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Boundedness of rough singular integral operators and commutators on Morrey-Herz spaces with variable exponents

Liwei Wang[1*], Meng Qu[2] and Lisheng Shu[2]

*Correspondence: wangliwei@ahpu.edu.cn
[1]Department of Applied Mathematics, School of Mathematics and Physics, Anhui Polytechnic University, Wuhu, 241000, The People's Republic of China
Full list of author information is available at the end of the article

Abstract

By decomposing functions, we establish some boundedness results for some rough singular integrals on the homogeneous Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$, where the two main indices are variable. The corresponding results as regards their commutators are also considered.

MSC: 42B20; 42B25

Keywords: variable exponent; Morrey-Herz spaces; commutators; singular integrals

1 Introduction

In recent years, function spaces with variable exponents have been intensively studied; see [1–6] for example. The origin of such spaces is the study of PDE with non-standard growth conditions, fluid dynamics and image restoration; see [7–9]. By virtue of the fundamental work [10] by Kováčik and Rákosník appearing in 1991, the Lebesgue spaces and various other function spaces have been investigated in the variable exponent setting. Meanwhile, the boundedness of some classical operators, such as the Hardy-Littlewood maximal operator, singular integrals and commutators, has been proved on these spaces; see [11–19] and the references therein.

Herz spaces have been playing a central role in harmonic analysis. After they were introduced in [20], the theory of these spaces had a remarkable development in part due to its useful applications. For instance, they are good substitutes of the ordinary Hardy spaces when considering the boundedness of non-translation invariant singular integral operators, they also appear in the characterization of multiplier on Hardy spaces and in the regularity theory for elliptic and parabolic equations in divergence form; see [21–23] for example.

One of the important problems on Herz spaces is the boundedness of sublinear operators satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy, \quad x \notin \operatorname{supp} f, \tag{1.1}$$

for integrable and compactly supported functions f. We mention that condition (1.1) was initially studied by Soria and Weiss [24] and it is satisfied by several classical operators, such as Calderón-Zymund operators, the Carleson maximal operator and the Hardy-Littlewood maximal operator. Hernández, Li, Lu, and Yang [25–27] proved that if a sublinear operator T is bounded on $L^p(\mathbb{R}^n)$ and satisfies the size condition (1.1), then T is bounded on the homogeneous Herz spaces $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ and on the non-homogeneous Herz spaces $K_p^{\alpha,q}(\mathbb{R}^n)$. This result is extended to the generalized Herz spaces $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ by Izuki [28]. In 2012, Almeida and Drihem [11] made a further step and gave boundedness results for T on $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$, where the exponent α is variable as well.

Denote by S^{n-1} the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) with normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^s(S^{n-1})$ for some $s \in (1,\infty]$ be homogeneous of degree zero. In this paper, we consider sublinear operators satisfying the size condition

$$\bigl|T_\Omega f(x)\bigr| \lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n}\bigl|f(y)\bigr|\,dy, \quad x \notin \operatorname{supp} f, \tag{1.2}$$

for integrable and compactly supported functions f and commutators defined by

$$[b, T_\Omega]f(x) := b(x)T_\Omega f(x) - T_\Omega(bf)(x), \quad b \in BMO\bigl(\mathbb{R}^n\bigr). \tag{1.3}$$

Lu *et al.* first proved the boundedness for both T_Ω and $[b, T_\Omega]$ on Herz spaces and Morrey-Herz spaces with constant exponent; see [29, 30]. Motivated by the work of Lu [29] and Almeida [11], we shall generalize these results to the case of variable exponent and also consider the boundedness on Morrey-Herz spaces with variable exponent. Our approach is mainly based on some properties of variable exponent and BMO norms obtained by the author [31].

For brevity, we denote $B := \{y \in \mathbb{R}^n : |x-y| < r\}$. f_B stands for the integral average of f on B, *i.e.* $f_B = \frac{1}{|B|}\int_B f(x)\,dx$. $p'(\cdot)$ means the conjugate exponent $1/p(\cdot) + 1/p'(\cdot) = 1$. C denotes a positive constant, which may have different values even in the same line. $f \lesssim g$ means that $f \leq Cg$, and $f \approx g$ means $f \lesssim g \lesssim f$.

