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Bounds for triple gamma functions and their ratios

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Abstract

In this work, in addition to the bounds for triple gamma function, bounds for the ratios of triple gamma functions are obtained. Similar bounds for the ratios of the double gamma functions are also obtained. These results and their consequences are obtained using the known results of the gamma function.

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Keywords: gamma function; multiple gamma function; logarithmically convex functions; monotonicity properties; bounds

1 Introduction

The multiple gamma functions denoted by Γ_n have applications in many areas of mathematics. For example, Γ_n are useful in computation of certain series in analytic number theory [1, 2]. The multiple gamma functions were first studied by Barnes [3–6]. The functions denoted by Γ_n are defined [2, 7] as:

$$\Gamma_n(z) = \big(G_n(z)\big)^{(-1)^{n-1}}, \quad n \in \mathbb{N},$$

where $G_m(z+1) = e^{g_m(z)}$ $(m \in \mathbb{N})$,

$$g_m(z) = -zP_m(1) + \sum_{l=1}^{m-1} \frac{q_l(z)}{l!}\big(g_{m-1}^{(l)}(0) - P_m^{(l)}(1)\big) + P_m(z),$$

$$P_m(z) = \sum_{r\in\mathbb{N}_0^{m-1}\times\mathbb{N}} \left[\frac{1}{m}\left(\frac{z}{M(r)}\right)^m - \frac{1}{m-1}\left(\frac{z}{M(r)}\right)^{m-1} + \cdots \right.$$
$$\left. + (-1)^{m-1}\frac{z}{M(r)} + (-1)^m \log\left(1 + \frac{z}{M(r)}\right)\right],$$

with $M(r) = r_1 + r_2 + \cdots + r_m$ if $r = (r_1, r_2, \ldots, r_m) \in \mathbb{N}_0^{m-1} \times \mathbb{N}$.

Here the polynomials $q_m(z)$ are defined as

$$q_m(z) := \begin{cases} \sum_{k=1}^{N-1} k^m & (z = N; N \in \mathbb{N} \setminus \{1\}), \\ \frac{B_{m+1}(z)-B_{m+1}}{m+1} & (z \in \mathbb{C}), \end{cases}$$

where $B_m(z)$ are Bernoulli polynomials of degree m in z.

Note that these polynomials $q_m(z)$ and $B_m(z)$ also obey the relation

$$q'_m(z) = \frac{B'_{m+1}(z)}{m+1} = B_m(z) \quad \text{and} \quad q_m(0) = 0.$$

Vignéras [7] characterized multiple gamma function with the following properties while introducing the notation $G_n(z)$:

(i) $G_n(z) = \frac{G_n(z+1)}{G_{n-1}(z)}$ for $z \in \mathbb{C}$,
(ii) $\frac{d^{n+1}}{dx^{n+1}} \log G_n(x+1) \geq 0$ for $x \geq 0$,
(iii) $G_n(1) = 1$,
(iv) $G_0(z) = z$.

It can be noted that the above conditions are the refinement of the Bohr-Morellup theorem and the multiple gamma function Γ_n of order n satisfies the following relations:

(i) $\Gamma_n(z) = \frac{\Gamma_{n+1}(z)}{\Gamma_{n+1}(z+1)}$ for $z \in \mathbb{C}$,
(ii) $\Gamma_n(1) = 1$,
(iii) $\Gamma_1(z) = \Gamma(z)$.

The double gamma function $G_2(z) = \frac{1}{\Gamma_2(z)}$ is the well-known Barnes G-function.

Problems for finding sharp bounds for gamma functions have always attracted researchers [8–11] since the 19th century. Recent research interest [2, 4, 6, 12–15] are in the bounds and asymptotic expansions for multiple gamma functions and their ratios. For an integral representation and asymptotic expansion of these functions we refer to [2, 14, 16, 17] and the references therein.

In 2008, Batir [12] obtained the bounds for the gamma function and extended these results in [13, 18] for the double gamma function. Recently, Chen [19] generalized the results of Batir [13].

