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Index of a bivariate mean and applications

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Abstract

Exploring some results of (Raïssouli in J. Math. Inequal. 10(1):83-99, 2016) from another point of view, we introduce here some power-operations for (bivariate) means. As application, we construct some classes of means in one or two parameters including some standard means. We also define a law between means which allows us to obtain, among others, a simple relationship involving the three familiar means, namely the first Seiffert mean, the second Seiffert mean, and the Neuman-Sándor mean. At the end, more examples of interest are discussed and open problems are derived as well.

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1 Introduction

By a (bivariate) mean we understand a map m between positive real numbers satisfying the following double inequality:

$$\forall a,b>0 \quad \min(a,b)\le m(a,b)\le \max(a,b).$$

As usual, continuous (resp. symmetric/homogeneous) means are defined in the habitual way. The standard examples of such means are given in the following:

$$A(a,b)=\frac{a+b}{2},\qquad G(a,b)=\sqrt{ab},\qquad H(a,b)=\frac{2ab}{a+b},\qquad C(a,b)=\frac{a^2+b^2}{a+b},$$
$$L(a,b)=\frac{b-a}{\ln b-\ln a},\qquad P(a,b)=\frac{b-a}{4\arctan\sqrt{b/a}-\pi}=\frac{b-a}{2\arcsin\frac{b-a}{b+a}},$$
$$T(a,b)=\frac{b-a}{2\arctan(b/a)-\pi/2}=\frac{b-a}{2\arctan\frac{b-a}{b+a}},\qquad M(a,b)=\frac{b-a}{2\operatorname{arcsinh}\frac{b-a}{b+a}},$$

with $L(a,a)=P(a,a)=T(a,a)=M(a,a)=a$, and they are known as the arithmetic mean, geometric mean, harmonic mean, contra-harmonic mean, logarithmic mean, first Seiffert mean [2], second Seiffert mean [3], and Neuman-Sándor mean [4], respectively. Other examples of means (not needed here) can be found in the literature; see [5] for instance and the references cited therein. As usual, we identify a mean m with its value at (a,b) by setting $m:=m(a,b)$ for the sake of simplicity. We write $m_1<m_2$ for meaning that $m_1(a,b)<m_2(a,b)$ for all $a,b>0$ with $a\ne b$. The notation m^* refers to the dual mean of

m defined by $m^*(a,b) = (m(a^{-1},b^{-1}))^{-1}$ for all $a,b > 0$. As is well known, if m is symmetric and homogeneous then $m^*(a,b) = ab/m(a,b)$, which we briefly write $m^* = G^2/m$.

In [1], the following result has been established (see Corollary 2.2).

Theorem A *Let m be a continuous homogeneous symmetric mean. Then the binary map m^σ, defined by $m^\sigma(a,a) = a$ and*

$$\bigl(m^\sigma(a,b)\bigr)^{-1} = \frac{1}{b-a}\int_1^{b/a} m\left(1,\frac{1}{t^2}\right)dt \tag{1.1}$$

for all $a,b > 0$ with $a \neq b$, is a continuous homogeneous symmetric mean (called the integral mean-transform of m).

For example, we have $A^\sigma = H$, $G^\sigma = L$ and $H^\sigma = T$ (see [1] for more details and examples). A continuous homogeneous symmetric mean will be called regular mean, for the sake of simplicity. A regular mean m will be called σ-regular if the map $x \longmapsto m(x,1)$ is continuously differentiable on $(0,\infty)$ and the function f_m (called the generated function of m) defined by

$$f_m(x) = \frac{d}{dx}\left(\frac{x-1}{m(x,1)}\right)$$

for all $x > 0$, with $f_m(1) = 1$, satisfies the double inequality

$$\min\bigl(1,1/x^2\bigr) \le f_m(x) \le \max\bigl(1,1/x^2\bigr)$$

for all $x > 0$. All the previous means are regular and σ-regular, except C, which is not σ-regular; see [1] (Examples 3.1, 3.2, 3.3).

With this, the following result has been proved in [1] (see Theorem 3.2).

Theorem B *Let m be a σ-regular mean with its generated function f_m. Then the binary map r_m, defined by*

$$r_m(a,b) = bf_m(\sqrt{b/a})$$

for all $a,b > 0$, is a regular mean with $r_m^\sigma = m$.

If we denote by $\mathcal{M}_r$ and $\mathcal{M}_\sigma$ the sets of all regular means and σ-regular means, respectively, then the mean-map $m \longmapsto m^\sigma$ is a bijection from $\mathcal{M}_r$ into $\mathcal{M}_\sigma$ and we can write

$$\bigl(r_m^\sigma = m, r_m \in \mathcal{M}_r\bigr) \quad\Longleftrightarrow\quad \bigl(r_m = m^{-\sigma}, m \in \mathcal{M}_\sigma\bigr). \tag{1.2}$$

For example, we have $H^{-\sigma} = A$, $L^{-\sigma} = G$ and $T^{-\sigma} = H$. See [1] for more examples and detail.

It is easy to see that the map $(a,b) \longmapsto z := A/G(a,b)$ is surjective from $(0,\infty)\times(0,\infty)$ into $[1,\infty)$, with $z = 1$ if and only if $a = b$. For $m \in \mathcal{M}_\sigma$, we put

$$F_m(z) := \frac{m^{-\sigma}}{G}, \quad \text{with } z = A/G \ge 1,$$

which we call the regularized function of m. It is easy to see that, if $m_1, m_2 \in \mathcal{M}_\sigma$ are such that $F_{m_1} = F_{m_2}$ then $m_1 = m_2$. Further, it is proved in [1] that

$$\big(\forall z > 1\ F_{m_1}(z) > F_{m_2}(z)\big) \quad \Longrightarrow \quad m_1 < m_2. \tag{1.3}$$

The following result has also been proved there (see Corollary 3.4).

Theorem C *The following relationships are met*:

$$F_L(z) = 1, \qquad F_H(z) = z, \qquad F_T(z) = \frac{1}{z}, \qquad F_A(z) = \frac{2}{z+1},$$
$$F_G(z) = \sqrt{\frac{z+1}{2}}, \qquad F_P(z) = \sqrt{\frac{2}{z+1}}, \qquad F_M(z) = \sqrt{\frac{2}{z^2+z}}.$$

By a simple observation, it is easy to see that, for all $z > 1$,

$$F_T(z) < F_M(z) < F_A(z) < F_P(z) < F_L(z) = 1 < F_G(z) < F_H(z). \tag{1.4}$$

This, with (1.3), immediately implies (simultaneously and in a fast way) the following well-known chain of mean-inequalities:

$$H < G < L < P < A < M < T.$$

See [1] for more detail.

The remainder of this paper will be organized as follows: after this introduction, Section 2 is devoted to a list of lemmas that will be needed throughout the paper. In Section 3 we define $m^{(q)}$ the mean-power of a mean m of order q, with $|q| \le 1$. As examples, we obtain $P = G^{(-1)}$, $T = H^{(-1)}$, and $P = A^{(1/2)}$. This allows us to construct, in Section 4, a family of means involving one parameter. Section 5 displays the definition of a new concept, so-called index of a mean, in the aim to define $m^{(q)}$ when $|q| > 1$. We obtain, among others, $P^{(2)} = A = G^{(-2)}$ and $L^{(q)} = L$ for each q real number. In Section 6, we introduce a law $\odot$ between means and we study its properties. As a first application, we obtain a simple relationship involving the three means P, M, and T, namely $M = T^{(1/2)} \odot P$. Further applications are discussed in Section 7 where we construct some families of means involving two parameters and including all the previous means. Finally, Section 8 is focused on giving more examples of applications as well as deriving open problems for future research.

2 Some lemmas needed

As already pointed out, we state here some lemmas that will be needed in the sequel. First, we mention that every homogeneous mean m can be written in the form $m = Gg(A/G)$ for some function g, since a and b can be both expressed in terms of A and G. The following lemma explains this situation.