2 Preliminaries and lemmas

Let $E \subset \mathbb{R}^n$ with the Lebesgue measure $|E| > 0$, $p(\cdot): E \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \biggl\{f \text{ is measurable} : \int_E \biggl(\frac{|f(x)|}{\lambda}\biggr)^{p(x)}\,dx < \infty \text{ for some constant } \lambda > 0\biggr\}.$$

This is a Banach space with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\biggl\{\lambda > 0 : \int_E \biggl(\frac{|f(x)|}{\lambda}\biggr)^{p(x)}\,dx \leq 1\biggr\}.$$

The space with variable exponent $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) = \bigl\{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E\bigr\}.$$

We denote

$$p_- = \operatorname{ess\,inf}\bigl\{p(x) : x \in E\bigr\}, \qquad p_+ = \operatorname{ess\,sup}\bigl\{p(x) : x \in E\bigr\},$$

$$\mathscr{P}(E) = \{p(\cdot) : p_- > 1 \text{ and } p_+ < \infty\},$$

and

$$\mathscr{B}(E) = \{p(\cdot) \in \mathscr{P}(E) : M \text{ is bounded on } L^{p(\cdot)}(E)\},$$

where the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r)\cap E} |f(y)|\, dy.$$

A function $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is called log-Hölder continuous at the origin (or has a log decay at the origin), if there exists a constant $C_{\log} > 0$ such that

$$|\alpha(x) - \alpha(0)| \le \frac{C_{\log}}{\log(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$. If, for some $\alpha_\infty \in \mathbb{R}$ and $C_{\log} > 0$, we have

$$|\alpha(x) - \alpha_\infty| \le \frac{C_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$, then $\alpha(\cdot)$ is called log-Hölder continuous at infinity (or has a log decay at infinity).

By $\mathscr{P}_0^{\log}(\mathbb{R}^n)$ and $\mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathscr{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively. It is worth noting that if $p(\cdot) \in \mathscr{P}_0^{\log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$, then we have $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$; see [31] or [32].

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $R_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{R_k}$ be the characteristic function of the set R_k for $k \in \mathbb{Z}$. Almeida and Direhem [11] first introduced the following Herz spaces with variable exponents.

Definition 2.1 Let $0 < q \le \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$.

(1) The homogeneous Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \backslash \{0\})$ such that

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} := \left(\sum_{k\in\mathbb{Z}} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} < \infty.$$

(2) The non-homogeneous Herz space $K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ consists of all $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} := \|f\chi_{B_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left(\sum_{k\ge1} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

Definition 2.2 Let $0 < q \le \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 \le \lambda < \infty$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous Morrey-Herz space $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ is defined as the set of all

$f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$ such that

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}\big\|2^{k\alpha(\cdot)}f\chi_k\big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

Remark 2.3 It is easy to see that $M\dot{K}^{\alpha(\cdot),0}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$. If $\alpha(\cdot)$ is constant, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$, which is first defined by Izuki [33]. If both $\alpha(\cdot)$ and $p(\cdot)$ are constant, then $M\dot{K}^{\alpha(\cdot),0}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)$ are classical Herz spaces in [22], and $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ are classical Morrey-Herz spaces in [30].

In [32, 34], Lu and Zhu obtained the following result:

Proposition 2.4 *Let $0 < q \le \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 \le \lambda < \infty$ and $\alpha \in L^\infty(\mathbb{R}^n)$. If α is log-Hölder continuous both at origin and at infinity, then*

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \approx \max\Bigg\{&\sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q},\\ &\sup_{L\ge0,L\in\mathbb{Z}}\left[2^{-L\lambda}\left(\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} + 2^{-L\lambda}\left(\sum_{k=0}^{L}2^{k\alpha_\infty q}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}\right]\Bigg\}.\end{aligned}$$

Before stating our main results, we introduce some key lemmas which will be used later.

Lemma 2.5 ([10]) (Generalized Hölder inequality) *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, if $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n}\big|f(x)g(x)\big|\,dx \le r_p\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $r_p = 1 + 1/p_- - 1/p_+$.

Lemma 2.6 ([35]) *Let $\Omega \in L^s(S^{n-1})$, $s \in [1,\infty]$. If $a > 0$, $d \in (0,s]$ and $-n + \frac{(n-1)d}{s} < \nu < \infty$, then*

$$\left(\int_{|y|\le a|x|}|y|^\nu\big|\Omega(x-y)\big|^d\,dy\right)^{1/d} \lesssim \|\Omega\|_{L^s(S^{n-1})}|x|^{(\nu+n)/d}.$$

Lemma 2.7 ([36]) *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. If $q \in (p_+,\infty)$ and $\frac{1}{p(x)} = \frac{1}{\tilde{q}(x)} + \frac{1}{q}$ $(x \in \mathbb{R}^n)$, then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$$

for all measurable functions f and g.

We remark that Lemmas 2.8-2.10 were shown in Izuki [14, 31], and Lemma 2.11 was considered by Almeida and Drihem in [11].

Lemma 2.8 *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then we see that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \lesssim 1.$$

Lemma 2.9 *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then we see that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_1}, \qquad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Lemma 2.10 *Let $b \in BMO(\mathbb{R}^n)$, $k > j$ $(k, j \in \mathbb{N})$, then we have*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\|(b - b_B)\chi_B\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \|b\|_{BMO}$$

and

$$\left\|(b - b_{B_j})\chi_{B_k}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim (k - j)\|b\|_{BMO}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.11 *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and $r_1 > 0$. If α is log-Hölder continuous both at origin and at infinity, then*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} (\frac{r_1}{r_2})^{\alpha_+}, & 0 < r_2 \le r_1/2, \\ 1, & r_1/2 < r_2 \le 2r_1, \\ (\frac{r_1}{r_2})^{\alpha_-}, & r_2 > 2r_1, \end{cases}$$

for any $x \in B(0, r_1)\backslash B(0, r_1/2)$ and $y \in B(0, r_2)\backslash B(0, r_2/2)$, with the implicit constant not depending on x, y, r_1, and r_2.