Choi and Srivastava [14] found the following inequality for the triple gamma function for $0 \leq x \leq 1$:

$$\begin{aligned} &\exp\left(c_{3,1}x + c_{3,2}x^2 - \left(\frac{1}{4} + \frac{\pi^2}{36} + \frac{\gamma}{6}\right)x^3\right) \\ &\quad < \Gamma_3(1+x) < \exp\left(c_{3,1}x + c_{3,2}x^2 + c_{3,3}x^3 + \frac{\pi^2}{48}x^4\right), \end{aligned} \tag{1}$$

where

$$\begin{aligned} c_{3,1} &= \frac{3}{8} - \frac{1}{4}\log(2\pi) - \log A; \qquad c_{3,2} = \frac{1}{8} + \frac{1}{4}\log(2\pi) + \frac{\gamma}{4}; \\ c_{3,3} &= -\frac{1}{4} - \frac{\gamma}{6}, \end{aligned}$$

and A is defined as [20]

$$\log A = \frac{1}{12} - \zeta'(-1), \tag{2}$$

known as the Glaisher-Kinkelin constant.

Here ζ is the well-known Riemann Zeta function. In Section 2, we generalize the results of Batir [13] for the triple gamma functions.

In 2007, Shabani [21] considered the ratio of gamma functions to find the following double inequality as a generalization of the independent results of Alsina and Tomás [22] and Sándor [23]:

$$\frac{\Gamma(\alpha+\beta)^p}{\Gamma(\alpha+\beta)^q} \geq \frac{\Gamma(\alpha+\beta x)^p}{\Gamma(\beta+\alpha x)^q} \geq \frac{\Gamma(\alpha)^p}{\Gamma(\beta)^q} \tag{3}$$

for $x \in [0,1]$, $\alpha \geq \beta > 0$ and $p,q > 0$ such that $\beta p \geq \alpha q > 0$ with $\Psi(\beta+\alpha x) > 0$, where $\Psi(x) = \frac{d}{dx}\log\Gamma(x)$. The double inequalities similar to (3) for the ratios of triple gamma function are obtained in Section 3.

Since these types of inequalities can be obtained for the ratios of the double gamma function using a similar procedure, a sample result on the ratio of the double gamma function is also mentioned in Section 3.

2 Bounds for triple gamma function

The Weierstrass canonical products for $G_2(x)$ and $G_3(x)$ are given, respectively, by [15], equations (1.1) and (1.10),

$$\frac{1}{\Gamma_2(x+1)} = (2\pi)^{\frac{x}{2}} e^{-\frac{1}{2}[(1+\gamma)x^2+x]} \prod_{k=1}^{\infty}\left(\left(1+\frac{x}{k}\right)^k e^{-x+\frac{x^2}{2k}}\right) \tag{4}$$

and

$$\begin{aligned}\Gamma_3(x+1) = \exp\biggl[&-\frac{x^3}{6}\left(\gamma+\frac{\pi^2}{6}+\frac{3}{2}\right)+\frac{x^2}{4}\left(\gamma+\log(2\pi)+\frac{1}{2}\right)\\ &+x\left(\frac{3}{8}-\frac{\log(2\pi)}{4}-\log A\right)\biggr] &(5)\\ &\times\prod_{k=1}^{\infty}\left(\left(1+\frac{x}{k}\right)^{-\frac{1}{2}k(k+1)}\exp\left[\frac{x}{2}(k+1)-\frac{x^2}{4}\left(1+\frac{1}{k}\right)+\frac{x^3}{6k}\left(1+\frac{1}{k}\right)\right]\right), &(6)\end{aligned}$$

where γ is the Euler constant and A is the Glaisher-Kinkelin constant as defined in (2).

Theorem 1 *The Barnes G-function $G(x+1) = G_2(x+1) = \frac{1}{\Gamma_2(x+1)}$ is logarithmically convex for all $x \geq 1$.*

Proof Let

$$g(x) = \log G(x+1) = \frac{x}{2}\left(\log 2\pi - 1 - (1+\gamma)x\right) + \sum_{l=1}^{\infty}\left[l\log\left(1+\frac{x}{l}\right) - x + \frac{x^2}{2l}\right].$$