Lemma 2.1 *Let* $m \in \mathcal{M}_\sigma$ *be given and we set*

$$F_m(z) = \frac{m^{-\sigma}}{G}, \quad \textit{with } z = A/G \ge 1.$$

Then we have

$$\forall z \geq 1 \quad l(z) := z - \sqrt{z^2-1} \leq F_m(z) \leq z + \sqrt{z^2-1} := u(z). \tag{2.1}$$

Proof Since $m^{-\sigma}$ is a mean

$$\forall a, b > 0 \quad \min(a,b) \leq m^{-\sigma}(a,b) \leq \max(a,b).$$

This, with $G(a,b) = \sqrt{ab}$ and the fact that $m^{-\sigma}$ is homogeneous, yields

$$\forall a, b > 0 \quad \min(\sqrt{a/b}, \sqrt{b/a}) \leq F_m(z) \leq \max(\sqrt{a/b}, \sqrt{b/a}),$$

where, with $A := (a+b)/2$, we put

$$z := A/G = (1/2)\sqrt{a/b} + (1/2)\sqrt{b/a}.$$

Setting $t = \sqrt{a/b}$ we obtain $t^2 - 2zt + 1 = 0$, which gives $t = z - \sqrt{z^2-1}$ or $t = z + \sqrt{z^2-1}$. The remainder of the proof is straightforward and therefore omitted here. □

We notice that $F_{\min}(z) = u(z)$ and $F_{\max}(z) = l(z)$, where min and max denote the trivial means $(a,b) \longmapsto \min(a,b)$ and $(a,b) \longmapsto \max(a,b)$, respectively.

A function g, defined from $[1,\infty)$ into $(0,\infty)$ and satisfying $l(z) \leq g(z) \leq u(z)$ for all $z \geq 1$, will be called here an admissible function. The functions l and u are the lower and upper admissible functions, respectively. To obtain other examples of admissible functions g, it is sufficient to take $g(z) = \eta(l(z), u(z))$ for some mean η (symmetric, homogeneous, or not). The following example explains the latter situation.

Example 2.2 Simple computation (with Theorem C) leads to:

(i) $g(z) := A(l(z), u(z)) = z = F_H(z)$.

(ii) $g(z) := G(l(z), u(z)) = 1 = F_L(z)$.

(iii) $g(z) := H(l(z), u(z)) = 1/z = F_T(z)$.

Inversely, let g be an admissible function, does $m \in \mathcal{M}_\sigma$ exist such that $g(z) = m^{-\sigma}/G$ whenever $z = A/G$? The following result answers affirmatively the latter question.

Lemma 2.3 *Let g be a continuous admissible function. Then there exists a unique $m \in \mathcal{M}_\sigma$ such that*

$$F_m(z) = g(z), \quad \textit{with } z = A/G.$$

Such a mean m is given by $m^{-\sigma} = Gg(A/G)$, or explicitly, for all $a, b > 0$, $a \neq b$,

$$\big(m(a,b)\big)^{-1} = \frac{1}{b-a}\int_1^{b/a} \frac{1}{t} g\left(\frac{t^2+1}{2t}\right) dt.$$

Proof If g is as assumed, it is easy to see that $Gg(A/G)$ is a regular mean. Detail is simple and therefore omitted here. □

Remark that, for all $z \geq 1$, we have $0 < l(z) \leq 1 \leq u(z)$ and $l(z) = 1/u(z)$. This implies that if g is admissible then its point-wise inverse g^{-1} *i.e.* $g^{-1}(z) = 1/g(z)$, is also admissible. More generally, let q be a real number and define the point-wise q-power of g by $g^q(z) := (g(z))^q$, with $g^0(z) = 1$, for all $z \geq 1$. If g is admissible, it is easy to see that g^2 is in general not admissible. A function g for which g^q is admissible, for some real number q, will be called q-admissible. It is immediate that, if g is q-admissible then so is $g^{-1} := 1/g$. With this, the following result may be stated.

Lemma 2.4 *Let g be an admissible function. Then g is q-admissible whenever $|q| \leq 1$.*

Proof Let g be admissible *i.e.*

$$\forall z \geq 1 \quad 0 < l(z) \leq g(z) \leq 1 \leq u(z).$$

Assume that $0 \leq q \leq 1$. It is easy to see that

$$\forall z \geq 1 \quad 0 < l(z) \leq \bigl(l(z)\bigr)^q \leq \bigl(g(z)\bigr)^q \leq 1 \leq \bigl(u(z)\bigr)^q \leq u(z),$$

which implies that g^q is admissible. Now, if $-1 \leq q \leq 0$ we write $g^q = 1/g^{-q}$, with $0 \leq -q \leq 1$. We then deduce that g^q is admissible, since g^{-q} is admissible as well. The proof is completed. □

The following lemma will also be needed in some situations below.

Lemma 2.5 *Let g be an admissible function and $q \geq 1$. Assume that $g(z) \leq 1$ (resp. $g(z) \geq 1$) for all $z \geq 1$. If g is q-admissible then g is s-admissible whenever $1 \leq s \leq q$.*

Proof If $g(z) \leq 1$ for all $z \geq 1$ and g is q-admissible, we have for $1 \leq s \leq q$,

$$l(z) \leq \bigl(g(z)\bigr)^q \leq \bigl(g(z)\bigr)^s \leq g(z) \leq u(z).$$

If $g(z) \geq 1$ then $g^{-1}(z) := 1/g(z) \leq 1$. We apply the above for g^{-1}. The proof is complete. □

We end this section by stating another needed lemma recited in the following.

Lemma 2.6 *Let $h(z) = j(z)/k(z) > 0$ for all $z \geq 1$, where j and k are two polynomial functions with $\deg(j) = c$ and $\deg(k) = d$. Let q be a real number. If the inequalities*

$$l(z) \leq \bigl(h(z)\bigr)^q \leq u(z)$$

hold for all $z \geq 1$ then $q|c - d| \leq 1$. In particular, $q \leq 1$ provided $c \neq d$.

Proof Since $l(z) = 1/u(z)$ we can, without loss the generality, assume that $c \geq d$. It is easy to see that

$$\bigl(h(z)\bigr)^q \sim \alpha z^{q(c-d)} \quad \text{and} \quad u(z) \sim 2z, \quad \text{when } z \to \infty$$

for some constant $\alpha > 0$. If $q(c-d) > 1 = \deg(2z)$ then $(h(z))^q \leq u(z)$ does not hold for z enough large. We then deduce $q(c-d) \leq 1$. If $c \neq d$ then $q \leq 1/(c-d) \leq 1$, since c and d are both integers. The proof is complete. □

3 Mean-power and mean-iterate

In this section, we will observe the previous results from another point of view in the aim to interpret them in service of means.

Let g be a continuous admissible function. By Lemma 2.3, there exists a unique $m \in \mathcal{M}_\sigma$ such that

$$F_m(z) := \frac{m^{-\sigma}}{G} = g(z), \quad z = A/G.$$

Due to Lemma 2.4, with Lemma 2.3 again, for each $q \in [-1,1]$ there exists $r_q \in \mathcal{M}_\sigma$ such that

$$F_{r_q}(z) := \frac{r_q^{-\sigma}}{G} = g^q(z), \quad z = A/G.$$

We then deduce that

$$F_{r_q}(z) = \big(F_m(z)\big)^q := F_m^q(z), \quad z = A/G.$$

Summarizing, we then obtain the following result.

Proposition 3.1 *Let $m \in \mathcal{M}_\sigma$ and $q \in [-1,1]$. Then there exists a unique $r_q \in \mathcal{M}_\sigma$ such that $F_{r_q}(z) = F_m^q(z)$, with $z = A/G$. Further, r_q is given by*

$$r_q = \left(G\left(\frac{m^{-\sigma}}{G}\right)^q\right)^\sigma, \quad \text{with } r_0 = G^\sigma = L. \tag{3.1}$$

We can then state the following definition.

Definition 3.2 Let $q \in [-1,1]$. The mean $r_q \in \mathcal{M}_\sigma$ defined by the previous proposition will be called the q-mean-power of m and we write $r_q = m^{(q)}$, with $m^{(0)} = L$. If $q = 1/n$, with $n \geq 2$ integer, $m^{(1/n)}$ will be called the n-mean-iterate of m. In particular, $m^{(1/2)}$ is the mean-root of m and $m^{(-1)}$ is the mean-inverse of m. Clearly, $m^{(1)} = m$.

Using (3.1), it is easy to see that $(m^{(q_1)})^{(q_2)} = m^{(q_1 q_2)}$ whenever $q_1, q_2 \in [-1,1]$. The following examples illustrate the previous concepts.

Example 3.3 According to Theorem C, with the previous definition, we immediately deduce the following statements.

(i) For each $q \in [-1,1]$, $L^{(q)} = L$.
(ii) $P = G^{(-1)}$ and $T = H^{(-1)}$.
(iii) $P = A^{(1/2)}$ and $G = A^{(-1/2)}$.