3 Main results and their proofs

In this section, we prove the boundedness of sublinear operators T_Ω satisfying the size condition (1.2) and the commutators $[b, T_\Omega]$ defined as in (1.3) on the homogeneous Morrey-Herz space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$, respectively.

Our main results in this paper can be stated as follows.

Theorem 3.1 *Suppose that $p(\cdot) \in \mathscr{P}^{\log}_0(\mathbb{R}^n) \cap \mathscr{P}^{\log}_\infty(\mathbb{R}^n)$ and $\Omega \in L^s(S^{n-1})$, $s > (p')_+$. Let $0 < \lambda < n$, $0 < q \le \infty$, $-1/s < \nu < \infty$ and let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and at infinity, such that*

$$\lambda - n\delta_1 - \nu - n/s < \alpha_- \le \alpha_+ < n\delta_2 - \nu - n/s, \tag{3.1}$$

where $0 < \delta_1, \delta_2 < 1$ are the constants appearing in Lemma 2.9. Then every sublinear operator T_Ω satisfying (1.2) which is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is also bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$.

Theorem 3.2 *Suppose that $b \in BMO(\mathbb{R}^n)$ and let $[b, T_\Omega]$ be defined as in* (1.3). *Under the assumptions in Theorem* 3.1, *if $[b, T_\Omega]$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then $[b, T_\Omega]$ is also bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$.*

Remark 3.3 According to Remark 2.3, Theorem 3.1 and Theorem 3.2 extend the corresponding results in [29] to a more generalized function space. Moreover, comparing with [11, 28], our main results generalize the integral kernel to the case of $\Omega \in L^s(S^{n-1})$.

Our proofs use partially some decomposition techniques already used in [30] where the constant exponent case was studied. We consider only $0 < q < \infty$, the arguments are similar in the case $q = \infty$.

Proof of Theorem 3.1 Let $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. We decompose

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Then we have

$$\begin{aligned}
\|T_\Omega(f)\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \|2^{k\alpha(\cdot)} T_\Omega(f)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left\| 2^{k\alpha(\cdot)} \left(\sum_{j=-\infty}^{k-2} |T_\Omega(f_j)(\cdot)| \right) \chi_k(\cdot) \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left\| 2^{k\alpha(\cdot)} \left(\sum_{j=k-1}^{k+1} |T_\Omega(f_j)(\cdot)| \right) \chi_k(\cdot) \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left\| 2^{k\alpha(\cdot)} \left(\sum_{j=k+2}^{\infty} |T_\Omega(f_j)(\cdot)| \right) \chi_k(\cdot) \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&=: U_1 + U_2 + U_3.
\end{aligned}$$

First we estimate U_1. Noting that if $x \in R_k, y \in R_j$ and $j \le k-2$, then $|x-y| \approx |x| \approx 2^k$, by Lemma 2.11, we get

$$\begin{aligned}
2^{k\alpha(x)} \left(\sum_{j=-\infty}^{k-2} |T_\Omega(f_j)(x)| \right) \chi_k(x) &\lesssim \sum_{j=-\infty}^{k-2} 2^{-kn} \int_{R_j} 2^{k\alpha(x)} |\Omega(x-y)| |f_j(y)|\, dy \cdot \chi_k(x) \\
&\lesssim \sum_{j=-\infty}^{k-2} 2^{-kn} 2^{(k-j)\alpha_+} \int_{R_j} 2^{j\alpha(y)} |\Omega(x-y)| |f_j(y)|\, dy \cdot \chi_k(x),
\end{aligned}$$

which combining with Lemma 2.5 yields

$$\begin{aligned}
&\left\| 2^{k\alpha(\cdot)} \sum_{j=-\infty}^{k-2} |T_\Omega(f_j)(\cdot)| \chi_k(\cdot) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \lesssim \sum_{j=-\infty}^{k-2} 2^{-kn} 2^{(k-j)\alpha_+} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \qquad (3.2)
\end{aligned}$$

We define a variable exponent $\widetilde{p}(\cdot)$ by $\frac{1}{p'(x)} = \frac{1}{\widetilde{p}'(x)} + \frac{1}{s}$, since $s > (p')_+$, by Lemma 2.7 and Lemma 2.6, we obtain

$$\begin{aligned}
&\bigl\|\Omega(x-\cdot)\chi_j(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim \bigl\|\Omega(x-\cdot)\chi_j(\cdot)\bigr\|_{L^s(\mathbb{R}^n)}\bigl\|\chi_j(\cdot)\bigr\|_{L^{\widetilde{p}'(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim 2^{-j\nu}\left(\int_{R_j}|y|^{s\nu}\bigl|\Omega(x-y)\bigr|^s\,dy\right)^{1/s}\bigl\|\chi_{B_j}(\cdot)\bigr\|_{L^{\widetilde{p}'(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim 2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\bigl\|\chi_{B_j}(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s},
\end{aligned}\tag{3.3}$$

where the last inequality is based on the fact that $\|\chi_{B_j}(\cdot)\|_{L^{\widetilde{p}'(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_j}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}$; see [37], p.258, for the details.

