation of $S$ is the collection of all sub-simplices $\sigma(y^1, \pi)$ with vertices $y^1, \dots, y^{n+1}$ in $S$ such that:

(i) each component of $y^1$ is a multiple of $\frac{1}{q}$;
(ii) $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ is a permutation of the elements of $I_n$;
(iii) $y^{i+1} = y^i + \frac{v(\pi_{i+1}) - v(\pi_i)}{q}$, where $v(i) = v^i$, $\forall i \in I_n$.

Note that the mesh of the standard triangulation of $S$ with the grid size $\frac{1}{q}$ is $\frac{\sqrt{n+1}}{q}$ or $\frac{\sqrt{n}}{q}$ if $n$ is odd or even, respectively. For a function $f \in C(S)$, a point $x$ in $S$ is labeled an integer $l(x) \in I_{n+1}$ where $l(x) = i$ if

$$i = \min\Bigl\{ j \Bigm| f_j(x) - x_j = \min_{h \in I_{n+1}} \bigl(f_h(x) - x_h\bigr) \Bigr\}.$$

Particularly, if $f(x) = x$ and $x_1 = 0$, then assign the label of $x$ as the first index $i$ such that $x_i > 0$, $i \in I_{n+1}$. We call $l: S \to I_{n+1}$ a standard integer-valued labeling function. Given the mesh of a standard triangulation on $S$, from Sperner's lemma, there exists at least one sub-simplex with complete integer labels (completely labeled simplex, with the meaning of vertices of the sub-simplex with totally different labels).

If a point $x$ in $S$ receives the $(n+1)$-vector $L(x)$, where

$$L(x) = -f(x) + x + e,$$

in this case, we call $L: S \to R^{n+1}$ a standard vector-valued labeling function. For a triangulation of $S$, a sub-simplex $\sigma(y^1, \dots, y^{n+1})$ with vector-valued labels is complete if $\sum_{i=1}^{n+1} \lambda_i L(y_i) = e$ has a solution $\lambda^* = (\lambda_1, \lambda_2, \dots, \lambda_{n+1})$ with $\lambda^* \in R_+^{n+1}$.

Given a grid size $\frac{1}{q}$, for each $f \in C(S)$, denote by $F(f,q)$ ($F'(f,q)$) the collection of all sub-simplices with complete integer-valued (vector-valued) labels in $S$, then we define a set-valued mapping from $C(S)$ to $S$ with $F(F'): C(S) \to 2^S$, in addition, for convenience of notation, we write $F(f)$ as $F(f,q)$. Note that each $x \in F(f,q)$ ($F'(f,q)$) is an approximation of fixed points of $f$ on $S$. In addition, from the connectedness, $F(f,q)$ ($F'(f,q)$) can be decomposed as $\bigcup_{i\in\Lambda} C_i$ with $C_i \cap C_j = \emptyset$ for any $i \neq j$, and $C_i$, $\forall i \in \Lambda$ is a connected component.

There are significant differences in relation to semi-continuities between $F$ and $F'$, and the following example shows that the set-valued mapping $F$ is not upper semi-continuous on $C(S)$; further results will be demonstrated in Section 3.

**Example 2.1** Let $S$ be standard simplex in $R^2$. A map $f \in C(S)$ is the identity, that is, $f(x) = x$, $\forall x \in S$. Given the grid size $\frac{1}{q}$ with $\frac{1}{q} = \frac{1}{2}$, then, for the integer labels of the sub-simplices of the triangulation with the grid size $\frac{1}{q}$, we have

$$l(x) = \begin{cases} 2, & x = (0,1), \\ 1, & x = (1,0), \\ 1, & x = (\frac{1}{2}, \frac{1}{2}). \end{cases}$$

It can be checked that $F(f,q) = \{(x_1, 1 - x_1) \mid 0 \leq x_1 \leq \frac{1}{2}\}$. For each $n = 1, 2, \ldots$, define $f^n \in C(S)$ satisfying

$$f^n(x_1, x_2) = \bigl((x_1)^{\frac{n}{1+n}}, 1 - (x_1)^{\frac{n}{1+n}}\bigr).$$

Then the corresponding integer labels using $f^n$ for each $n = 1, 2, \ldots$ is the same as

$$l(x) = \begin{cases} 2, & x = (0,1), \\ 1, & x = (1,0), \\ 2, & x = (\frac{1}{2}, \frac{1}{2}). \end{cases}$$

We can calculate that $F(f^n, q) = \{(x_1, 1 - x_1) \mid \frac{1}{2} \leq x_1 \leq 1\}$. Obviously, for small enough open set $U$ with $F(f,q) \subset U$, however close $f^n$ is to $f$, we have $F(f^n, q) \not\subset U$. Therefore, $F$ is not upper semi-continuous on $C(S)$, hence, the graph of $F$ is not closed. In fact, clearly, $F$ is also not lower semi-continuous.