Example 3.4 By using (3.1), elementary computations lead to (with $|q| \leq 1$ and $a, b > 0$, $a \neq b$)

$$(\min)^{(q)}(a,b) = q\frac{b-a}{b^q - a^q}\max\big(a^q, b^q\big)$$

and

$$(\max)^{(q)}(a,b) = q\frac{b-a}{b^q-a^q}\min\bigl(a^q,b^q\bigr).$$

In particular, $(\min)^{(-1)} = \max$ and $(\max)^{(-1)} = \min$. This also follows from the fact that $F_{\min}(z) = u(z)$ and $F_{\max}(z) = l(z)$ with $l(z) = 1/u(z)$.

Other various examples of interest will be discussed throughout the following sections. Now, we state the following result summarizing the elementary properties of the mean-map $m \longmapsto m^{(q)}$ for $q \in (0,1)$.

Proposition 3.5

(i) *Let $m_1, m_2 \in \mathcal{M}_\sigma$ be such that $F_{m_1}(z) < F_{m_2}(z)$ for all $z > 1$. Then $m_1^{(q)} > m_2^{(q)}$ for each $q \in (0,1)$.*

(ii) *Let $m \in \mathcal{M}_\sigma$ be such that $F_m(z) < 1$ (resp. $F_m(z) > 1$) for all $z > 1$. Then we have $m^{(q_1)} < m^{(q_2)} < m$ (resp. $m < m^{(q_2)} < m^{(q_1)}$) whenever $0 \le q_1 < q_2 \le 1$.*

Proof It is straightforward. We therefore omit it for the reader. □

If $q, q_1, q_2 \in (-1,0)$ then the mean-inequalities in the previous proposition are reversed. Now, we will discuss the comparison between m and $m^{(q)}$ when m belongs to the set of the previous standard means A, G, H, L, P, T, and M. First, by (1.4) with Proposition 3.5, we immediately deduce

$$\begin{aligned}(\min)^{(q)} &< H^{(q)} < G^{(q)} < L^{(q)} = L < P^{(q)}\\ &< A^{(q)} < M^{(q)} < T^{(q)} < (\max)^{(q)}\end{aligned} \tag{3.2}$$

and

$$H < H^{(q)}, \qquad G < G^{(q)}, \qquad P^{(q)} < P, \qquad A^{(q)} < A, \qquad M^{(q)} < M, \qquad T^{(q)} < T. \tag{3.3}$$

Further, by virtue of the relation $A^{(1/2)} = P$, with Proposition 3.5(ii), we deduce

$$\text{if } 0 < q < 1/2, \text{then } A^{(q)} < P \quad \text{and} \quad \text{if } 1/2 < q < 1, \text{then } P < A^{(q)}. \tag{3.4}$$

Now, we state the following results which give more information as regards comparison between the previous means.

Proposition 3.6 *Let $q \in (0,1)$. Then we have*:

(i) *If $0 < q \le 1/4$ then $G < H^{(q)}$ and $T^{(q)} < P$.*

(ii) *If $1/2 \le q < 1$ then $H^{(q)} < G$.*

(iii) *If $0 < q < 2/3$ then $M^{(q)} < A$ and so $M^{(q/2)} < P$.*

Proof We use (1.4) and Theorem C. We have to compare $F_G(z) = \sqrt{(z+1)/2}$ and $F_{H^{(q)}}(z) := (F_H(z))^q = z^q$. Setting $\Phi(z) = 2z^{2q} - z - 1$, $z > 1$, and studying the monotonicity of Φ we deduce the desired result about G and $H^{(q)}$ in a simple way. Similar method for the other mean-inequalities. Detail is simple and therefore omitted here. □

The previous proposition when combined with (3.2), (3.3), and (3.4) yields the following corollary.

Corollary 3.7 *If* $0 < q \le 1/4$ *then we have*

$$\begin{aligned}(H <)\ G < H^{(q)} < G^{(q)} < L^{(q)} = L < P^{(q)} \\ < A^{(q)} < M^{(q)} < T^{(q)} < P\ (< A < M < T).\end{aligned} \tag{3.5}$$

In particular, for all integer $n \ge 4$ *we have*

$$\begin{aligned}(H <)\ G < H^{(1/n)} < G^{(1/n)} < L^{(1/n)} = L < P^{(1/n)} \\ < A^{(1/n)} < M^{(1/n)} < T^{(1/n)} < P\ (< A < M < T).\end{aligned} \tag{3.6}$$

Remark 3.8 For some values of q, like $q \in (1/4, 1/2)$, Proposition 3.6 does not give any information as regards comparison of G and $H^{(q)}$. In fact, $F_G(z)$ and $(F_H(z))^q$ are not comparable, because the related function Φ is not monotonic and satisfies $\Phi(1) = 0$, $\lim_{z\uparrow\infty} \Phi(z) = -\infty$. Of course, we cannot deduce any conclusion as for comparison of G and $H^{(q)}$, since (1.4) is just an implication. This is the reason why analogous of (3.5) when $q \notin (0, 1/4]$ cannot be stated in a similar manner as previous.

Now, let us go back to the relationships of Example 3.3. They tell us that the two means P and T can be written in a short and simple form involving the simplest means A, G and H. Some questions arise from this observation:

(a) What about the mean M? See Theorem 6.3, Section 6 below.
(b) What is the expression of $A^{(q)}$, for each $q \in [-1, 1]$, extending the relationship $A^{(1/2)} = P$? The answer to this question will be the aim of the next section. We also give the expression of $T^{(q)}$.
(c) State a reciprocal study of the previous one *i.e.* let $r \in \mathcal{M}_\sigma$ and $q \in [-1, 1]$. Does $m \in \mathcal{M}_\sigma$ exist such that $m^{(q)} = r$? This will be the purpose of Section 5 below.

4 On a family of 1-power means

As pointed before, this section will be devoted to an expression of $A^{(q)}$ for $q \in [-1, 1]$. Such an expression should coincide with that of P when we take $q = 1/2$, since $A^{(1/2)} = P$. Precisely, the following result may be stated.

Theorem 4.1 *Let* $q \in [-1, 1]$. *Then* $A^{(q)}$ *is given by*

$$A^{(q)}(a, b) = \frac{b - a}{4^q(\Theta_q(b/a) - \Theta_q(1))} \tag{4.1}$$

for all $a, b > 0$, $a \ne b$, *where for all* $x > 0$ *we set*

$$\Theta_q(x) = \int_0^x \frac{t^{q-1}}{(t+1)^{2q}}\, dt = \frac{1}{q} \int_0^{x^q} \frac{dt}{(t^{1/q} + 1)^{2q}}. \tag{4.2}$$

Further, $A^{(q)}$ *is strictly increasing* (*resp. decreasing*) *in* $q \in (0, 1)$ (*resp. in* $q \in (-1, 0)$).

Proof By (3.1), with Definition 3.2 and the fact that $A^{-\sigma} = 2G^2/(A+G)$ (see [1], Theorem 3.3) we have

$$A^{(q)} = \bigl(G(A^{-\sigma}/G)^q\bigr)^\sigma = \left(G\left(\frac{2G}{A+G}\right)^q\right)^\sigma.$$

Simple computation leads to (for all $t > 0$)

$$G\bigl(1,1/t^2\bigr) = 1/t \quad \text{and} \quad \frac{2G}{A+G}\bigl(1,1/t^2\bigr) = \frac{4t}{(t+1)^2}.$$

These, when substituting in (1.1), with $m := G(2G/(A+G))^q$, yield (for all $a, b > 0$, $a \neq b$)

$$\bigl(A^{(q)}(a,b)\bigr)^{-1} = \frac{4^q}{b-a}\int_1^{b/a} \frac{t^{q-1}}{(t+1)^{2q}}\,dt.$$

We then deduce (4.1) after simple manipulations and some elementary topics of integration. Detail is simple and omitted here. Now, by Proposition 3.5(ii) and the fact that $F_A(z) < 1$, for all $z > 1$, we deduce that $A^{(q)}$ is strictly increasing in $q \in (0,1)$. The case $q \in (-1,0)$ is similar. The proof is complete. □

The above theorem immediately gives (again) $A^{(1)} = A$ and $A^{(1/2)} = P$. Taking $q = 1/n$, with $n \geq 1$ integer, in the previous theorem we obtain a sequence of regular means. Such sequence satisfies interesting properties as summarized in the following result.