Then, for $x \geq 1$, a simple computation gives

$$\begin{aligned} g''(x) &= -(1+\gamma) + \sum_{l=1}^{\infty}\left(-\frac{l}{(x+l)^2}+\frac{1}{l}\right)\\ &\geq -(1+\gamma) + \sum_{l=1}^{\infty}\left(-\frac{l}{(1+l)^2}+\frac{1}{l}\right),\end{aligned}$$

which, by using partial fraction and the usual summation, leads to

$$g''(x) \geq -(1+\gamma) + \sum_{l=1}^{\infty} \frac{1}{l^2} = -(1+\gamma) + \frac{\pi^2}{6} \simeq 0.0677184019\ldots,$$

which proves the theorem. □

Theorem 2 *For all* $x \geq 1$,

$$\begin{aligned} &\left(\frac{(2\pi)^{\frac{x+1}{8}}}{A}\right)^{x-1} e^{-\frac{1}{24}(x-1)(2x^2+2x-1)} \cdot \bigl(G(x+1)\bigr)^{\frac{x-1}{4}} \\ &\quad < \Gamma_3(x+1) < \left(\frac{(2\pi)^{\frac{x+1}{8}}}{A}\right)^{x-1} e^{-\frac{1}{24}(x-1)(2x^2+2x-1)} \cdot \left(\frac{G(x+1)}{G(\frac{x+3}{2})}\right)^{\frac{x-1}{2}}. \end{aligned} \tag{7}$$

Proof Let $h : [\alpha, \beta] \to \mathbb{R}$ be a convex function. Then by the Hadamard inequality [24] we have

$$h\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} h(t)\,dt \leq \frac{h(\alpha)+h(\beta)}{2}. \tag{8}$$

Note that by Theorem 1, $G(x+1)$ is logarithmically convex for all $x \geq 1$. Therefore taking $h(t) = \log G(t+1)$, we get

$$\begin{aligned} &\log G\left(1+\frac{x+1}{2}\right) < \frac{1}{x-1}\int_1^x \log G(t+1)\,dt < \frac{1}{2}\log G(x+1) \\ &\quad \Rightarrow \quad \int_0^1 \log G(t+1)\,dt + (x-1)\log G\left(\frac{x+3}{2}\right) \\ &\qquad\qquad < \int_0^x \log G(t+1)\,dt < \int_0^1 \log G(t+1)\,dt + \frac{x-1}{2}\log G(x+1). \end{aligned}$$

Now from [15], equation (4.13), we obtain

$$\begin{aligned} \int_0^x \log G(t+1)\,dt = {}& (1/4 - 2\log A)x + \frac{1}{4}\log 2\pi - \frac{x^3}{6} + (x-1)\log G(x+1) \\ & - 2\log\Gamma_3(x+1). \end{aligned} \tag{9}$$

Hence we have

$$\begin{aligned} &-(1/4 - 2\log A)x - \frac{x^2}{4}\log 2\pi + \frac{x^3}{6} - (x-1)\log G(x+1) + 1/12 \\ &\qquad + 1/4\log 2\pi - 2\log A + (x-1)\log G\left(\frac{x+3}{2}\right) \\ &\quad < -2\log\Gamma_3(x+1) \\ &\quad < -(1/4 - 2\log A)x - \frac{x^2}{4}\log 2\pi - (x-1)\log G(x+1) \\ &\qquad + \frac{x^3}{6} + 1/12 + 1/4\log 2\pi - 2\log A + \frac{x-1}{2}\log G(x+1) \end{aligned}$$

$$\begin{aligned}\Rightarrow\quad & 2\log A\left(x^2-1\right)-\frac{(x-1)}{4}\log 2\pi+\left(\frac{x^3}{6}-\frac{x}{4}+\frac{1}{12}\right)+(x-1)\log\left(\frac{G(\frac{x+3}{2})}{G(x+1)}\right)\\ & <-2\log\Gamma_3(x+1)\\ & <2\log A(x-1)-\frac{1}{4}\log 2\pi\left(x^2-1\right)+\left(\frac{x^3}{6}-\frac{x}{4}+\frac{1}{12}\right)\\ & \quad+\frac{x-1}{2}\log G(x+1),\end{aligned}$$