For each $f \in C(S)$, denote by $\mathrm{Fix}(f)$ the fixed point set of $f$ on $S$. Note the fact of Example 2.1 about $F$, the following definitions consider a kind of description for stability of $F'$ and subsets of $\mathrm{Fix}(f)$.

**Definition 2.1** Given the grid size $\frac{1}{q}$, for each $f \in C(S)$, a closed subset $e(f)$ of $F'(f,q)$ is called an essential set with respect to $C(S)$ if for any open set $U$ with $U \supset e(f)$, there exists an open neighborhood $O(f)$ of $f$ in $C(S)$ such that $F'(f', q) \cap U \neq \emptyset$, $\forall f' \in O(f)$. If a connected component $C \subset F'(f,q)$ is an essential set, $C$ is called an essential connected component of $F'(f,q)$ with respect to $C(S)$.

**Definition 2.2** Let $f \in C(S)$, $e(f)$ be a closed subset of $\mathrm{Fix}(f)$. We call $e(f)$ an approximate essential set if for each $\varepsilon$ neighborhood $B(e(f), \varepsilon)$ of $e(f)$, there exists a $\delta > 0$ such that, for each $f' \in C(S)$ with $\|f - f'\| < \delta$, we can find a number $Z$, such that $F'(f', q) \cap B(e(f), \varepsilon) \neq \emptyset$, $\forall q > Z$.

**Lemma 2.1** (see [36]) *Let $Y$ be a metric space, $E$ be a Baire space, and $F : E \to 2^Y$ be an upper semi-continuous mapping with compact values. Then there is a dense residual subset $Q$ of $E$ such that $F$ is lower semi-continuous at each $x \in Q$.*

## 3 Stability results under function perturbations

**Theorem 3.1** *Given a triangulation of $S$ with vertices $v^1, v^2, \dots, v^{n+1}$ with a grid size $\frac{1}{q}$, the graph of the set-valued mapping $F'$,*

$$\operatorname{Gr} F' = \{(f,x) \mid f \in C(S), x \in F'(f,q)\},$$

*is closed.*

*Proof* Let $(f^m, x^m) \in \operatorname{Gr} F'$ with $(f^m, x^m) \to (f^0, x^0)$, $m = 1, 2, \dots$. It is clear that $(f^0, x^0) \in C(S) \times S$. We need to show that $x^0$ is a point of a complete sub-simplex $\sigma_{f_0}$ with vector-valued labels in $S$. For each $m = 1, 2, \dots$, since $(f^m, x^m) \in \operatorname{Gr} F'$, there exists a complete labeled sub-simplex $\sigma_{f_m}$ such that $x^m \in \sigma_{f^m} \subset F'(f^m, q) \subset S$, hence, denote $\sigma_{f^m}$ as $\sigma_{f^m}(y_m^1, y_m^2, \dots, y_m^{n+1}) = \sigma_{f^m}(y_m^1, \pi_m)$.

Since $\{\pi_m^1\}$ belongs in the finite set $I_{n+1}$, $\{\pi_m^1\}$ has a convergence subsequence $\{\pi_{mk}^1\}$, such that $\pi_{mi}^1 = \pi_{mj}^1$ for large enough $i$ and $j$ with $i \neq j$. For $\{\pi_{mk}^2\}$, we can also find such a convergence subsequence which is denoted $\{\pi_{mk}^2\}$ for convenience of notation. Then, following this method, we will get a convergence subsequence $\{\pi_{mk}^i\}$ of $\{\pi_m^i\}$ which can be unified as one $\{\pi^i\}$ with $\pi^i \neq \pi^j$, $\forall i \neq j$, that is, $\sigma_{f^{mk}}(y_{mk}^1, \pi_{mk}) = \sigma_{f^{mk}}(y_{mk}^1, \pi)$. Since $\{y_{mk}^1\} \subset X$, there is a sequence, being its convergence subsequence, that, without loss of generality, we also denote it by $\{y_{mk}^1\}$ with $y_{mk}^1 \to y_0^1$ $(k \to \infty)$. So far, by choosing some real numbers $p_{mk}^i$, $i \in I_{n+1}$, we can write $\sigma_{f^{mk}}(y_{mk}^1, \pi)$ as

$$y_{mk}^1 = (p_{mk}^1, p_{mk}^2, \dots, p_{mk}^{n+1})/q$$

and

$$y_{mk}^{i+1} = y_{mk}^i + (v^{\pi^{i+1}} - v^{\pi^i})/q, \quad \forall i \in I_n.$$

Then we have a point $y_0^i$ such that $y_{mk}^i \to y_0^i \in S$ for each $i \in I_{n+1}$. That is, $\sigma(y_0^1, \pi) = \sigma(y_0^1, y_0^2, \dots, y_0^{n+1})$ is definitely a sub-simplex in the triangulation of $S$ given the grid size $\frac{1}{q}$.