From (3.2), (3.3), Lemma 2.8, and Lemma 2.9, we deduce that

$$\begin{aligned}
&\left\|2^{k\alpha(\cdot)}\sum_{j=-\infty}^{k-2}\bigl|T_\Omega(f_j)(\cdot)\bigr|\chi_k(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim \sum_{j=-\infty}^{k-2}2^{-kn}2^{(k-j)(\alpha_++\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\bigl\|2^{j\alpha(\cdot)}f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)}\bigl\|\chi_{B_j}(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\bigl\|\chi_{B_k}(\cdot)\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim \sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha_++\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\bigl\|2^{j\alpha(\cdot)}f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \\
&\quad\lesssim \sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\alpha_+-\nu-n/s)}\bigl\|2^{j\alpha(\cdot)}f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.4}$$

For convenience below we put $\sigma = n\delta_2 - \alpha_+ - \nu - n/s$. It follows from the condition (3.1) that $\sigma > 0$. Now we can distinguish two cases as follows:

Case 1°: If $0 < q \leq 1$, using the well-known inequality

$$\left(\sum_{j=1}^{\infty}a_j\right)^q \leq \sum_{j=1}^{\infty}a_j^q \quad (a_j > 0, j = 1, 2, \ldots),\tag{3.5}$$

we have

$$\begin{aligned}
U_1 &\lesssim \sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-2}2^{(j-k)\sigma q}\bigl\|2^{j\alpha(\cdot)}f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\
&\lesssim \sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L-2}\bigl\|2^{j\alpha(\cdot)}f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\sum_{k=j+2}^{L}2^{(j-k)\sigma q} \\
&\lesssim \sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L-2}\bigl\|2^{j\alpha(\cdot)}f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\
&\lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q.
\end{aligned}$$

Case 2°: If $1 < q < \infty$, by Hölder's inequality, we have

$$\begin{aligned}
U_1 &\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)\sigma q/2} \left\| 2^{j\alpha(\cdot)} f_j \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right) \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)\sigma q'/2} \right)^{q/q'} \\
&\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)\sigma q/2} \left\| 2^{j\alpha(\cdot)} f_j \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right) \\
&\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L-2} \left\| 2^{j\alpha(\cdot)} f_j \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=j+2}^{L} 2^{(j-k)\sigma q/2} \\
&\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L-2} \left\| 2^{j\alpha(\cdot)} f_j \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\
&\lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q.
\end{aligned}$$

Next we estimate U_2. By Proposition 2.4 and the hypothesis T_Ω is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, so that

$$\begin{aligned}
U_2 &\approx \max\left\{ \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left\| 2^{k\alpha(0)} \left(\sum_{j=k-1}^{k+1} \left| T_\Omega(f_j)(\cdot) \right| \right) \chi_k(\cdot) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q, \right. \\
&\quad \sup_{L\geq 0,L\in\mathbb{Z}} \left[2^{-L\lambda q} \sum_{k=-\infty}^{-1} \left\| 2^{k\alpha(0)} \left(\sum_{j=k-1}^{k+1} \left| T_\Omega(f_j)(\cdot) \right| \right) \chi_k(\cdot) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right. \\
&\quad \left.\left. + 2^{-L\lambda q} \sum_{k=0}^{L} \left\| 2^{k\alpha_\infty} \left(\sum_{j=k-1}^{k+1} \left| T_\Omega(f_j)(\cdot) \right| \right) \chi_k(\cdot) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right] \right\} \\
&\lesssim \max\left\{ \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left\| 2^{k\alpha(0)} |f\chi_k| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q, \right. \\
&\quad \left. \sup_{L\geq 0,L\in\mathbb{Z}} \left[2^{-L\lambda q} \sum_{k=-\infty}^{-1} \left\| 2^{k\alpha(0)} |f\chi_k| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q + 2^{-L\lambda q} \sum_{k=0}^{L} \left\| 2^{k\alpha_\infty} |f\chi_k| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right] \right\} \\
&\lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q.
\end{aligned}$$