which can easily be reduced to (7). □

Theorem 3 *For* $x \geq 0$,

$$\mathcal{L}(x)<\Gamma_3(1+x)<\mathcal{U}(x), \tag{10}$$

where

$$\begin{aligned}\mathcal{L}(x)&=A^{-x}\left(G(x+1)\right)^{\frac{x-1}{2}}\exp\left(\frac{1}{24}\left(3x^2+3x+2\gamma x^3\right)\right.\\ &\quad\left.-\frac{x^4}{24}\left(\Psi'\left(q(x)\right)+\frac{1}{2}\left(q(x)-1\right)\Psi''\left(q(x)\right)\right)\right);\\ \mathcal{U}(x)&=A^{-x}\left(G(x+1)\right)^{\frac{x-1}{2}}\exp\left(\frac{1}{24}\left(3x^2+3x+2\gamma x^3\right)\right.\\ &\quad\left.-\frac{x^4}{24}\left(\Psi'\left(p(x)\right)+\frac{1}{2}\left(p(x)-1\right)\Psi''\left(p(x)\right)\right)\right)\end{aligned}$$

with

$$\begin{aligned}p(x)&=1+\frac{x}{4},\\ q(x)&=\left(\frac{3}{x^3}\log(x+1)-\frac{3}{x^2}+\frac{3}{2x}\right)^{-1/3},\end{aligned}$$

and A being the Glaisher-Kinkelin constant defined as in (2).

Proof With the help of a Taylor series, Batir and Cancan [18], equation (2.3), proved that

$$\log G(x+1)=\frac{x}{2}\left(\log 2\pi-1-(\gamma+1)x\right)+\frac{x^3}{3}\sum_{m=0}^{\infty}\frac{m+1}{(m+1+\lambda(m+1))^3},$$

where $\lambda(m)$ is given by

$$\lambda(m)=\left(\frac{3}{x^3}\log\left(1+\frac{x}{m}\right)-\frac{3}{mx^2}+\frac{3}{2m^2x}\right)^{-1/3}-m$$

and, for all $m \geq 1$ and $x \geq 0$, $\lambda(m)$ is strictly increasing with

$$\begin{aligned}\lambda(1)&=\left(\frac{3}{x^3}\log(x+1)-\frac{3}{x^2}+\frac{3}{2x}\right)^{-1/3}-1=q(x)-1,\\ \lambda(\infty)&=\lim_{m\to\infty}\lambda(m)=\frac{x}{4}=p(x)-1.\end{aligned}$$

Hence,

$$\int_0^x \log G(t+1)\,dt = \frac{x^2}{4}\log 2\pi - \frac{1}{2}\bigl(x^2/2 + (\gamma+1)x^3/3\bigr) + \frac{x^4}{12}\sum_{m=0}^{\infty}\biggl(\frac{1}{(m+1+\lambda(m+1))^2} - \frac{\lambda(m+1)}{(k+1+\lambda(k+1))^3}\biggr). \tag{11}$$

Since for all $m \geq 1$ and $x \geq 0$, $\lambda(m)$ is strictly increasing. Therefore,

$$\begin{aligned}
&\frac{x^2}{4}\log 2\pi - \frac{(3x^2 + 2(1+\gamma)x^3)}{12} + \frac{x^4}{12}\biggl(\Psi'\bigl(\lambda(1)+1\bigr) + \frac{1}{2}\lambda(1)\Psi''\bigl(\lambda(1)+1\bigr)\biggr) \\
&\quad < \int_0^x \log G(t+1)\,dt \\
&\quad < \frac{x^2}{4}\log 2\pi - \frac{(3x^2 + 2(1+\gamma)x^3)}{12} + \frac{x^4}{12}\biggl(\Psi'\bigl(\lambda(\infty)+1\bigr) + \frac{1}{2}\lambda(\infty)\Psi''\bigl(\lambda(\infty)+1\bigr)\biggr).
\end{aligned}$$