Note that $(f^{mk}, x^{mk}) \in \operatorname{Gr} F'$ with $(f^{mk}, x^{mk}) \to (f^0, x^0)$ as $k$ tends to infinity. Since $\sigma(y_{mk}^1, y_{mk}^2, \dots, y_{mk}^{n+1})$ is a complete sub-simplex with vector-valued labels, there exists a nonnegtive vector $(\lambda_{mk}^1, \lambda_{mk}^2, \dots, \lambda_{mk}^{n+1})$ such that

$$\sum_{i=1}^{n+1} \lambda_{mk}^i \bigl(-f^{mk}(y_{mk}^i) + y_{mk}^i + e\bigr) = e. \tag{1}$$

There is a convergence subsequences $\{\lambda_{mkj}^i\}$ of $\{\lambda_{mk}^i\}$ with $\lambda_{mkj}^i \to \lambda_0^i \geq 0$ $(j \to \infty)$, $\forall i \in I_{n+1}$. Then, by substituting $mk$ with $mkj$ in equation (1), as $j \to \infty$, we have

$$\sum_{i=1}^{n+1} \lambda_0^i \bigl(-f^0(y_0^i) + y_0^i + e\bigr) = e. \tag{2}$$

Therefore, $\sigma_{f^0} = \sigma(y_0^1, y_0^2, \dots, y_0^{n+1})$ is a complete sub-simplex with vector-valued labels.

Next, we have $x^0 \in \sigma_{f^0}$. Since $x^{mkj} \in \sigma_{f^{mkj}}$, there exists $\beta_{mkj}^i \geq 0$ such that

$$x^{mkj} = \sum_{i=1}^{n+1} \beta_{mkj}^i y_{mkj}^i \tag{3}$$

with $\sum_{i=1}^{n+1} \beta_{mkj}^i = 1$. Without loss of generality, we can assume that $\beta_{mkj}^i$ is convergent with the limit $\beta_0^i$, that is, $\beta_{mkj}^i \to \beta_0^i$ $(j \to \infty)$. Then, as $j \to \infty$, for equation (3), we have $x^0 = \sum_{i=1}^{n+1} \beta_0^i y_0^i \in \sigma_{f^0}$. □

From Theorem 3.1, we have the following direct corollary.

**Corollary 3.1** *Given a triangulation of $S$ with a grid size $\frac{1}{q}$, the set-valued mapping $F'$ is upper semi-continuous on $C(S)$.*

The following example shows that $F'$ does not possess the property of being lower semi-continuous on $C(S)$.

**Example 3.1** Let $S, f \in C(S)$ be the same as Example 2.1. With the grid size $\frac{1}{q}$ with $\frac{1}{q} = \frac{1}{4}$, for the vector-valued labels of the sub-simplices of the triangulation with $f$, we have for each grid point $x = (1/4, 3/4), (1/2, 1/2)$, $L(x) = (1, 1)$. Then the sub-simplex $\sigma = \{(x_1, x_2) \in S : 1/4 \leq x_1 \leq 1/2\}$ is complete and $\sigma \subset F'(f, q)$. We take a point $\bar{x} = (3/8, 5/8) \in \sigma$. For each $n = 1, 2, \ldots$, we define $f^n \in C(S)$ such that

$$f^n(x_1, x_2) = \left((x_1)^{\frac{n+1}{n}}, 1 - (x_1)^{\frac{n+1}{n}}\right).$$

Then, for each $n = 1, 2, \ldots$, the vector-valued labels for the sub-simplex $\sigma$ using $f^n$ is

$$L(x) = \begin{cases} \left(\frac{5}{4} - \left(\frac{1}{4}\right)^{\frac{n+1}{n}}, \left(\frac{1}{4}\right)^{\frac{n+1}{n}} + \frac{3}{4}\right) = (a, c), & x = (1/4, 3/4), \\ \left(\frac{3}{2} - \left(\frac{1}{2}\right)^{\frac{n+1}{n}}, \left(\frac{1}{2}\right)^{\frac{n+1}{n}} + \frac{1}{2}\right) = (b, d), & x = (1/2, 1/2), \end{cases}$$

hence, the right-hand side of the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d - b \\ a - c \end{bmatrix}$$

is the solution $(\lambda_1^*, \lambda_2^*)$ of the equations $\lambda_1 L(1/4, 3/4) + \lambda_2 L(1/2, 1/2) = e$, then $\lambda_2^* = \frac{a-c}{ad-bc}$. By a straightforward calculation, for each $n = 1, 2, \ldots$, we have $a - c = \frac{4^{\frac{1}{n}} - 1}{2 \cdot 4^{\frac{1}{n}}} > 0$, while $ad - bc = \frac{(2 - 2^{\frac{1}{n}}) 2^{\frac{1}{n}} - 1}{2 \cdot 4^{\frac{1}{n}}} < 0$. Then $\lambda_2^* < 0$, $\forall n = 1, 2, \ldots$. Thus, by labeling the sub-simplex $\sigma$ using $f^n$, one sees that $\sigma$ is not complete. Therefore, for a small enough open neighborhood $U$ of $\bar{x}$, we have $F'(f_n, q) \cap U = \emptyset$, for each $n = 1, 2, \ldots$, that is, $F'$ is not lower semi-continuous on $C(S)$.