Proposition 4.2 *Let $P_n := A^{(1/n)}$ for each integer $n \geq 1$. Then the mean-sequence (P_n) is strictly point-wisely decreasing. Further, (P_n) converges point-wisely to L, the logarithmic mean i.e.*

$$L = P_\infty < \cdots < P_4 < P_3 < P_2 = P < P_1 = A.$$

Proof The fact that (P_n) is strictly point-wisely decreasing follows from Proposition 3.5. This, with the fact that P_n is a mean *i.e.* $\min(a,b) \leq P_n(a,b) \leq \max(a,b)$ for all $a, b > 0$ and $n \geq 1$, implies that (P_n) converges point-wisely to a mean, which we call P_∞. We need to prove $P_\infty = L$. Such a result follows from the next lemma and the proof of the theorem will then be completed. □

Lemma 4.3 *Let $q \in (0,1)$. For $x > 0$ we set*

$$\alpha_q(x) = \frac{1}{q}\int_1^{x^q} \frac{dt}{(t^{1/q}+1)^{2q}}.$$

Then we have

$$\lim_{q \downarrow 0} \alpha_q(x) = \ln x.$$

Proof By the classical mean value theorem we can write

$$\alpha_q(x) = \frac{x^q - 1}{q}\frac{1}{(s^{1/q}+1)^{2q}}$$

for some $s := s_q(x)$ between 1 and x^q. Assume that $x^q \geq 1$ then $2 \leq s^{1/q} + 1 \leq x + 1$. We then obtain

$$\frac{x^q - 1}{q} \frac{1}{(x+1)^{2q}} \leq \alpha_q(x) \leq \frac{1}{4^q} \frac{x^q - 1}{q},$$

with reversed inequalities if $x^q \leq 1$. Since $\lim_{q\downarrow 0} \frac{x^q-1}{q} = \ln x$ (by Hopital's rule, for example) we deduce the desired result. □

By (3.6) we deduce $\lim_{n\uparrow\infty} P^{(1/n)}(a,b) = L(a,b)$. We can also see this by writing $P^{(1/n)} = (A^{(1/2)})^{(1/n)} = A^{(1/2n)}$.

Now, let us compute $T^{(q)}$ for $0 < q \leq 1$. By similar way as for $A^{(q)}$, we obtain the following result (details are omitted here).

Theorem 4.4 *Let $q \in [-1,1]$. For all $a, b > 0$ with $a \neq b$, we have*

$$T^{(q)}(a,b) = \frac{b-a}{2^{q-1}(\Theta_{q/2}(b/a)^2 - \Theta_{q/2}(1))}, \tag{4.3}$$

where $\Theta_{q/2}(x)$, for $x > 0$, is defined as in (4.2). *Moreover, $T^{(q)}$ is strictly increasing* (*resp. decreasing*) *in $q \in (0,1)$* (*resp. in $q \in (-1,0)$*).

We also have $\lim_{n\uparrow\infty} T^{(1/n)}(a,b) = L(a,b)$, by the same arguments as previously. This, with (3.6), immediately gives $\lim_{n\uparrow\infty} M^{(1/n)}(a,b) = L(a,b)$. Otherwise, from (4.1) with (4.3), it is easy to see that, for all $a, b > 0$ and each $q \in [0,1]$,

$$A^{(q/2)}\left(a^2, b^2\right) = A(a,b)T^{(q)}(a,b).$$

Taking $q = 0$ in the latter equality we obtain $L(a^2, b^2) = A(a,b)L(a,b)$ while if we take $q = 1$ we find $A^{(1/2)}(a^2,b^2) = P(a^2,b^2) = A(a,b)T(a,b)$. These latter relations are well known and can be obtained from the definitions of A, L, P, and T.

Since $H = T^{(-1)}$ we deduce $H^{(q)} = T^{(-q)}$ for each $|q| \leq 1$. The expression of $H^{(q)}$ follows from (4.3). For another way of computation of $G^{(q)}$ and $P^{(q)}$, see the next section (Examples 5.7, 5.9). For $M^{(q)}$, see Section 7 below.

5 Index of a mean

Let q be a real number. A mean $m \in \mathcal{M}_\sigma$ will be called q-coherent if its regularized function F_m is q-admissible. By Lemma 2.4, we deduce that every $m \in \mathcal{M}_\sigma$ is q-coherent for each $|q| \leq 1$. We can then introduce the following definition.

Definition 5.1 Let $m \in \mathcal{M}_\sigma$. We set

$$\operatorname{ind}(m) := \sup\{s, m \text{ is } q\text{-coherent for each } 0 \leq q \leq s\},$$

which we will call the index of m.

It is clear that $\operatorname{ind}(m) \geq 1$ for every $m \in \mathcal{M}_\sigma$, since m is q-coherent for $|q| \leq 1$. The following result may be stated as well.

Proposition 5.2 *Let $m \in \mathcal{M}_\sigma$ be such that* $\operatorname{ind}(m) = i$. *Then the following assertions hold true*:

(i) $\operatorname{ind}(m^{(-1)}) = i$.

(ii) $\operatorname{ind}(m^{(q)}) = i/|q|$ *for every* $|q| \leq 1$ *with* $q \neq 0$.

(iii) *There exists* $r \in \mathcal{M}_\sigma$ *such that* $r^{(1/q)} = m$ *whenever* $1 \leq q \leq i$.

Proof (i) follows from the definition of index with the fact that $l(z) = 1/u(z)$.

(ii) It is easy to see that $m^{(q)}$ is s-coherent if and only if m is qs-coherent. If $q \in (0,1]$, we can then write

$$\begin{aligned}\operatorname{ind}\bigl(m^{(q)}\bigr) &= \sup\bigl\{s, m^{(q)} \text{ is } p\text{-coherent for each } 0 \leq p \leq s\bigr\}\\ &= \sup\{s, m \text{ is } pq\text{-coherent for each } 0 \leq p \leq s\}\\ &= \frac{1}{q}\sup\{qs, m \text{ is } pq\text{-coherent for each } 0 \leq pq \leq sq\} = \frac{i}{q}.\end{aligned}$$

If $q \in [-1,0)$ we write $m^{(q)} = (m^{(-1)})^{(-q)}$ and the desired result follows from the previous case with (i).

(iii) If $\operatorname{ind}(m) = i$ then, by definition, $F_m^q(z) := (F_m(z))^q$ is admissible for each $1 \leq q \leq i$. Then, by Lemma 2.3, there exists $r \in \mathcal{M}_\sigma$ such that $(F_m(z))^q = F_r(z)$ or again $F_m(z) = (F_r(z))^{1/q} := F_{r^{(1/q)}}(z)$, where $r^{(1/q)}$ is defined by (3.1). We then deduce $m = r^{(1/q)}$ and the proof is complete. □

The following example illustrates the previous concepts.

Example 5.3

(i) Since $F_L(z) = 1$, L is q-coherent for all real number q and so $\operatorname{ind}(L) = \infty$. This rejoins Proposition 5.2(ii) with $m^{(0)} = L$, by adopting the convention $i/0 = \infty$.

(ii) Since $F_{\min}(z) = u(z)$ we deduce that the largest $q \geq 1$ such that (for all $z \geq 1$)

$$l(z) \leq \bigl(F_{\min}(z)\bigr)^q = \bigl(u(z)\bigr)^q \leq u(z)$$

is $q = 1$. This means that $\operatorname{ind}(\min) = 1$. Similarly, we verify that $\operatorname{ind}(\max) = 1$.

The following proposition gives more examples of interest.

Proposition 5.4 *The following assertions hold true*:

(i) $\operatorname{ind}(A) = \operatorname{ind}(H) = \operatorname{ind}(T) = \operatorname{ind}(M) = 1$.

(ii) $\operatorname{ind}(G) = \operatorname{ind}(P) = 2$.

(iii) *L is the unique σ-regular mean such that* $\operatorname{ind}(L) = \infty$.

Proof (i) Let q be the index of A (resp. H, T or M). Lemma 4.3 (with Theorem C) immediately implies that $q \leq 1$. Since $\operatorname{ind}(m) \geq 1$ for every mean $m \in \mathcal{M}_\sigma$, we deduce the desired result.

(ii) Since $P = A^{(1/2)}$ we deduce by Proposition 5.2(ii), $\operatorname{ind}(P) = 2\operatorname{ind}(A) = 2$. The relation $P = G^{(-1)}$, with Proposition 5.2(i), yields $\operatorname{ind}(G) = \operatorname{ind}(P) = 2$.

(iii) Let $m \in \mathcal{M}_\sigma$ be such that $\mathrm{ind}(m) = \infty$. This means that m is q-coherent for every real number q. By definition we then have, for all $z \geq 1$,

$$l(z) \leq \big(F_m(z)\big)^q \leq u(z).$$

We would like to show that $m = L$ or equivalently $1 = F_L(z) = F_m(z)$ for all $z \geq 1$. Assume that $F_m(z_0) \neq 1$ for some $z_0 \geq 1$. Since $l(z) = 1/u(z)$ we can assume that $F_m(z_0) > 1$. Writing

$$l(z_0) \leq \big(F_m(z_0)\big)^q \leq u(z_0),$$

which must be valid for arbitrary real number q, we then obtain a contradiction by letting $q \uparrow \infty$. The proof is completed. □

The following example is also of interest.