Using (9) we have

$$\begin{aligned}
&-(1/4 - 2\log A)x - \frac{x^2}{4}\log 2\pi + \frac{x^3}{6} - (x-1)\log G(x+1) + \frac{x^2}{4}\log 2\pi \\
&\qquad - \frac{1}{12}\bigl(3x^2 + 2(1+\gamma)x^3\bigr) + \frac{x^4}{12}\biggl(\Psi'\bigl(p(x)\bigr) + \frac{1}{2}\bigl(p(x)-1\bigr)\Psi''\bigl(p(x)\bigr)\biggr) \\
&\quad < -2\log\Gamma_3(1+x) \\
&\quad < -\biggl(\frac{1}{4} - 2\log A\biggr)x - \frac{x^2}{4}\log 2\pi + \frac{x^3}{6} - (x-1)\log G(x+1) + \frac{x^2}{4}\log 2\pi \\
&\qquad - \frac{1}{12}\bigl(3x^2 + 2(1+\gamma)x^3\bigr) + \frac{x^4}{12}\biggl(\Psi'\bigl(q(x)\bigr) + \frac{1}{2}\bigl(q(x)-1\bigr)\Psi''\bigl(q(x)\bigr)\biggr),
\end{aligned}$$

which implies

$$\begin{aligned}
&2x\log A - \frac{1}{12}\bigl(3x^2 + 3x + 2\gamma x^3\bigr) - (x-1)\log G(x+1) \\
&\qquad + \frac{x^4}{12}\biggl(\Psi'\bigl(p(x)\bigr) + \frac{1}{2}\bigl(p(x)-1\bigr)\Psi''\bigl(p(x)\bigr)\biggr) \\
&\quad < -2\log\Gamma_3(x+1) \\
&\quad < 2x\log A - \frac{1}{12}\bigl(3x^2 + 3x + 2\gamma x^3\bigr) - (x-1)\log G(x+1) \\
&\qquad + \frac{x^4}{12}\biggl(\Psi'\bigl(q(x)\bigr) + \frac{1}{2}\bigl(q(x)-1\bigr)\Psi''\bigl(q(x)\bigr)\biggr).
\end{aligned}$$

Reversing the above inequality by changing the sign and keeping the logarithmic components together in each part of the inequality give the required result. □

For the purpose of graphical illustration given below, we denote $\Gamma_3(1+x)$ as $y(x)$ and L_2, L_3 respectively as the lower bounds of Theorem 2 and Theorem 3 and U_2, U_3, respectively, as the upper bounds of Theorem 2 and Theorem 3.

Figure 1 Comparison of upper bounds between (7) and (10).

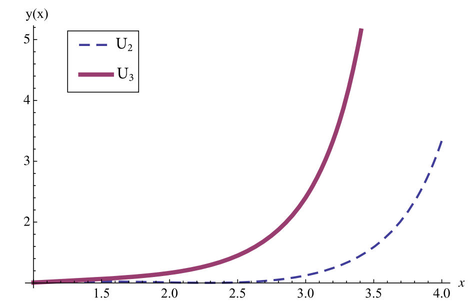


Figure 2 Comparison of lower bounds between (7) and (10).

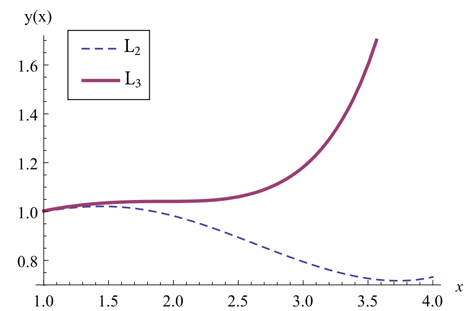
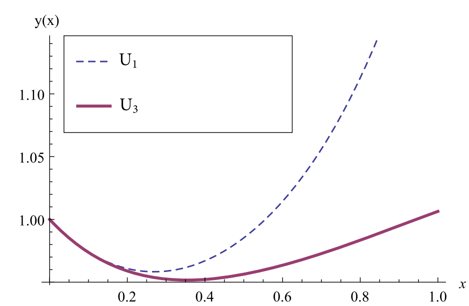


Figure 3 Comparison of upper bounds between (1) and (10).



Remark 1 From the graphical illustrations we observe that:

(i) Although Theorem 2 is valid only for $x \geq 1$, the upper bound in Theorem 2 is better than the upper bound in Theorem 3 for $x \geq 1$. However, the lower bound given in Theorem 3 is better than the lower bound of Theorem 2. Figure 1 and Figure 2 support the claim.

(ii) It can be noted that the upper bound U_3 in Theorem 3 is sharper than the upper bound U_1 in (1). Figure 3 supports the claim.