From Theorem 3.1, the set-valued mapping $F'$ with $F' : C(S) \to 2^S$ is upper semi-continuous. If the set-valued mapping $F'$ is lower semi-continuous at a point $f^0$, then, given a grid size $\frac{1}{q}$, clearly, each point in $F'(f^0, q)$ is essential. Thus, by Fort's lemma (Lemma 2.1) and Definition 2.1, we can obtain the following generic stability result.

**Corollary 3.2** *Given a grid size $\frac{1}{q}$, there exists a dense residual set $Q$ in $C(S)$ such that for each $f \in C(S)$, each point in $F'(f, q)$ is essential with respect to $C(S)$.*

**Theorem 3.2** *Given a triangulation of $S$ with a grid size $\frac{1}{q}$, for each $f \in C(S)$, there exist finite essential connected components in $F'(f, q)$ with respect to $C(S)$.*

*Proof* From Theorem 3.1, the set-valued mapping $F'$ is upper semi-continuous on $S$. Then the set $F'(f,q)$ itself is an essential set with respect to $C(S)$. Let $\Phi$ denote the collection of all essential sets in $F'(f,q)$. Note that each decreasing chain in $\Phi$ with the set inclusion order has its intersection as a lower bound. Therefore, there exists a minimal element $e(f)$ in $\Phi$, which is an essential set in $F'(f,q)$. Hence, it is clear that each connected component $C$ with $C \supset e(f)$ is an essential connected component by Definition 2.1. Then the remaining problem is to show that each $e(f)$ is connected.

If not, let $e(f) = D^1 \cup D^2$. Nonessential closed sets $D^1$ and $D^2$ can be separated by two open sets $U^1$ and $U^2$ with $D^i \subset U^i$, $i = 1,2$. For each $i = 1,2$ and $\varepsilon > 0$, there exists an open set $W^i$ and $f^i \in C(S)$ with $D^i \subset W^i \subset \overline{W}^i \subset U^i$ such that $\|f - f^i\| < \frac{\varepsilon}{3}$ but $F'(f^i,q) \cap W^i = \emptyset$; meanwhile, for any $f' \in C(S)$ with $\|f' - f\| < \varepsilon$, we have $F'(f',q) \cap (W^1 \cup W^2) \neq \emptyset$. Construct a special $f' \in C(S)$ by defining

$$f'(x) = \lambda(x)f^1(x) + \bigl(1 - \lambda(x)\bigr)f^2(x), \quad \forall x \in S,$$

where $\lambda(x) = d(x,\overline{W}^2)/(d(x,\overline{W}^1) + d(x,\overline{W}^2))$. Routinely, we can check that $\|f' - f\| < \varepsilon$, this means that there is at least a point $x$ such that $x \in F'(f',q) \cap (W^1 \cup W^2)$. For each $i = 1,2$, if $x \in W^i$, such that $f'(x) = f^i(x)$, then the labels for the sub-simplices' vertices in $W^i$ using $f^i$ or $f$ are no different. Therefore, $F'(f^i,q) \cap W^i = F'(f',q) \cap W^i$, from which one deduces the fact that $x \notin F'(f',q)$, a contradiction.

Finally, from the finiteness of the complete labeled simplex in $S$, the result follows. □

The following result shows that essential connected components under the grid size $\frac{1}{q}$ can be close to an approximate fixed point set as $q$ tends to infinity.

**Theorem 3.3** *Given a continuous function $f \in C(S)$, for each grid size $\frac{1}{q}$, let $C^q \subset F'(f,q)$ be an essential connected component with respect to $C(S)$, there exists a subsequence $\{C^{q_k}\}$ of $\{C^q\}$ with $C^{q_k} \xrightarrow{h} C^0$ and $C^0$ is an approximate essential connected set in* $\mathrm{Fix}(f)$, *where $h$ is the Hausdorff metric induced by the Euclidean metric on $R^{n+1}$.*

*Proof* Since $\{C^q\}$ is a sequence in $K(S)$, where $K(S)$ is the collection of nonempty compact subsets of $S$, from the compactness of $S$, there is a subsequence $\{C^{q_k}\}$ of $\{C^q\}$ with the limit $C^0 \in K(S)$. We denote the subsequence just as $\{C^q\}$ for convenience. For each $x^0 \in C^0$, there is a sequence $\{x^q\}$ with $x^q \in C^q$ and $x^q \to x^0$. For each $\varepsilon > 0$, since $f$ is continuous, there exists a number $N$ such that, for each sub-simplex $\sigma_f$ in the triangulation of $S$ under the grid size $\frac{1}{N}$, we have

$$\max_{i \in I_{n+1}} \bigl\{\bigl|f_i(x) - f_i(y)\bigr|\bigr\} < \frac{1}{N} < \frac{\varepsilon}{3\sqrt{n+1}}, \quad \forall x,y \in \sigma_f.$$