For U_3, once again by Proposition 2.4, we have

$$\begin{aligned}
U_3 &\approx \max\left\{ \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left\| 2^{k\alpha(0)} \left(\sum_{j=k+2}^{\infty} \left| T_\Omega(f_j)(\cdot) \right| \right) \chi_k(\cdot) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q, \right. \\
&\quad \sup_{L\geq 0,L\in\mathbb{Z}} \left[2^{-L\lambda q} \sum_{k=-\infty}^{-1} \left\| 2^{k\alpha(0)} \left(\sum_{j=k+2}^{\infty} \left| T_\Omega(f_j)(\cdot) \right| \right) \chi_k(\cdot) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right. \\
&\quad \left.\left. + 2^{-L\lambda q} \sum_{k=0}^{L} \left\| 2^{k\alpha_\infty} \left(\sum_{j=k+2}^{\infty} \left| T_\Omega(f_j)(\cdot) \right| \right) \chi_k(\cdot) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right] \right\} \\
&=: \max\{I, J\}. \qquad (3.6)
\end{aligned}$$

To estimate I, observe that if $x \in R_k$, $y \in R_j$ and $j \geq k+2$, then $|x-y| \approx |y| \approx 2^j$. We apply Lemma 2.5 and obtain

$$\begin{aligned} \left|T_\Omega(f_j)(x)\right| &\lesssim 2^{-jn} \int_{R_j} \left|\Omega(x-y)\right| \left|f_j(y)\right| dy \\ &\lesssim 2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.7}$$

An application of (3.7), (3.3), and Lemma 2.9 gives

$$\begin{aligned} &\left\| \sum_{j=k+2}^{\infty} \left|T_\Omega(f_j)(\cdot)\right| \chi_k(\cdot) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\quad\lesssim \sum_{j=k+2}^{\infty} 2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left\|\Omega(x-\cdot)\chi_j(\cdot)\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \left\|\chi_{B_k}(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\quad\lesssim \sum_{j=k+2}^{\infty} 2^{(k-j)(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ &\quad\lesssim \sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_1+\nu+n/s)} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.8}$$

In the sequel, we put $\eta = n\delta_1 + \nu + n/s$ for short. If $0 < q \leq 1$, in view of $\eta > 0$, from (3.6), (3.8), and (3.5), we conclude that

$$\begin{aligned} I &\lesssim \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \sum_{j=k+2}^{\infty} 2^{(k-j)\eta q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ &\lesssim \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \sum_{j=k+2}^{L-1} 2^{(k-j)\eta q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ &\quad + \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \sum_{j=L}^{\infty} 2^{(k-j)\eta q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q =: I_1 + I_2. \end{aligned}$$

For I_1, noting that $\eta + \alpha(0) > \eta + \alpha_- > 0$, hence we have

$$\begin{aligned} I_1 &\lesssim \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L-1} 2^{j\alpha(0)q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=-\infty}^{j-2} 2^{(k-j)(\eta+\alpha(0))q} \\ &\lesssim \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L-1} 2^{j\alpha(0)q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^q. \end{aligned}$$

For I_2, since $\lambda - \eta - \alpha(0) < \lambda - \eta - \alpha_- < 0$, we get

$$\begin{aligned} I_2 &\lesssim \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \sum_{j=L}^{\infty} 2^{(k-j)\eta q} 2^{-j\alpha(0)q} 2^{j\lambda q} \\ &\quad \times 2^{-j\lambda q} \sum_{l=-\infty}^{j} 2^{l\alpha(0)q} \|f_l\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q}\left(\sum_{k=-\infty}^{L} 2^{k(\alpha(0)+\eta)q}\right)\left(\sum_{j=L}^{\infty} 2^{j(\lambda-\eta-\alpha(0))q}\right)\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q}2^{L(\alpha(0)+\eta)q}2^{L(\lambda-\eta-\alpha(0))q}\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

If $1 < q < \infty$, we have

$$
\begin{aligned}
I &\lesssim \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left(\sum_{j=k+2}^{L} 2^{(k-j)\eta}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q \\
&\quad + \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left(\sum_{j=L+1}^{\infty} 2^{(k-j)\eta}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q \\
&=: I_a + I_b.
\end{aligned}
$$

For I_a, by Hölder's inequality, we get

$$
\begin{aligned}
I_a &\lesssim \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L}\sum_{j=k+2}^{L} 2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(\eta+\alpha(0))q/2} \\
&\quad \times \left(\sum_{j=k+2}^{L} 2^{(k-j)(\eta+\alpha(0))q'/2}\right)^{q/q'} \\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} 2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j-2} 2^{(k-j)(\eta+\alpha(0))q/2} \\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} 2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