These observations lead to the problem of improving the lower bound for Γ_3, in comparison with (1), so that it can supplement Theorem 2. This can be obtained by establishing the logarithmic convexity of $G(x+1)$ for $x \geq 0$, which requires a different approach. Otherwise, an improved bound for (11) can also suffice the requirement.

3 Inequalities for the ratio of triple gamma functions

Similar to the di-gamma function $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, let $\Psi_2(x) := \frac{\Gamma_2'(x)}{\Gamma_2(x)}$ and $\Psi_3(x) := \frac{\Gamma_3'(x)}{\Gamma_3(x)}$, $x > 0$, denote the di-double gamma and di-triple gamma function, respectively.

In this section, some inequalities for the ratio of the double gamma functions and the triple gamma functions are obtained with the help of the series representation of di-double gamma and di-triple gamma function. For this purpose, techniques given in [21] will be utilized.

First we establish some new inequalities for $\Psi_2(x)$ and $\Psi_3(x)$. Taking the logarithmic derivative of the double gamma function and triple gamma function, respectively, the following result is immediate.

Lemma 1 *For all $x > 0$ one has the series representation*

(i)

$$\Psi_2(x) = \frac{1}{2}(1 - \log 2\pi) + (1 + \gamma)(x - 1) - (x - 1)^2 \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+x)}. \tag{12}$$

(ii)

$$\begin{aligned}\Psi_3(x) = {}& -\frac{(x-1)^2}{2}\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right) + \frac{(x-1)}{2}\left(\gamma + \log(2\pi) + \frac{1}{2}\right) \\ & + \left(\frac{3}{8} - \frac{\log(2\pi)}{4} - \log A\right) + \frac{(x-1)^3}{2} \sum_{k=0}^{\infty} \frac{k+2}{(k+1)^2(k+x)}.\end{aligned} \tag{13}$$

Lemma 2 *Let α and β be two positive real numbers such that $\alpha \geq \beta$, then*:

(i) *for all $x \in [0,1]$*,

$$\begin{aligned}&\Psi_2(\alpha + \beta x) \leq \Psi_2(\beta + \alpha x), \\ &\Psi_3(\alpha + \beta x) \geq \Psi_3(\beta + \alpha x),\end{aligned}$$

(ii) *for all $x \geq 1$*,

$$\begin{aligned}&\Psi_2(\alpha + \beta x) \geq \Psi_2(\beta + \alpha x), \\ &\Psi_3(\alpha + \beta x) \leq \Psi_3(\beta + \alpha x).\end{aligned}$$

Proof It is enough to prove for Ψ_3, as the result for Ψ_2 will follow in a similar fashion.

Let $x > 0$, $y \geq 0$, and $x \geq y$, then

$$\begin{aligned}&\Psi_3(x) - \Psi_3(y) \\ &\quad = -\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right)\left(\frac{x-y}{2}\right)\bigl[2 - (x+y)\bigr] + \frac{(x-y)}{2}\left(\gamma + \log(2\pi) + \frac{1}{2}\right) \\ &\qquad + \frac{1}{2}\sum_{k=0}^{\infty} \frac{k+2}{(k+1)^2}\left[\frac{(x-1)^3}{k+x} - \frac{(y-1)^3}{k+y}\right] \\ &\quad = -\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right)\left(\frac{x-y}{2}\right)\bigl[2 - (x+y)\bigr] + \frac{(x-y)}{2}\left(\gamma + \log(2\pi) + \frac{1}{2}\right) \\ &\qquad + \frac{1}{2}\sum_{k=0}^{\infty} \frac{k+2}{(k+1)^2}\left[\frac{\mathcal{A}(x,y,k)}{(k+x)(k+y)}\right] \geq 0,\end{aligned}$$

where

$$\mathcal{A}(x,y,k) = k\big[\big(x^3 - y^3\big) - 3\big(x^2 - y^2\big) + 3(x-y)\big] + \big[xy\big(x^2 - y^2\big) - 3xy(x-y) - (x-y)\big].$$

So $\Psi_2(x) \geq \Psi_2(y)$.

Since $\alpha + \beta x > 0$, $\beta + \alpha x > 0$, it can be verified that, for $x \in [0,1]$, $\alpha \geq \beta > 0$, we obtain $\alpha + \beta x \geq \beta + \alpha x$, which implies $\Psi_3(\alpha + \beta x) \geq \Psi_3(\beta + \alpha x)$.