For each $q > N$, since $x^q \in C^q \subset F'(f,q)$, we have $\|f(x^q) - x^q\| < \frac{\varepsilon}{3}$. Then we can find a large enough $q$ such that the following inequality holds:

$$\begin{aligned}\bigl\|f\bigl(x^0\bigr) - x^0\bigr\| &\le \bigl\|f\bigl(x^0\bigr) - f\bigl(x^q\bigr)\bigr\| + \bigl\|f\bigl(x^q\bigr) - x^q\bigr\| + \bigl\|x^q - x^0\bigr\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Therefore, we assert that $x^0 \in \mathrm{Fix}(f)$, hence, $C^0 \subset \mathrm{Fix}(f)$.

Assume that $C^0$ is not connected, then $C^0$ can be decomposed as two disjoint compact sets like $C^0 = C' \cup C''$ with two open sets $W'$ and $W''$ such that $C' \subset W'$, $C'' \subset W''$, and $W' \cap W'' = \emptyset$. By the compactness of $C'$ and $C''$, there are two open sets $U'$ and $U''$ such that $C' \subset U' \subset \bar{U}' \subset W'$ and $C'' \subset U'' \subset \bar{U}'' \subset W''$. Since $C^q$ is connected, we have $C^q \subset U'$ or $C^q \subset U''$ as $q$ large enough. Then the limit of $C^q$ is in $W'$ or $W''$, which contradicts the fact that $C^q \xrightarrow{h} C' \cup C''$ and $C' \subset W'$, $C'' \subset W''$, and $W' \cap W'' = \emptyset$. Therefore, $C^0$ is connected.

Finally, we show that $C^0$ is an approximate essential set of $\mathrm{Fix}(f)$. If not, then there exists a $\bar{\varepsilon} > 0$ and $f^j$ ($j = 1, 2, \ldots$) with $f^j \to f$, such that for each number $q$, $F'(f^j, q) \cap B(C^0, \bar{\varepsilon}) = \emptyset$, $j = 1, 2, \ldots$. Since $C^q \xrightarrow{h} C^0$, there is a number $N$ such that $C^q \subset B(C^0, \bar{\varepsilon})$ when $q \geq N$. Because $C^N$ is essential, for the open set $B(C^0, \varepsilon)$, there is a $\delta > 0$ such that for any $f'$ with $\|f - f'\| < \delta$, we have $F(f', N) \cap B(C^0, \varepsilon) \neq \emptyset$. From the fact that $f^j \to f$, for large enough $j$, we have $F(f^j, N) \cap B(C^0, \varepsilon) \neq \emptyset$, a contradiction. □

## 4 Stability results under perturbations of simplices and functions

In order to analyze the perturbation of domains, let $X \subset R^{n+1}$ be a $n$ dimensional compact set, $M$ the collection of all $n$-simplex in $X$. For any two $S_1(v_1^1, v_1^2, \ldots, v_1^{n+1})$ and $S_2(v_2^1, v_2^2, \ldots, v_2^{n+1})$ in $M$, define

$$\rho(S_1, S_2) = \min_{\pi} \sum_{k=0}^{n+1} \| v_1^k - v_2^{\pi_k} \|.$$

**Lemma 4.1** *$\rho$ is a metric on $M$.*

*Proof* (i) For any $S_1(v_1^1, v_1^2, \ldots, v_1^{n+1}), S_2(v_2^1, v_2^2, \ldots, v_2^{n+1}) \in M$, we have $\rho(S_1, S_2) = \rho(S_2, S_1)$. Let $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_{n+1})$ match $\rho(S_1, S_2)$. We have $\rho(S_1, S_2) = \sum_{k=0}^{n+1} \| v_1^k - v_2^{\bar{\pi}_k} \|$. Then

$$\begin{aligned}
\rho(S_2, S_1) &= \min_{\pi} \sum_{k=0}^{n+1} \| v_2^{\bar{\pi}_k} - v_1^{\pi_k} \| \\
&= \sum_{k=0}^{n+1} \| v_2^{\bar{\pi}_k} - v_1^k \| = \rho(S_1, S_2).
\end{aligned}$$

(ii) For any $S_1(v_1^1, v_1^2, \ldots, v_1^{n+1}), S_2(v_2^1, v_2^2, \ldots, v_2^{n+1}) \in M$, we have $\rho(S_1, S_2) = 0 \Leftrightarrow S_1 = S_2$. From the definition of $\rho$, one needs only the proof of the necessity. Let $\rho(S_1, S_2) = 0$, then there exists $\bar{\pi}$ such that $\sum_{k=0}^{n+1} \| v_1^k - v_2^{\bar{\pi}_k} \| = 0$, which means that $\| v_1^k - v_2^{\bar{\pi}_k} \| = 0$, $\forall k \in I_{n+1}$. That is, $S_1 = S_2$.