Example 5.5 Let $q \geq 1$. Since $\mathrm{ind}(A) = 1$, Proposition 5.2(ii) implies that $\mathrm{ind}(A^{1/q}) = q$, where $A^{(1/q)}$ is defined by (4.1). Similarly, $\mathrm{ind}(T^{1/q}) = q$, where $T^{(1/q)}$ is defined by (4.3). If follows that, for each $q \geq 1$, there exists $m \in \mathcal{M}_\sigma$ (not unique) such that $\mathrm{ind}(m) = q$. In another way, the map $m \longmapsto \mathrm{ind}(m)$ defined from $\mathcal{M}_\sigma$ into $[1, \infty)$ is surjective but not injective.

Now, we are in a position to state the following definition.

Definition 5.6 Let $m \in \mathcal{M}_\sigma$ be such that $\mathrm{ind}(m) = i$ and $1 \leq q \leq i$. Then the mean $r \in \mathcal{M}_\sigma$ defined by Proposition 5.2(ii) will be denoted by $r := m^{(q)}$ and called the q-mean-power of m. In particular, if $i \geq 2$ and $q = 2$ then $m^{(2)}$ is called the mean-square of m.

Definition 3.2 introduces $m^{(q)}$ when $|q| \leq 1$ for all $m \in \mathcal{M}_\sigma$, while Definition 5.6 defines $m^{(q)}$ when $1 \leq q \leq i$ provided $\mathrm{ind}(m) = i$. We can then define $m^{(q)}$ for $q \leq -i$ when $\mathrm{ind}(m) = i$. In fact, we write $m^{(q)} := (m^{(-1)})^{(-q)}$, with $\mathrm{ind}(m) = \mathrm{ind}(m^{(-1)})$.

We now observe the following question: how could one compute $m^{(q)}$ when $\mathrm{ind}(m) = i$ and $1 \leq q \leq i$. Following the previous definition, with the help of (1.2) and (3.1), we have (with $q \geq 1$)

$$\begin{aligned} r := m^{(q)} \quad &\Longleftrightarrow \quad m = r^{(1/q)} = \big(G\big(r^{-\sigma}/G\big)^{1/q}\big)^\sigma \\ &\Longleftrightarrow \quad m^{-\sigma}/G = \big(r^{-\sigma}/G\big)^{1/q} \\ &\Longleftrightarrow \quad r^{-\sigma} = G\big(m^{-\sigma}/G\big)^q \\ &\Longleftrightarrow \quad r := m^{(q)} = \big(G\big(m^{-\sigma}/G\big)^q\big)^\sigma. \end{aligned}$$

Summarizing, $m^{(q)}$ can be, in all cases, computed by the same formulas (3.1) whenever $|q| \leq \mathrm{ind}(m)$.

Now, let us observe the following example explaining the previous discussion.

Example 5.7

(i) By $\mathrm{ind}(L) = \infty$ we then deduce that $L^{(q)}$ is also defined for each $q \geq 1$. We then have $L^{(q)} = L$ for every real number q.

(ii) Since $\mathrm{ind}(P) = 2$, the relationship $P = A^{(1/2)}$ is, following the previous definition, equivalent to $P^{(2)} = A$.

(iii) By $\mathrm{ind}(A) = 1$, the relation $P_n := A^{(1/n)}$ can then be written as $P_n^{(n)} = A$, since $\mathrm{ind}(A^{(1/n)}) = n$ according to Proposition 5.2(ii).

(iv) Since $\mathrm{ind}(G) = 2$, $G^{(q)}$ exists (as a mean) for each $|q| \le 2$. We can show that (we omit detail here)

$$G^{(2)} = \left(\frac{A+G}{2}\right)^{\sigma} = \frac{2G^2L}{G^2 + AL}.$$

See similar examples in Section 8 (Theorem 8.3), where some details are presented.

We end this section by stating the following result, which summarizes some elementary properties of the index.

Proposition 5.8 *Let $m \in \mathcal{M}_\sigma$. Then the following assertions are met*:

(i) $(m^{(q_1)})^{(q_2)} = m^{(q_1q_2)}$ *for all* $|q_1| \le 1$ *and* $|q_2| \le 1$.

(ii) $(m^{(q_1)})^{(q_2)} = m^{(q_1q_2)}$ *for all* q_1, q_2 *such that* $|q_1| \le \mathrm{ind}(m)$ *and* $|q_1q_2| \le \mathrm{ind}(m)$. *In particular*, $(m^{(1/q)})^{(q)} = m$ *for each* $|q| \ge 1$ *and* $(m^{(q)})^{(1/q)} = m$ *for every* $|q| \le \mathrm{ind}(m)$.

(iii) $\mathrm{ind}(m^{(q)}) = \mathrm{ind}(m)/|q|$ *for each* $|q| \le \mathrm{ind}(m)$.

Proof It is straightforward and we therefore omit all detail here. □

Now, we present the following example, which illustrates the previous results.

Example 5.9 Since $P = A^{(1/2)}$ and $\mathrm{ind}(P) = 2$, $P^{(q)}$ exists (as a mean) for every $|q| \le 2$. By Proposition 5.8(ii), we can write $P^{(q)} = (A^{(1/2)})^{(q)} = A^{(q/2)}$. This, with (4.1), immediately gives (for all $a, b > 0$, $a \neq b$, $|q| \le 2$),

$$P^{(q)}(a,b) = \frac{b-a}{2^q(\Theta_{q/2}(b/a) - \Theta_{q/2}(1))}. \tag{5.1}$$

As for $A^{(q)}$ and $T^{(q)}$, we can easily see that $P^{(q)}$ is strictly increasing in $p \in (0, 2]$ and strictly decreasing in $q \in [-2, 0)$.

Before ending this section, we will give an explanation as regards the interest of the index concept. Indeed, Proposition 5.2(iii) does not hold for $q > i = \mathrm{ind}(m)$. That is, there is no $r \in \mathcal{M}_\sigma$ such that $r^{(1/q)} = m$ for $q > \mathrm{ind}(m)$. In fact, let us choose m such that $\mathrm{ind}(m) = 1$, $m = H$ for fixing the idea. Assume that there exists $r \in \mathcal{M}_\sigma$ such that $r^{(1/q)} = H$ for some $q > \mathrm{ind}(H) = 1$. By (3.1), such r should satisfy $(G(r^{-\sigma}/G)^{1/q})^{\sigma} = H$ or equivalently, since H is σ-regular and $H^{-\sigma} = A$,

$$r^{-\sigma} = G\bigl(H^{-\sigma}/G\bigr)^q = G(A/G)^q = A^qG^{1-q}.$$

Now, we will show that, for $q > 1$, A^qG^{1-q} is not a mean. Assume that, for all $a, b > 0$, we have

$$\min(a,b) \le A^q(a,b)G^{1-q}(a,b) \le \max(a,b).$$

Taking $b = 1$ and $a \geq 1$, it is then necessary to have

$$1 \leq \left(\frac{a+1}{2}\right)^q a^{(1-q)/2} \leq a \quad \text{or equivalently} \quad a^{q-1} \leq \left(\frac{a+1}{2}\right)^{2q} \leq a^{q+1}.$$

Since $2q > q + 1$, the latter inequality is false for $a \geq 1$ large enough. Summarizing, our desired claim is justified.

6 On a law between means

We preserve the same notation as in the above sections. Inspired by the previous study, we will construct here an internal operation (law) between σ-regular means.

Let $\mathcal{C}$ be defined by

$$\mathcal{C} = \left\{(m_1, m_2) \in \mathcal{M}_\sigma \times \mathcal{M}_\sigma, l(z) \leq F_{m_1}(z)F_{m_2}(z) \leq u(z) \text{ for all } z \geq 1\right\}.$$

If $\operatorname{ind}(m) \geq i$ then $(m^{(i_1)}, m^{(i_2)}) \in \mathcal{C}$ whenever $i_1 + i_2 = i$. In particular, $(m, m) \in \mathcal{C}$ for every $m \in \mathcal{M}_\sigma$ such that $\operatorname{ind}(m) \geq 2$.