Again, $x \geq 1 \Rightarrow \alpha + \beta x \leq \beta + \alpha x$ for $\alpha \geq \beta > 0$.

Therefore, $\Psi_3(\alpha + \beta x) \leq \Psi_3(\beta + \alpha x)$. □

Alternative proof of Lemma 2 Clearly, $x \in [0,1]$, $\alpha, \beta > 0 \Rightarrow \alpha + \beta x > 0$, $\beta + \alpha x > 0$. Then by (13), we obtain

$$\begin{aligned}
&\Psi_3(\alpha + \beta x) - \Psi_3(\beta + \alpha x) \\
&\quad = -\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right)(\alpha - \beta)\big[(\alpha + \beta) - (\alpha + \beta)x^2 + 2x - 2\big] + \frac{(\alpha - \beta)}{2}(1 - x) \\
&\qquad \times \left(\gamma + \log(2\pi) + \frac{1}{2}\right) + \frac{1}{2}\sum_{k=0}^{\infty} \frac{k+2}{(k+1)^2}\left[\frac{(\alpha + \beta x - 1)^3}{k + \alpha + \beta x} - \frac{(\beta + \alpha x - 1)^3}{k + \beta + \alpha x}\right] \\
&\quad = -\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right)(\alpha - \beta)\big[(\alpha + \beta) - (\alpha + \beta)x^2 + 2x - 2\big] + \frac{(\alpha - \beta)}{2}(1 - x) \\
&\qquad \times \left(\gamma + \log(2\pi) + \frac{1}{2}\right) + \frac{1}{2}\sum_{k=0}^{\infty} \frac{k+2}{(k+1)^2}\frac{\mathcal{A}(\alpha, \beta, k, x)}{(k + \alpha + \beta x)(k + \beta + \alpha x)} \geq 0,
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{A}(\alpha, \beta, k, x) \\
&\quad = -x^4\big(\alpha^2 - \beta^2\big)\alpha\beta + x^3\big[3\alpha\beta(\alpha - \beta) - \big(\alpha^4 - \beta^4\big) - k\big(\alpha^3 - \beta^3\big)\big] \\
&\qquad + x^2\big[3\big(\alpha^2 - \beta^2\big) - 6\alpha\beta(\alpha - \beta) + 3\big(\alpha^3 - \beta^3\big) + 3k\big(\alpha^2 - \beta^2\big) - 3k\alpha\beta(\alpha - \beta)\big] \\
&\qquad + x\big[\big(\alpha^4 - \beta^4\big) - 3\big(\alpha^3 - \beta^3\big) + 3\big(\alpha^2 - \beta^2\big) - (\alpha - \beta) + 6\alpha\beta(\alpha - \beta) \\
&\qquad + 3k\alpha\beta(\alpha - \beta) - 3k(\alpha - \beta)\big] + \big[(\alpha - \beta) - 3\alpha\beta(\alpha - \beta) + \alpha\beta\big(\alpha^2 - \beta^2\big) \\
&\qquad + 3k(\alpha - \beta) - 3k\big(\alpha^2 - \beta^2\big) + k\big(\alpha^2 - \beta^2\big)\big].
\end{aligned}$$

□

Lemma 3 *Let α, β, p, and q be positive real numbers. Further suppose that $\beta p - \alpha q$ and $\Psi_3(\alpha + \beta x)$ have the same sign. If for $0 \leq x \leq 1$, $\alpha \geq \beta$, and for $x \geq 1$, $\alpha \leq \beta$. Then*

$$\beta p \Psi_3(\alpha + \beta x) - \alpha q \Psi_3(\beta + \alpha x) \geq 0.$$

Proof We only prove the case where $x \in [0,1]$, $\alpha \geq \beta$, $\beta p - \alpha q \geq 0$, and $\Psi_3(\beta + \alpha x) > 0$.