(iii) For any $S_1(v_1^1, v_1^2, \ldots, v_1^{n+1}), S(v^1, v^2, \ldots, v^{n+1}), S_2(v_2^1, v_2^2, \ldots, v_2^{n+1}) \in M$, we have $\rho(S_1, S_2) \leq \rho(S_1, S) + \rho(S, S_2)$. Let $\rho(S_1, S) = \sum_{k=0}^{n+1} \| v_1^k - v^{\bar{\pi}_k} \|$. Then we have

$$\begin{aligned}
\rho(S_1, S_2) &= \min_{\pi} \sum_{k=0}^{n+1} \| v_1^k - v_2^{\pi_k} \| \\
&\leq \min_{\pi} \left( \sum_{k=0}^{n+1} \| v_1^k - v^{\bar{\pi}_k} \| + \sum_{k=0}^{n+1} \| v^{\bar{\pi}_k} - v_2^{\pi_k} \| \right) \\
&= \sum_{k=0}^{n+1} \| v_1^k - v^{\bar{\pi}_k} \| + \min_{\pi} \sum_{k=0}^{n+1} \| v^{\bar{\pi}_k} - v_2^{\pi_k} \|
\end{aligned}$$

$$= \rho(S_1, S) + \min_{\pi} \sum_{k=0}^{n+1} \| v^k - v_2^{\pi_k} \|$$

$$= \rho(S_1, S) + \rho(S, S_2). \qquad \square$$

Concerning a stability analysis of approximate fixed points, we intend to restrain domains to avoiding a domain perturbed in a large-scale range. Let $\Delta$ be an $n$ dimensional subset of a compact set $X$ in $R^{n+1}$. Let $M' \subset M$ satisfy $M' = \{S \in M : \Delta \subset S \subset X\}$.

**Lemma 4.2** *The metric space* $(M', \rho)$ *is complete.*

*Proof* Take a Cauchy sequence $\{S_m(v_m^1, v_m^2, \ldots, v_m^{n+1})\}$ in $M'$. Then, for each $\varepsilon > 0$, there exists a number $N$ such that $\rho(S_s, S_t) < \varepsilon$ for any $s, t > N$. Without loss of generality, we can assume that $\rho(S_s, S_t) = \sum_{k=0}^{n+1} \|v_s^k - v_t^k\|$. Therefore, $\{v_m^k\}$ is a Cauchy sequence with the limit $v_0^k$, $\forall k \in I_{n+1}$. Denote by $S_0$ the simplex $S_0(v_0^1, v_0^2, \ldots, v_0^{n+1})$. Then we have $\rho(S_m, S_0) \to 0$. Since $\Delta \subset \bigcap_{m=1}^{\infty} S_m \subset X$, it follows that $\Delta \subset S_0 \subset X$, hence $S_0$ is an $n$-simplex in $M'$. $\square$

Let $P$ be the set of pairs $(f, S)$ such that

$$P = \{(f, S) \in C(X) \times M' : f(x) \in S, \forall x \in S\}.$$

Define the metric $d$ between two $u_1 = (f_1, S_1)$ and $u_2 = (f_2, S_2)$ in $M'$ as

$$d(u_1, u_2) = \max_{x \in X} \|f_1(x) - f_2(x)\| + \rho(S_1, S_2).$$

Given a grid size $\frac{1}{q}$, for each $u = (f, S) \in P$, let $T(u, q)$ be the set of all sub-simplices with complete vector-valued labels with the function $f$ in the triangulation of $S$ under the grid size $\frac{1}{q}$, then we define a set-valued mapping $T$ from $P$ to $X$.

Similar to Definition 2.1, we consider the essential stability of approximate fixed points under both perturbations of functions and domains.

**Definition 4.1** Given the grid size $\frac{1}{q}$, for each $u = (f, S) \in P$, we call a closed subset $e(f)$ in $T(u, q)$ an essential set with respect to $P$ if, for any open set $U$ with $U \supset e(f)$, there is an open $O(u)$ of $u$ in $P$ such that $U \cap T(u', q) \neq \emptyset$, $\forall u' \in O(u)$. A minimal element in the collection (ordered by set inclusion) of essential sets in $T(u, q)$ is called a minimal essential set with respect to $P$.