For I_b, as argued in I_2, we obtain

$$
\begin{aligned}
I_b &\lesssim \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L}\left(\sum_{j=L+1}^{\infty} 2^{j\alpha(0)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(\eta+\alpha(0)+\lambda)/2}2^{(k-j)(\eta+\alpha(0)-\lambda)/2}\right)^q \\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L}\sum_{j=L+1}^{\infty} 2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(\eta+\alpha(0)+\lambda)q/2} \\
&\quad \times \left(\sum_{j=L+1}^{\infty} 2^{(k-j)(\eta+\alpha(0)-\lambda)q'/2}\right)^{q/q'} \\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L}\sum_{j=L+1}^{\infty} 2^{(k-j)(\eta+\alpha(0)+\lambda)q/2}2^{j\lambda q}
\end{aligned}
$$

$$\times 2^{-j\lambda q}\sum_{l=-\infty}^{j}2^{l\alpha(0)q}\|f_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\lambda q}\sum_{j=L+1}^{\infty}2^{(k-j)(\eta+\alpha(0)-\lambda)q/2}\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

Hence, we arrive at the desired inequality,

$$I\lesssim I_1+I_2\lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

We omit the estimation of J since it is essentially similar to that of I. Consequently, the proof of Theorem 3.1 is complete. □

Proof of Theorem 3.2 Let $f\in M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. We decompose

$$f(x)=\sum_{j=-\infty}^{\infty}f(x)\chi_j(x)=\sum_{j=-\infty}^{\infty}f_j(x).$$

Then we have

$$\begin{aligned}\big\|[b,T_\Omega](f)\big\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} &= \sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\big\|2^{k\alpha(\cdot)}[b,T_\Omega](f)\chi_k\big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\lesssim \sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\left(\sum_{j=-\infty}^{k-2}\big|[b,T_\Omega](f_j)(\cdot)\big|\right)\chi_k(\cdot)\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad+\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\left(\sum_{j=k-1}^{k+1}\big|[b,T_\Omega](f_j)(\cdot)\big|\right)\chi_k(\cdot)\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad+\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\left(\sum_{j=k+2}^{\infty}\big|[b,T_\Omega](f_j)(\cdot)\big|\right)\chi_k(\cdot)\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&=: V_1+V_2+V_3.\end{aligned}$$

First we estimate V_1. Noting that $|x-y|\approx|x|\approx 2^k$ for $x\in R_k, y\in R_j$ and $j\le k-2$, then, from Lemma 2.11 and Lemma 2.5, it follows that

$$\begin{aligned}&2^{k\alpha(x)}\sum_{j=-\infty}^{k-2}\big|[b,T_\Omega](f_j)(x)\big|\chi_k(x)\\
&\lesssim\sum_{j=-\infty}^{k-2}2^{-kn}\int_{R_j}2^{k\alpha(x)}\big|b(x)-b(y)\big|\big|\Omega(x-y)\big|\big|f_j(y)\big|\,dy\cdot\chi_k(x)\\
&\lesssim\sum_{j=-\infty}^{k-2}2^{-kn}2^{(k-j)\alpha_+}\int_{R_j}2^{j\alpha(y)}\big|b(x)-b(y)\big|\big|\Omega(x-y)\big|\big|f_j(y)\big|\,dy\cdot\chi_k(x)\end{aligned}$$

$$\lesssim \sum_{j=-\infty}^{k-2} 2^{-kn} 2^{(k-j)\alpha_+} \biggl(\bigl|b(x) - b_{B_j}\bigr| \int_{R_j} 2^{j\alpha(y)} \bigl|\Omega(x-y)\bigr| \bigl|f_j(y)\bigr|\,dy$$
$$+ \int_{R_j} 2^{j\alpha(y)} \bigl|b_{B_j} - b(y)\bigr| \bigl|\Omega(x-y)\bigr| \bigl|f_j(y)\bigr|\,dy \biggr). \tag{3.9}$$

Similar to (3.3), we define a variable exponent $\widetilde{p}(\cdot)$ by $\frac{1}{p'(x)} = \frac{1}{\widetilde{p}'(x)} + \frac{1}{s}$, since $s > (p')_+$, by Lemma 2.7, Lemma 2.6, and Lemma 2.10, we get

$$\begin{aligned}
&\bigl\| \bigl(b_{B_j} - b(\cdot)\bigr)\Omega(x-\cdot)\chi_j(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim \bigl\|\Omega(x-\cdot)\chi_j(\cdot)\bigr\|_{L^s(\mathbb{R}^n)} \bigl\| \bigl(b_{B_j} - b(\cdot)\bigr)\chi_j(\cdot)\bigr\|_{L^{\widetilde{p}'(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim \|b\|_{BMO} \bigl\|\chi_{B_j}(\cdot)\bigr\|_{L^{\widetilde{p}'(\cdot)}(\mathbb{R}^n)} \bigl\|\Omega(x-\cdot)\chi_j(\cdot)\bigr\|_{L^s(\mathbb{R}^n)} \\
&\quad\lesssim \|b\|_{BMO} 2^{(k-j)(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \bigl\|\chi_{B_j}(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim 2^{(k-j)(\nu+n/s)} \bigl\|\chi_{B_j}(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.10}$$