Then by part (i) of Lemma 2, it is clear that $\Psi_3(\alpha + \beta x)$ is also positive. Since $\beta p \geq \alpha q$, using Lemma 2, we have

$$\beta p \Psi_3(\alpha + \beta x) \geq \alpha q \Psi_3(\alpha + \beta x) \geq \alpha q \Psi_3(\beta + \alpha x),$$

which establishes the result. □

Theorem 4 *Define* $g : [0,\infty) \to (0,\infty)$ *by*

$$g(x) = \frac{\Gamma_3(\alpha+\beta x)^p}{\Gamma_3(\beta+\alpha x)^q},$$

where $\alpha \geq \beta > 0$, $p > 0$, $q > 0$ *such that* $\beta p \geq \alpha q > 0$ *and* $\Psi_3(\beta+\alpha x) > 0$, *then the following are true*:

(i) $g(x)$ *is increasing on* $0 \leq x \leq 1$ *and*

(ii)

$$\frac{\Gamma_3(\alpha)^p}{\Gamma_3(\beta)^q} \leq \frac{\Gamma_3(\alpha+\beta x)^p}{\Gamma_3(\beta+\alpha x)^q} \leq \frac{\Gamma_3(\alpha+\beta)^p}{\Gamma_3(\alpha+\beta)^q}, \quad 0 \leq x \leq 1.$$

Proof Let $h(x) = \log g(x)$. Then

$$\begin{aligned} & h(x) = p\log\Gamma_3(\alpha+\beta x) - q\log\Gamma_3(\beta+\alpha x) \\ \Rightarrow \quad & h'(x) = \beta p\frac{\Gamma_3'(\alpha+\beta x)}{\Gamma_3(\alpha+\beta x)} - \alpha q\frac{\Gamma_3'(\beta+\alpha x)}{\Gamma_3(\beta+\alpha x)} \\ & \qquad = \beta p\Psi_3(\alpha+\beta x) - \alpha q\Psi_3(\beta+\alpha x). \end{aligned}$$

By part (i) of Lemma 3, we get $h'(x) \geq 0$, which implies $h(x)$ is increasing on $0 \leq x \leq 1$. This indicates that $g(x)$ is increasing on $0 \leq x \leq 1$.

So for $x \in [0,1]$ we have $g(0) \leq g(x) \leq g(1)$ or

$$\frac{\Gamma_3(\alpha)^p}{\Gamma_3(\beta)^q} \leq \frac{\Gamma_3(\alpha+\beta x)^p}{\Gamma_3(\beta+\alpha x)^q} \leq \frac{\Gamma_3(\alpha+\beta)^p}{\Gamma_3(\alpha+\beta)^q}. \qquad \square$$

The following theorem is immediate. We omit the proof.

Theorem 5 *Define* $f : [0,\infty) \to (0,\infty)$ *by*

$$f(x) = \frac{\Gamma_3(\alpha+\beta x)^p}{\Gamma_3(\beta+\alpha x)^q},$$

where $\alpha,\beta,p,q > 0$. *Further suppose that* $\beta-\alpha$, $\beta p-\alpha q$, *and* $\Psi_3(\beta+\alpha x)$ *have the same sign. Then for all* $x \geq 0$, f *is an increasing function.*

Along similar lines, with the help of Lemma 2, the inequalities for the ratio of the double gamma function can be obtained. For the sake of brevity we provide only one result without proof.

Theorem 6 *Define* $f : [0,\infty) \to (0,\infty)$ *by*

$$f(x) = \frac{\Gamma_2(\alpha+\beta x)^p}{\Gamma_2(\beta+\alpha x)^q},$$

where $\alpha \geq \beta > 0$, $p > 0$, $q > 0$ *such that* $\beta p \geq \alpha q > 0$ *and* $\Psi_2(\beta+\alpha x) < 0$. *Then the following are true*:

(i) $f(x)$ *is decreasing on* $0 \leq x \leq 1$ *and*

(ii)

$$\frac{\Gamma_2(\alpha)^p}{\Gamma_2(\beta)^q} \geq \frac{\Gamma_2(\alpha+\beta x)^p}{\Gamma_2(\beta+\alpha x)^q} \geq \frac{\Gamma_2(\alpha+\beta)^p}{\Gamma_2(\alpha+\beta)^q}, \quad 0 \leq x \leq 1.$$

Remark 2 Unlike Theorem 5, information about the monotonicity of $f(x)$ in Theorem 6 for $x > 1$ is not explicitly clear. However, further analysis of the monotonicity of $f(x)$ in both Theorem 5 and Theorem 6 is expected to provide interesting consequences.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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