**Theorem 4.1** *Given a grid size* $\frac{1}{q}$ *and a continuous function* $f \in C(X)$, *the graph of the set-valued mapping* $T$, $\operatorname{Gr} T = \{(u, x) \mid u \in P, x \in T(u, q)\}$, *is closed.*

*Proof* Let $\{(u_m, x_m)\} \subset \operatorname{Gr} T$ with $(u_m, x_m) \to (u_0, x_0)$, where $u_m = (f_m, S_m)$, $u_0 = (f_0, S_0)$, and $S_m$ is the simplex with $v_m^1, v_m^2, \ldots, v_m^{n+1}$ as its vertices for each $m = 1, 2, \ldots$. Since $(u_m, x_m) \in \operatorname{Gr} T$, there exists a complete sub-simplex $\sigma_{f_m}(y_m^1, y_m^2, \ldots, y_m^{n+1})$ with vector-valued labels such that $x_m \in \sigma_{f_m} \subset T(u_m, q) \subset S_m$, $m = 1, 2, \ldots$.

Denote $\sigma_{f_m}(y_m^1, y_m^2, \ldots, y_m^{n+1})$ as $\sigma_{f_m}(y_m^1, \pi_m)$. Similar to Theorem 3.1, there exists a subsequence $\{mk\}$ of $\{m\}$ and a permutation $\pi$ such that $\sigma_{f_{mk}}(y_{mk}^1, \pi_{mk}) = \sigma_{f_{mk}}(y_{mk}^1, \pi)$. There is a convergent subsequence of $\{y_{mk}^1\} \subset X$, which is also denoted by $\{y_{mk}^1\}$ with $y_{mk}^1 \to y_0^1$

($k \to \infty$). So far, for each $mk$, by choosing some real numbers $p^i_{mk}$ ($i \in I_{n+1}$) with $p^i_{mk} \to p^i_0$ ($k \to \infty$), the sub-simplex $\sigma_{f_{mk}}(y^1_{mk}, \pi)$ can be written as

$$y^1_{mk} = \left(p^1_{mk}, p^2_{mk}, \ldots, p^{n+1}_{mk}\right)/q$$

and

$$y^{i+1}_{mk} = y^i_{mk} + \left(v^{\pi^{i+1}}_{mk} - v^{\pi^i}_{mk}\right)/q, \quad \forall i \in I_n.$$

Since $u_m \xrightarrow{d} u_0$, which means that $S_m \xrightarrow{\rho} S_0 \in M'$, then, by the definition of $\rho$, we have $v^{\pi^i}_{mk} \to v^{\pi^i}_0$, $\forall i \in I_{n+1}$. Noting that $y^1_{mk} \to y^1_0$, we have $\sigma_{f_{mk}}(y^1_{mk}, \pi) \xrightarrow{\rho} \sigma(y^1_0, \pi)$ as $k \to \infty$. Clearly, $\sigma_{f_0}(y^1_0, \pi)$ is a simplex in the triangulation of $S_0(v^1_0, v^2_0, \ldots, v^{n+1}_0)$ with the grid size $\frac{1}{q}$.

To finish the proof that $x_0 \in \sigma(y^1_0, \pi)$ and $\sigma(y^1_0, \pi)$ is a complete sub-simplex with vector-valued labels by function $f_0$, we can adopt the corresponding part of Theorem 3.1. □

From Theorem 4.1, $T$ is upper semi-continuous on $P$. Following the proof of Theorem 3.2 for the part of the existence of minimal element of essential sets, we obtain the following result.

**Theorem 4.2** *For each $u = (f, S) \in P$, given a triangulation of $S$ with a grid size $\frac{1}{q}$, there exists a minimal essential set in $T(u, q)$ with respect to $P$.*

**Remark 4.1** Theorems 3.1, 3.2 and Theorem 4.1 obtain the stability for approximate fixed points with simplicial methods. From the point of applications, this may facilitate the stability analysis of equilibrium problems, such as Nash equilibria and $\varepsilon$-approximate Nash equilibria.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors read and approved the manuscript.

**Author details**
[1]College of Science, Guizhou University, Guiyang, 550025, China. [2]College of Science, Guilin University of Technology, Guilin, 541004, China.

**Acknowledgements**
This project is supported by the National Natural Science Foundation of China (11271098, 11661030), the China Postdoctoral Science Foundation (2016M590905), and the Doctoral Research Fund of Guilin University of Technology.

Received: 24 June 2016 Accepted: 4 September 2016 Published online: 13 September 2016