Now, let $(m_1, m_2) \in \mathcal{C}$ and define the following law:

$$m_1 \odot m_2 = m \quad \text{if and only if} \quad F_{m_1}(z)F_{m_2}(z) = F_m(z) \quad \text{for all } z \geq 1.$$

Remark that, if $\operatorname{ind}(m) \geq 2$ then $m \odot m = m^{(2)}$, where $m^{(2)}$ was introduced in Definition 5.1. More generally, it is not hard to show that $m^{(p)} \odot m^{(q)} = m^{(p+q)}$ whenever $p, q \geq 0$ and $p + q \leq \operatorname{ind}(m)$, where $m^{(q)}$ was defined in the previous sections.

A triplet (m_1, m_2, m_3) of σ-regular means will be called $\odot$-compatible if $(m_1, m_2) \in \mathcal{C}$, $(m_2, m_3) \in \mathcal{C}$, $(m_1 \odot m_2, m_3) \in \mathcal{C}$ and $(m_1, m_2 \odot m_3) \in \mathcal{C}$. If $\operatorname{ind}(m) \geq 3$ then (m, m, m) is $\odot$-compatible. More generally, if $\operatorname{ind}(m) \geq i$ then $(m^{(i_1)}, m^{(i_2)}, m^{(i_3)})$ is $\odot$-compatible whenever $i_1 + i_2 + i_3 = i$.

The following result summarizes some properties of the law $\odot$.

Proposition 6.1 *With the above, the following assertions are met*:

(i) *For all* $(m_1, m_2) \in \mathcal{C}$, $(m_2, m_1) \in \mathcal{C}$ *and* $m_1 \odot m_2 = m_2 \odot m_1$.

(ii) $(m_1 \odot m_2) \odot m_3 = m_1 \odot (m_2 \odot m_3)$ *for all* $\odot$*-compatible triplet* (m_1, m_2, m_3).

(iii) *For all* $m \in \mathcal{M}_\sigma$, $(m, L) \in \mathcal{C}$ *and* $m \odot L = L \odot m = m$.

(iv) *For all* $m \in \mathcal{M}_\sigma$, $(m, m^{(-1)}) \in \mathcal{C}$ *and*

$$m \odot m^{(-1)} = m^{(-1)} \odot m = L,$$

where $m^{(-1)}$ *was defined in Definition* 3.2 *and is given by*

$$m^{(-1)} = \left(\left(m^{-\sigma}\right)^*\right)^\sigma.$$

Proof (i) and (ii) are immediate from the definition of $\odot$, while (iii) follows from Lemma 2.1 with the fact that $F_L(z) = 1$. We then need to show (iv). In fact, let $m \in \mathcal{M}_\sigma$. We first search $m' \in \mathcal{M}_\sigma$ such that $m \odot m' = L$ *i.e.* $F_m(z)F_{m'}(z) = F_L(z) = 1$, and consequently $(m, m') \in \mathcal{C}$. According to the definition of $m^{(-1)}$, we then should have

$$F_{m'}(z) = 1/F_m(z) := F_m^{-1}(z) = F_{m^{(-1)}}(z),$$

from which we deduce $m' = m^{(-1)}$. Now, by (3.1), we have

$$m^{(-1)} = \left(G\left(\frac{m^{-\sigma}}{G}\right)^{-1}\right)^{\sigma} = \left(\frac{G^2}{m^{-\sigma}}\right)^{\sigma} = \left(\left(m^{-\sigma}\right)^*\right)^{\sigma},$$

so completing the proof. □

The operation $\odot$ is not a law as in the usual sense, but it is defined between elements of $\mathcal{M}_\sigma$ subject to a convenient condition. Following Proposition 6.1, such an operation satisfies properties similar to those of an abelian group. A remedy to this situation will be discussed in the following. Indeed, let us put

$$\mathcal{I}_2 = \left\{m \in \mathcal{M}_\sigma, \operatorname{ind}(m) \geq 2\right\}.$$

Following the previous study, G, L, P belong to $\mathcal{I}_2$ but min, max, A, H, T, and M do not. A simple observation on the statement of Example 5.5 implies that $\mathcal{I}_2$ is an infinite and uncountable set. Otherwise, it is easy to see that, if $(m_1, m_2) \in \mathcal{I}_2 \times \mathcal{I}_2$ then $(m_1, m_2) \in \mathcal{C}$, that is, $\mathcal{I}_2 \times \mathcal{I}_2$ is a subset of $\mathcal{C}$. The law $\odot$ is still stable on $\mathcal{I}_2$ in the sense that, if $m_1, m_2 \in \mathcal{I}_2$ then $m_1 \odot m_2 \in \mathcal{I}_2$. This follows from a simple manipulation. After this, the following result may be stated.

Proposition 6.2 *With the above*, $(\mathcal{I}_2, \odot)$ *is an abelian group.*

Proof First, as previously pointed, $\odot$ is stable on $\mathcal{I}_2$. Further, it is easy to verify that if $m_1, m_2, m_3 \in \mathcal{I}_2$ then the triplet (m_1, m_2, m_3) is $\odot$-compatible. This, with Proposition 6.1(i), (ii), asserts that $\odot$ is commutative and associative on $\mathcal{I}_2$. Now, we have $L \in \mathcal{I}_2$ and so L is the unit element of $\odot$ on $\mathcal{I}_2$. Lastly, if $m \in \mathcal{I}_2$ then $m^{(-1)} \in \mathcal{I}_2$, since $\operatorname{ind}(m) = \operatorname{ind}(m^{(-1)})$. The proof is completed. □

We now are in a position to state the following result giving a simple and nice relationship between the three familiar means P, T and M in terms of the law $\odot$.

Theorem 6.3 *We have*

$$M = T^{(1/2)} \odot A^{(1/2)} = H^{(-1/2)} \odot A^{(1/2)} = T^{(1/2)} \odot P.$$

Proof By Theorem C, with the definition of $m^{(1/2)}$, it is easy to see that

$$F_M(z) = \left(F_T(z)\right)^{1/2}\left(F_A(z)\right)^{1/2} = F_{T^{(1/2)}}(z) F_{A^{(1/2)}}(z).$$

This, with the definition of $\odot$, implies that $M = T^{(1/2)} \odot A^{(1/2)}$. The other equalities follow then, since $T = H^{(-1)}$ and $P = A^{(1/2)}$. □

We notice that the relation $M = T^{(1/2)} \odot P$ is not equivalent to $M^{(2)} = T \odot P^{(2)}$, since $\operatorname{ind}(M) = 1$ and so M is not 2-coherent. Also, we cannot write $M = T^{(1/2)} \odot A^{(1/2)}$ in the form $M = (T \odot A)^{(1/2)}$, since $(T, A) \notin \mathcal{C}$ and so $T \odot A$ is not defined.

Otherwise, the previous theorem is interesting from theoretical point of view as well as in practical purposes. First, it gives M in a short form involving the simplest means A

and H. Second, it regroups P, T and M in a unified expression, simple to prove and easy to remember. Third, we can derive some families of means involving two parameters and including all the previous familiar means A, G, H, L, P, T and M. In the next section, we will give some clarification as regards the latter situation.

We end this section by stating the following result.

Theorem 6.4 *Let $m_1, m_2 \in \mathcal{M}_\sigma$ be such that $(m_1, m_2) \in \mathcal{C}$. Then we have*

$$\frac{1}{\operatorname{ind}(m_1 \odot m_2)} \leq \frac{1}{\operatorname{ind}(m_1)} + \frac{1}{\operatorname{ind}(m_2)}. \tag{6.1}$$

Proof Let us put $s = \operatorname{ind}(m_1)$ and $p = \operatorname{ind}(m_2)$. Let $q \geq 1$ be defined by $1/q = 1/s + 1/p$. We can write

$$\big(F_{m_1 \odot m_2}(z)\big)^q := \big(F_{m_1}(z)\big)^q \big(F_{m_2}(z)\big)^q = \big(\big(F_{m_1}(z)\big)^s\big)^{q/s} \big(\big(F_{m_2}(z)\big)^p\big)^{q/p}.$$

This, with the fact that $s = \operatorname{ind}(m_1)$ and $p = \operatorname{ind}(m_2)$, implies that

$$l(z) = \big(l(z)\big)^{q/s+q/p} \leq \big(F_{m_1 \odot m_2}(z)\big)^q \leq \big(u(z)\big)^{q/s+q/p} = u(z).$$

We then deduce, by definition of the index, that $\operatorname{ind}(m_1 \odot m_2) \geq q$. The desired result follows. □

If $m_2 = m_1^{(-1)}$ then inequality (6.1) gives $0 \leq 2/\operatorname{ind}(m_1)$, since $m_1 \odot m_1^{(-1)} = L$ and $\operatorname{ind}(L) = \infty$. This means that (6.1) is not, in general, an equality. It is the best possible in the sense that if $m_1 = m_2$, it remains an equality. We can then state the following open question.