Then by (3.9), Lemma 2.5, (3.3), (3.10), Lemma 2.10, and Lemma 2.9, we obtain

$$\begin{aligned}
&\Biggl\| 2^{k\alpha(\cdot)} \sum_{j=-\infty}^{k-2} \bigl|[b, T_\Omega](f_j)(\cdot)\bigr| \chi_k(\cdot) \Biggr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim \sum_{j=-\infty}^{k-2} 2^{-kn} 2^{(k-j)\alpha_+} \bigl\|2^{j\alpha(\cdot)} f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \bigl(\bigl\| \bigl(b_{B_j} - b(\cdot)\bigr)\chi_k(\cdot)\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \bigl\|\Omega(x-\cdot)\chi_j(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\qquad + \bigl\| \bigl(b_{B_j} - b(\cdot)\bigr)\Omega(x-\cdot)\chi_j(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \bigl\|\chi_k(\cdot)\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \bigr) \\
&\quad\lesssim \sum_{j=-\infty}^{k-2} 2^{-kn} 2^{(k-j)\alpha_+} \bigl\|2^{j\alpha(\cdot)} f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\qquad \times \bigl((k-j)\|b\|_{BMO} 2^{(k-j)(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \bigl\|\chi_{B_j}(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \bigl\|\chi_{B_k}(\cdot)\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\qquad + \|b\|_{BMO} 2^{(k-j)(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \bigl\|\chi_{B_j}(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \bigl\|\chi_{B_k}(\cdot)\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \bigr) \\
&\quad\lesssim \sum_{j=-\infty}^{k-2} (k-j) 2^{-kn} 2^{(k-j)\alpha_+} \bigl\|2^{j\alpha(\cdot)} f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(\nu+n/s)} \bigl\|\chi_{B_j}(\cdot)\bigr\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \bigl\|\chi_{B_k}(\cdot)\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\lesssim \sum_{j=-\infty}^{k-2} (k-j) 2^{(k-j)(\alpha_+ + \nu + n/s)} \bigl\|2^{j\alpha(\cdot)} f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \\
&\quad\lesssim \sum_{j=-\infty}^{k-2} (k-j) 2^{(j-k)(n\delta_2 - \alpha_+ - \nu - n/s)} \bigl\|2^{j\alpha(\cdot)} f_j\bigr\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.11}$$

By comparing (3.11) with (3.4), after applying the same arguments used in the estimation of U_1 in Theorem 3.1, we can easily get, for all $0 < q < \infty$,

$$V_1 \lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

Next we estimate V_2. Using Proposition 2.4 and the boundedness of $[b, T_\Omega]$ on $L^{p(\cdot)}(\mathbb{R}^n)$, we derive the estimate

$$\begin{aligned}
V_2 &\approx \max\Bigg\{\sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\bigg\|2^{k\alpha(0)}\bigg(\sum_{j=k-1}^{k+1}\big|[b,T_\Omega](f_j)(\cdot)\big|\bigg)\chi_k(\cdot)\bigg\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
&\quad \sup_{L\geq 0,L\in\mathbb{Z}}\Bigg[2^{-L\lambda q}\sum_{k=-\infty}^{-1}\bigg\|2^{k\alpha(0)}\bigg(\sum_{j=k-1}^{k+1}\big|[b,T_\Omega](f_j)(\cdot)\big|\bigg)\chi_k(\cdot)\bigg\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + 2^{-L\lambda q}\sum_{k=0}^{L}\bigg\|2^{k\alpha_\infty}\bigg(\sum_{j=k-1}^{k+1}\big|[b,T_\Omega](f_j)(\cdot)\big|\bigg)\chi_k(\cdot)\bigg\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg]\Bigg\} \\
&\lesssim \max\Bigg\{\sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\big\|2^{k\alpha(0)}|f\chi_k|\big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
&\quad \sup_{L\geq 0,L\in\mathbb{Z}}\Bigg[2^{-L\lambda q}\sum_{k=-\infty}^{-1}\big\|2^{k\alpha(0)}|f\chi_k|\big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} + 2^{-L\lambda q}\sum_{k=0}^{L}\big\|2^{k\alpha_\infty}|f\chi_k|\big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg]\Bigg\} \\
&\lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For V_3, we apply Proposition 2.4 again and obtain

$$\begin{aligned}
V_3 &\approx \max\Bigg\{\sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\bigg\|2^{k\alpha(0)}\bigg(\sum_{j=k+2}^{\infty}\big|[b,T_\Omega](f_j)(\cdot)\big|\bigg)\chi_k(\cdot)\bigg\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
&\quad \sup_{L\geq 0,L\in\mathbb{Z}}\Bigg[2^{-L\lambda q}\sum_{k=-\infty}^{-1}\bigg\|2^{k\alpha(0)}\bigg(\sum_{j=k+2}^{\infty}\big|[b,T_\Omega](f_j)(\cdot)\big|\bigg)\chi_k(\cdot)\bigg\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + 2^{-L\lambda q}\sum_{k=0}^{L}\bigg\|2^{k\alpha_\infty}\bigg(\sum_{j=k+2}^{\infty}\big|[b,T_\Omega](f_j)(\cdot)\big|\bigg)\chi_k(\cdot)\bigg\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg]\Bigg\} \\
&=: \max\{G,H\}.
\end{aligned} \tag{3.12}$$