**References**

1. Brouwer, LEJ: Über abbildung von mannigfaltigkeiten. Math. Ann. **71**(1), 97-115 (1911)
2. Nash, JF: Equilibrium points in *n*-person games. Proc. Natl. Acad. Sci. USA **36**(1), 48-49 (1950)
3. Arrow, K, Debreu, G: Existence of an equilibrium for a competitive economy. Econometrica **22**, 265-290 (1954)
4. Papadimitriou, CH: On graph-theoretic lemmata and complexity classes. In: Proceedings of the 31st Annual Symposium on Foundations of Computer Science, pp. 794-801. IEEE Press, New York (1990)
5. Cole, R, Dodis, Y, Roughgarden, T: Pricing network edges for heterogeneous selfish users. In: Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing, pp. 521-530. ACM, New York (2003)
6. Low, SH: A duality model of TCP and queue management algorithms. IEEE/ACM Trans. Netw. **11**(4), 525-536 (2003)
7. Meinardus, G: Invarianz bei linearen Approximationen. Arch. Ration. Mech. Anal. **14**(1), 301-303 (1963)
8. Spielmat, DA, Teng, SH: Spectral partitioning works: planar graphs and finite element meshes. In: Proceedings of the 37th Annual Symposium on Foundations of Computer Science, pp. 96-105. IEEE Press, New York (1996)
9. Scarf, H: The approximation of fixed points of a continuous mapping. SIAM J. Appl. Math. **15**(5), 1328-1343 (1967)

10. Kuhn, HW: Simplicial approximation of fixed points. Proc. Natl. Acad. Sci. USA **61**(4), 1238-1242 (1968)
11. Kuhn, HW, MacKinnon, JG: Sandwich method for finding fixed points. J. Optim. Theory Appl. **17**(3), 189-204 (1975)
12. Merrill, OH: Applications and extensions of an algorithm that computes fixed points of certain upper semi-continuous point to set mappings. Technical report 71-7, University of Michigan (1972)
13. Van der Laan, G, Talman, A: A restart algorithm for computing fixed points without an extra dimension. Math. Program. **17**(1), 74-84 (1979)
14. Van der Laan, G, Talman, AJJ: A class of simplicial restart fixed point algorithms without an extra dimension. Math. Program. **20**(1), 33-48 (1981)
15. Talman, AJJ: Variable dimension fixed point algorithms and triangulations. Stat. Neerl. **35**(1), 59 (1981)
16. Eaves, BC: Homotopies for computation of fixed points. Math. Program. **3**(1), 1-22 (1972)
17. Herings, PJ-J, Peeters, R: Homotopy methods to compute equilibria in game theory. Econ. Theory **42**(1), 119-156 (2010)
18. Fort, MK: Essential and nonessential fixed points. Am. J. Math. **72**, 315-322 (1950)
19. Kinoshita, S: On essential component of the set of fixed points. Osaka Math. J. **4**, 19-22 (1952)
20. O'Neill, B: Essential sets and fixed points. Am. J. Math. **75**, 497-509 (1953)
21. McLennan, A: Selected topics in the theory of fixed points. University of Minnesota, Minneapolis (1989)
22. Tan, KK, Yu, J, Yuan, XZ: The stability of coincident points for multivalued mappings. Nonlinear Anal., Theory Methods Appl. **25**, 163-168 (1995)
23. Isac, G, Yuan, GXZ: The essential components of coincident points for weakly inward and outward set-valued mappings. Appl. Math. Lett. **12**, 121-126 (1999)
24. Song, QQ: On essential sets of fixed points for functions. Numer. Funct. Anal. Optim. **36**, 942-950 (2015)
25. Yu, J, Xiang, SW: The stability of the set of KKM points. Nonlinear Anal., Theory Methods Appl. **54**, 839-844 (2003)
26. Khanh, PQ, Quan, NH: Generic stability and essential components of generalized KKM points and applications. J. Optim. Theory Appl. **148**, 488-504 (2011)
27. Yu, J: Essential equilibria of *N*-person noncooperative games. J. Math. Econ. **31**, 361-372 (1999)
28. Govindan, S, Wilson, R: Essential equilibria. Proc. Natl. Acad. Sci. USA **102**, 15706-15711 (2005)
29. Carbonell-Nicolau, O: Essential equilibria in normal-form games. J. Econ. Theory **145**, 421-431 (2010)
30. Yang, H, Xiao, X: Essential components of Nash equilibria for games parametrized by payoffs and strategies. Nonlinear Anal., Theory Methods Appl. **71**, e2322-e2326 (2009)
31. Song, QQ, Wang, LS: On the stability of the solution for multiobjective generalized games with the payoffs perturbed. Nonlinear Anal., Theory Methods Appl. **73**, 2680-2685 (2010)
32. Yang, Z, Pu, YJ: Essential stability of solutions for maximal element theorem with applications. J. Optim. Theory Appl. **150**, 284-297 (2011)
33. Hung, NV: Sensitivity analysis for generalized quasi-variational relation problems in locally *G*-convex spaces. Fixed Point Theory Appl. **2012**, 158 (2012)
34. Yang, Z: On existence and essential stability of solutions of symmetric variational relation problems. J. Inequal. Appl. **2014**, 5 (2014)
35. Hung, NV, Kieu, PT: On the existence and essential components of solution sets for systems of generalized quasi-variational relation problems. J. Inequal. Appl. **2014**, 250 (2014)
36. Fort, MK: Points of continuity of semicontinuous function. Publ. Math. (Debr.) **2**, 100-102 (1951)