Problem 1 *Under what condition between m_1 and m_2, is* (6.1) *an equality*? *We conjecture that* (6.1) *is an equality if and only if $m_2 = m_1^{(q)}$ for some $q \geq 0$.*

7 On some families of 2-power means

We start this section by stating the following needed lemma.

Lemma 7.1 *Let $p, q \geq 0$ be such that $p + q \leq 1$. Then $A^{(p)}$ and $T^{(q)}$ satisfy $(A^{(p)}, T^{(q)}) \in \mathcal{C}$ i.e. $A^{(p)} \odot T^{(q)}$ is well defined.*

Proof By definition of $\odot$, with Theorem C, we have

$$F_{A^{(p)} \odot T^{(q)}}(z) = F_{A^{(p)}}(z) F_{T^{(q)}}(z) = \big(F_A(z)\big)^p \big(F_T(z)\big)^q = \frac{2^p}{z^q (z+1)^p}.$$

By the definition of $\mathcal{C}$, we then need to show that the double inequality

$$l(z) \leq \frac{2^p}{z^q (z+1)^p} \leq u(z)$$

holds for all $z \geq 1$. For the right inequality, we write $z^q(z+1)^p u(z) \geq (1+1)^p = 2^p$, since $u(z) \geq 1$ and $z^q \geq 1$. For the left inequality, it is equivalent to $z^q(z+1)^p \leq 2^p u(z)$, since

$l(z) = 1/u(z)$. Using $p + q \leq 1$ *i.e.* $q \leq 1 - p$ and $u(z) \geq z$ we can then write

$$z^q(z+1)^p \leq z^{1-p}(z+1)^p = \left(1 + \frac{1}{z}\right)^p z \leq 2^p z \leq 2^p u(z).$$

The lemma is completely proved. □

Notice that $A^{(p)}$ and $T^{(q)}$ were previously computed and are given by (4.1) and (4.3), respectively. We then set

$$W_{p,q} := A^{(p)} \odot T^{(q)}.$$

It is clear that

$$W_{0,0} = L, \qquad W_{1,0} = A, \qquad W_{0,1} = T, \qquad W_{1/2,0} = A^{(1/2)} = P, \qquad W_{1/2,1/2} = M.$$

We can extend $W_{p,q}$ by setting

$$W_{p,0} = A^{(p)} \quad \text{if } |p| \leq 1 \quad \text{and} \quad W_{0,q} = T^{(q)} \quad \text{if } |q| \leq 1. \tag{7.1}$$

With this, we have $W_{0,-1} = T^{(-1)} = H$ and $W_{-1/2,0} = A^{(-1/2)} = G$.

We can give an expression of $W_{p,q}$ as recited in the following result.

Theorem 7.2 *Let $p, q \geq 0$ be such that $p + q \leq 1$. Then we have, for all $a, b > 0$, $a \neq b$,*

$$W_{p,q}(a,b) = \frac{b-a}{2^{2p+q}(\Theta_{p,q}(b/a) - \Theta_{p,q}(1))}, \tag{7.2}$$

where $\Theta_{p,q}(x)$ is defined for all $x > 0$ by

$$\Theta_{p,q}(x) = \int_0^x \frac{t^{p+q-1}}{(t+1)^{2p}(t^2+1)^q}\, dt.$$

Proof By definition of $W_{p,q}$ (with Theorem C), we have

$$\frac{W_{p,q}^{-\sigma}}{G} := F_{W_{p,q}}(z) = \big(F_A(z)\big)^p \big(F_T(z)\big)^q = \frac{2^p}{z^q(z+1)^p}, \quad \text{with } z = A/G. \tag{7.3}$$

Replacing z by A/G in the latter equality we obtain after a simple reduction

$$W_{p,q} = \left(2^p \frac{G^{p+q+1}}{A^q(A+G)^p}\right)^\sigma. \tag{7.4}$$

We then use (1.1). Simple computation leads to

$$G^{p+q+1}\big(1, 1/t^2\big) = \frac{1}{t^{p+q+1}}, \qquad A^q\big(1, 1/t^2\big) = \frac{(t^2+1)^q}{2^q t^{2q}}, \qquad (A+G)^p\big(1, 1/t^2\big) = \frac{(t+1)^p}{2^p t^{2p}}.$$

Substituting these expressions in (7.4), again with (1.1), we obtain (7.2) after simple manipulation and reduction. □

If in (7.2) we take $p = 0$ and $|q| \leq 1$, or $q = 0$ and $|p| \leq 1$, we obtain (after simple computation) equations (4.1) and (4.3), respectively. This means that (7.2) is also valid for the particular situation (7.1). In another way, (7.2) includes all the previous means A, G, H, L, P, T, and M.

In order to give more example of construction of 2-power means, we need the next lemma.

Lemma 7.3 *Let $p, q \geq 0$ be such that $q + p/2 \leq 1$. Then we have*

$$P^{(p)} \odot M^{(q)} = T^{(q/2)} \odot A^{(p/2+q/2)}.$$

Proof According to Theorem C and similarly to the proof of Lemma 7.1 (we omit the routine detail here), we have

$$F_{P^{(p)}}(z)F_{M^{(q)}}(z) = \big(F_P(z)\big)^p\big(F_M(z)\big)^q = \frac{2^{p/2+q/2}}{z^{q/2}(z+1)^{p/2+q/2}}.$$

This, when compared with (7.3), yields the desired result after a simple manipulation. □

The previous lemma implies that $P^{(p)} \odot M^{(q)}$ exists. It further implies, with Theorem 7.2, an expression of $Z_{p,q} := P^{(p)} \odot M^{(q)}$ as recited in the following result.

Theorem 7.4 *Let $p, q \geq 0$ be such that $q + p/2 \leq 1$. Then we have, for all $a, b > 0$, $a \neq b$,*

$$Z_{p,q}(a,b) = \frac{b-a}{2^{\frac{p+3q}{2}}(\Theta_{q/2,(p+q)/2}(b/a) - \Theta_{q/2,(p+q)/2}(1))}. \tag{7.5}$$

When $q = 0$ and $0 \leq p \leq 2$, (7.5) coincides with (5.1). For $p = 0$ and $0 \leq q \leq 1$, it immediately yields an expression of $M^{(q)}$. All the above expressions are uncomputable exactly, except for a few trivial values of p and q.

8 Further examples

As already pointed out before, this section displays some other examples of interest in the aim to illustrate more the previous concepts as well as their related results.

The two means

$$U := U(a,b) = \frac{b-a}{\sqrt{2}\arctan(\frac{b-a}{\sqrt{2ab}})}, \qquad U(a,a) = a,$$

and

$$V := V(a,b) = \frac{b-a}{\sqrt{2}\operatorname{arcsinh}(\frac{b-a}{\sqrt{2ab}})}, \qquad V(a,a) = a,$$

were introduced in [6] (see p.9 and p.24, respectively). These two means are included in the so-called Seiffert type means discussed in [7]. For recent developments about the means U and V and their optimal bounds in terms of the (power) standard means, see for instance [6, 8–13] and the related references cited therein.

Applying the above theoretical study to the means U and V, we obtain the following result.

Theorem 8.1 *The following assertions are met*:

(i) *The means U and V are σ-regular, with*

$$F_U(z)=\frac{1}{z}\sqrt{\frac{z+1}{2}} \quad \textit{and} \quad F_V(z)=\sqrt{\frac{z+1}{2z}}; \quad z=A/G.$$

(ii) $U=G\odot T=G\odot H^{(-1)}=P^{(-1)}\odot T$, $V=G\odot T^{(1/2)}$ *and* $U=V\odot T^{(1/2)}$.

(iii) $\operatorname{ind}(U)=2$ *and* $\operatorname{ind}(V)\geq 6$.

Proof (i) We leave it to the reader as an interesting exercise. We can also consult [1] for similar situations.

(ii) follows from (i), with the definition of $\odot$.