To estimate G, we note that if $x \in R_k, y \in R_j$ and $j \geq k+2$, then $|x-y| \approx |y| \approx 2^j$. By Lemma 2.5, we have

$$\begin{aligned}
\big|[b,T_\Omega](f_j)(x)\big| &\lesssim 2^{-jn}\int_{R_j}\big|b(x)-b(y)\big|\big|\Omega(x-y)\big|\big|f_j(y)\big|\,dy \\
&\lesssim 2^{-jn}\bigg(\big|b(x)-b_{B_k}\big|\int_{R_j}\big|\Omega(x-y)\big|\big|f_j(y)\big|\,dy \\
&\quad + \int_{R_j}\big|b(y)-b_{B_k}\big|\big|\Omega(x-y)\big|\big|f_j(y)\big|\,dy\bigg) \\
&\lesssim 2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\big(\big|b(x)-b_{B_k}\big|\big\|\Omega(x-\cdot)\chi_j(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad + \big\|\Omega(x-\cdot)\big(b(\cdot)-b_{B_k}\big)\chi_j(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\big).
\end{aligned} \tag{3.13}$$

As argued for (3.10), we get actually

$$\begin{aligned}
&\big\|\Omega(x-\cdot)\big(b(\cdot)-b_{B_k}\big)\chi_j(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\quad\lesssim \big\|\Omega(x-\cdot)\chi_j(\cdot)\big\|_{L^s(\mathbb{R}^n)}\big\|\big(b(\cdot)-b_{B_k}\big)\chi_j(\cdot)\big\|_{L^{\widetilde{p}'(\cdot)}(\mathbb{R}^n)}\\
&\quad\lesssim (j-k)\|b\|_{BMO}\big\|\chi_{B_j}(\cdot)\big\|_{L^{\widetilde{p}'(\cdot)}(\mathbb{R}^n)}\big\|\Omega(x-\cdot)\chi_j(\cdot)\big\|_{L^s(\mathbb{R}^n)}\\
&\quad\lesssim (j-k)\|b\|_{BMO}2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\big\|\chi_{B_j}(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}\\
&\quad\lesssim (j-k)2^{(k-j)(\nu+n/s)}\big\|\chi_{B_j}(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.14}$$

From (3.13) and (3.14), it follows that

$$\begin{aligned}
&\big\|[b,T_\Omega](f_j)(\cdot)\chi_k(\cdot)\big\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\lesssim 2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big(2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\big\|\chi_{B_j}(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\big\|\big(b(\cdot)-b_{B_k}\big)\chi_k(\cdot)\big\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad+(j-k)\|b\|_{BMO}2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\big\|\chi_{B_j}(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\big\|\chi_{B_k}(\cdot)\big\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\
&\quad\lesssim 2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big(2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\big\|\chi_{B_j}(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|b\|_{BMO}\big\|\chi_{B_k}(\cdot)\big\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad+(j-k)\|b\|_{BMO}2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(S^{n-1})}\big\|\chi_{B_j}(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\big\|\chi_{B_k}(\cdot)\big\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\
&\quad\lesssim (j-k)\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}2^{-jn}2^{(k-j)(\nu+n/s)}\big\|\chi_{B_j}(\cdot)\big\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\big\|\chi_{B_j}(\cdot)\big\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\\
&\quad\lesssim (j-k)2^{(k-j)(n\delta_1+\nu+n/s)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.15}$$

Hence we have

$$\Bigg\|\sum_{j=k+2}^{\infty}\big|[b,T_\Omega](f_j)(\cdot)\big|\chi_k(\cdot)\Bigg\|_{L^{p(\cdot)}(\mathbb{R}^n)}\lesssim\sum_{j=k+2}^{\infty}(j-k)2^{(k-j)(n\delta_1+\nu+n/s)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.16}$$

By comparing (3.16) with (3.8), as long as we repeat the same procedure as the estimation of I in Theorem 3.1, we can immediately get for all $0<q<\infty$,

$$G\lesssim\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

We omit the estimation of H since it is essentially similar to that of G. Consequently, the proof of Theorem 3.2 is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors worked jointly in drafting and approved the final manuscript.

Author details
[1]Department of Applied Mathematics, School of Mathematics and Physics, Anhui Polytechnic University, Wuhu, 241000, The People's Republic of China. [2]Department of Applied Mathematics, School of Mathematics and Computer Science, Anhui Normal University, Wuhu, 241003, The People's Republic of China.

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