(iii) Let us prove that $\operatorname{ind}(U)=2$. If we put $\operatorname{ind}(U)=q\geq 1$, Lemma 4.3 immediately implies that $q\leq 2$. We first show that U is 2-coherent *i.e.*

$$z-\sqrt{z^2-1}\leq F_U^2(z)=\frac{z+1}{2z^2}\leq z+\sqrt{z^2-1}$$

for all $z\geq 1$. For the right inequality, it is sufficient to write

$$2z^2\bigl(z+\sqrt{z^2-1}\bigr)=2z^3+2z^2\sqrt{z^2-1}\geq 2z^3=z^3+z^3\geq z+1.$$

For the left inequality, it is equivalent (after simple manipulation) to the following one

$$z(z-1)\leq (z+1)\sqrt{z^2-1}$$

or, by squaring,

$$z^2(z-1)\leq (z+1)^3,$$

which is obviously satisfied. Now, to prove that U is s-coherent for each $1\leq s\leq 2$, it is sufficient to remark that $F_U(z)\leq 1$ for all $z\geq 1$ and then to apply Lemma 2.5. It follows that $\operatorname{ind}(U)=2$. We now show that $\operatorname{ind}(V)\geq 6$. Since $F_V(z)\leq 1$ for all $z\geq 1$, by Lemma 2.5, we have to show that V is 6-coherent. Remark that $F_V(z)\leq 1$ implies $(F_V(z))^q\leq 1\leq u(z)$, for all $z\geq 1$ and every $q\geq 1$. Summarizing, we have to show that the inequality

$$l(z):=z-\sqrt{z^2-1}\leq \bigl(F_V(z)\bigr)^6:=\left(\frac{z+1}{2z}\right)^3$$

holds for all $z\geq 1$. This inequality is equivalent to (after an elementary manipulation)

$$8z^3-(z+1)^3z\leq (z+1)^3\sqrt{z^2-1}.$$

Remarking that

$$8z^3-(z+1)^3z=z(z-1)\bigl(-z^2+4z+1\bigr),$$

it is then sufficient to prove that the inequality

$$z^2(z-1)\left(z^2-4z-1\right)^2 \le (z+1)^7$$

holds for all $z \ge 1$. If we write

$$(z+1)^7 = (z+1)^2(z+1)(z+1)^4 \ge z^2(z-1)(z+1)^4,$$

it is sufficient that

$$(z+1)^2 \ge \left|z^2-4z+1\right|$$

holds for all $z \ge 1$. It is a simple verification which we omit here. The proof is then complete. □

We also mention that, although $\operatorname{ind}(U) = \operatorname{ind}(G) = 2$, the relation $U = G \odot T$ is not equivalent to $U^{(2)} = G^{(2)} \odot T^{(2)}$ since $\operatorname{ind}(T) = 1$ and so T is not 2-coherent. Otherwise, it is easy to see that, for all $z > 1$, we have

$$F_T(z) = \frac{1}{z} < F_U(z) < F_P(z) = \sqrt{\frac{2}{z+1}}.$$

This, with (1.3), immediately yields (simultaneously and in a fast way) the double mean-inequality $P < U < T$, which was differently proved in [6]. Since $P < M < T$, the double inequality $P < U < T$ does not give any information as regards a comparison between U and M. About this, the following result may be stated.

Proposition 8.2 *We have*

$$L < V < P < U < M.$$

Proof A simple verification asserts that $F_U(z) > F_M(z)$ for all $z > 1$. By (1.3) again, we then deduce that $U < M$. Similarly, it is easy to verify that $F_P(z) < F_V(z)$ and $F_V(z) < 1 = F_L(z)$, for all $z > 1$. We then deduce $L < V < P$. Summarizing, the desired inequalities are obtained. □

Since $\operatorname{ind}(U) = 2$ and $\operatorname{ind}(V) \ge 6$, $U^{(2)}$ and $V^{(2)}$ both exist (as means). Then the two relations $V = G \odot T^{(1/2)}$ and $U = V \odot T^{(1/2)}$ are equivalent to the following ones: $V^{(2)} = G^{(2)} \odot T$ and $U^{(2)} = V^{(2)} \odot T$, respectively. We might also be interested by computing the explicit forms of $U^{(2)}$ and $V^{(2)}$. The following result answers the latter claim.

Theorem 8.3 *The σ-regular means $U^{(2)}$ and $V^{(2)}$ are given through*

$$\frac{2}{U^{(2)}} = \frac{1}{T} + \frac{1}{C},$$
$$\frac{2}{V^{(2)}} = \frac{1}{T} + \frac{1}{L}.$$

Proof As in similar situations above, elementary computations lead to, for $a,b>0$, $a\neq b$,

$$\bigl(U^{(2)}(a,b)\bigr)^{-1}=\frac{1}{b-a}\int_1^{b/a}\frac{(t+1)^2}{(t^2+1)^2}\,dt$$

and

$$\bigl(V^{(2)}(a,b)\bigr)^{-1}=\frac{1}{2(b-a)}\int_1^{b/a}\frac{(t+1)^2}{t(t^2+1)}\,dt.$$

The two last integrals can be computed in an elementary way. After all computation and reduction, we obtain

$$U^{(2)}(a,b)=\frac{b-a}{\arctan(b/a)-\pi/4+\frac{b^2-a^2}{2(b^2+a^2)}}$$

and

$$V^{(2)}(a,b)=\frac{2(b-a)}{\ln b-\ln a+2\arctan(b/a)-\pi/2},$$

which, after simple manipulations, can be reduced to the desired forms, thus completing the proof. □

In fact, $V^{(6)}$ also exists, since $\operatorname{ind}(V)\geq 6$. In a similar way to the previous one, an elementary computation leads to, for $a,b>0$, $a\neq b$,

$$\bigl(V^{(6)}(a,b)\bigr)^{-1}=\frac{1}{2^3(b-a)}\int_1^{b/a}\frac{(t+1)^6}{t(t^2+1)^3}\,dt.$$

The latter integral is computable explicitly, by decomposing its related rational function. We could then write $V^{(6)}$ in terms of the previous familiar means. We omit all details as regards the latter point.

We left to the reader the routine task for finding the integral expression of $V^{(q)}$, for $1\leq q\leq 6$, and then deduce some writings of the means $V^{(3)}$, $V^{(4)}$ and $V^{(5)}$ in terms of the previous familiar means.

The index of V is not simple to find exactly. One reason of this difficulty is that Lemma 4.3 does not give here any information as regards the upper bound of q. We then should solve the question directly, if possible, by using the definition of index. Since the inequality $(F_V(z))^q\leq 1\leq u(z)$ holds for all $z\geq 1$ and every $q\geq 1$, we must determine the largest $q\geq 1$ such that, for each $s\leq q$, the inequality $l(z)\leq (F_V(z))^s$ holds for all $z\geq 1$. By Lemma 2.5, it is then sufficient to determine the largest $q\geq 1$ such that $l(z)\leq (F_V(z))^q$ for all $z\geq 1$. Such $q=\operatorname{ind}(V)$ is given by

$$\operatorname{ind}(V)=\inf_{z>1}\frac{2\ln(z+\sqrt{z^2-1})}{\ln(2z)-\ln(z+1)}\approx 8.693\ldots.$$

We now put the following as an open question.

Problem 2 *Find the exact value of* $\operatorname{ind}(V)$.

In [1], the author introduced the following means:

$$R_1(a,b)=\frac{b-a}{\sinh i(\ln(b/a))},\qquad R_1(a,a)=a,\quad \text{with } \sinh i(x):=\int_0^x\frac{\sinh(t)}{t}\,dt,$$
$$R_2(a,b)=\frac{b-a}{\tanh i(\ln(b/a))},\qquad R_2(a,a)=a,\quad \text{with } \tanh i(x):=\int_0^x\frac{\tanh(t)}{t}\,dt.$$

As pointed out in [1], such means are σ-regular and satisfy $R_1^{-\sigma}=L$ and $R_2^{-\sigma}=GL/A$. The following result is not hard to establish (we omit its proof here with the aim of making this paper not too lengthy).

Proposition 8.4 *With $z=A/G$, the following hold*:

$$F_{R_1}(z)=\frac{\sqrt{z^2-1}}{\ln(z+\sqrt{z^2-1})},\qquad F_{R_1}(1)=1,$$
$$F_{R_2}(z)=\frac{\sqrt{z^2-1}}{z\ln(z+\sqrt{z^2-1})},\qquad F_{R_2}(1)=1.$$

Consequently, we have $R_2=T\odot R_1$.

The expressions of $F_{R_1}(z)$ and $F_{R_2}(z)$ are relatively complicated and involve the transcendent logarithm function, while those of $F_m(z)$, for

$$m\in\{A,H,G,L,P,T,M,U,V\},$$

are simple and involve only algebraic functions. This can be in fact studied from a general point of view, which we leave to a future paper.

We end this section by stating the following open question.

Problem 3 *Evaluate* $\operatorname{ind}(R_1)$ *and* $\operatorname{ind}(R_2)$.

Competing interests
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