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Some new generalized Volterra-Fredholm type discrete fractional sum inequalities and their applications

Haidong Liu* and Fanwei Meng

*Correspondence:
tomlhd983@163.com
School of Mathematical Sciences, Qufu Normal University, Qufu, 273165, P.R. China

Abstract

In this paper, we present some new Volterra-Fredholm-type discrete fractional sum inequalities. These inequalities can be used as handy and powerful tools in the study of certain fractional sum-difference equations. Some applications are also presented to illustrate the usefulness of our results.

Keywords: discrete fractional sum inequality; Volterra-Fredholm type; fractional sum-difference equation; boundedness

1 Introduction

It is well known that Gronwall-Bellman-type inequalities and their various generalizations have historically great importance in the qualitative analysis of differential equations, difference equations, and fractional differential equations. During the past few years, there are a lot of mathematical results about the generalized Gronwall-Bellman-type inequalities and their applications (see, *e.g.*, [1–13] and the references therein).

Recently, there has been an increase in study in the theory of discrete fractional calculus, and many interesting researches have been devoted to many topics of the fractional difference equations (see, *e.g.*, [14–29] and the references therein). However, compared with integer-order equations and fractional differential equations, Gronwall-Bellman-type inequalities for discrete fractional calculus receive less attention (see, *e.g.*, [18–20] and the references therein).

In this paper, we employ the Riemann-Liouville definition of the fractional difference initiated by Miller and Ross [30, 31], and developed by Atici and Eloe [14–16, 18] to establish some Volterra-Fredholm-type discrete fractional sum inequalities, which are generalizations of Gronwall-Bellman forms. These inequalities can be used as handy and powerful tools in the analysis of certain classes of Volterra-Fredholm-type fractional sum-difference equations.

The paper is organized as follows. Some important definitions and results on discrete fractional calculus are collected in Section 2. Some new nonlinear Volterra-Fredholm-type discrete fractional sum inequalities are presented in Section 3. In the last section, as an application of the inequalities obtained, the boundedness and uniqueness of the solutions of certain Volterra-Fredholm fractional sum-difference equation are established.

2 Preliminaries

Throughout this paper, we denote $\mathbf{N}_t = \{t, t+1, t+2, \ldots\}$, $I_t = [t, T] \cap \mathbf{N}_t$, where $t \in \mathbf{N}_t$, and $T \in \mathbf{N}_t$ is a constant. Let $\mathbf{R}_+ = [0, \infty)$, and $\sum_{s=k}^{n} f(s) = 0$ for $k > n$. Denote by $C^i(M, N)$ the class of all i times continuously differentiable functions defined from a set M into a set N for $i = 1, 2, \ldots$. As usual, let z be a real-valued function on $\mathbf{N}_t$, and the difference operator Δ on z be defined as $\Delta z(n) = z(n+1) - z(n)$, $n \in \mathbf{N}_t$.

Next, we list some important definitions and results on discrete fractional calculus.

Definition 2.1 ([16]) Let a be any real number, α be any positive real number, and $\sigma(s) = s + 1$. The α-th fractional sum (α-sum) of f is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} \bigl(t - \sigma(s)\bigr)^{(\alpha-1)} f(s).$$

Here f is defined for $s = a \pmod 1$, and $\Delta_a^{-\alpha} f$ is defined for $t = a + \alpha \pmod 1$; in particular, $\Delta_a^{-\alpha}$ maps functions defined on $\mathbf{N}_a$ to functions defined on $\mathbf{N}_{a+\alpha}$. We recall that the falling factorial is defined as $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$.

Definition 2.2 ([16]) The μ-th fractional difference is defined as

$$\Delta^{\mu} u(t) = \Delta^{m-\nu} u(t) = \Delta^{m}\bigl(\Delta^{-\nu} u(t)\bigr),$$

where $\mu > 0$ and $m - 1 < \mu < m$, where m denotes a positive integer, and $-\nu = \mu - m$.

Lemma 2.1 (Pachpatte [1], p.103) *Let $u(t)$ and $b(t)$ be nonnegative functions defined for $t \in \mathbf{N}_0$, and c be a nonnegative constant. Let $g(u)$ be a nondecreasing continuous function defined on $\mathbf{R}_+$ with $g(u) > 0$ for $u > 0$. If*

$$u(t) \le c + \sum_{s=0}^{t-1} b(s) g\bigl(u(s)\bigr)$$

for $t \in \mathbf{N}_0$, then, for $0 \le t \le t_1$, $t, t_1 \in \mathbf{N}_0$,

$$u(t) \le G^{-1}\left[G(c) + \sum_{s=0}^{t-1} b(s)\right],$$

where $G(r) = \int_{r_0}^{r} \frac{1}{g(s)}\,\mathrm{d}s$, $r > 0$, $r_0 > 0$ is arbitrary, G^{-1} is the inverse of G, and $t_1 \in \mathbf{N}_0$ is chosen so that $G(c) + \sum_{s=0}^{t-1} b(s) \in \operatorname{Dom}(G^{-1})$ for all $t \in \mathbf{N}_0$ such that $0 \le t \le t_1$.

For other important properties on the discrete fractional calculus, we refer the reader to [14, 15, 17].

3 Main results

Theorem 3.1 *Assume that $0 < \alpha \le 1$ is a constant, $u : \mathbf{N}_{\alpha-1} \to \mathbf{R}_+$, $f, g : \mathbf{N}_0 \to \mathbf{R}_+$ are functions, $k \ge 0$ is a constant, and $p > q > 0$ are constants. Suppose that u satisfies*

$$\begin{aligned} u^p(n) \le{}& k + \Delta_0^{-\alpha}\bigl[f(n) u^q(n+\alpha-1)\bigr] \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T-\alpha} (T-s-1)^{(\alpha-1)} g(s) u^p(s+\alpha-1), \quad n \in I_{\alpha-1}. \end{aligned} \tag{3.1}$$

If

$$\lambda = 2^{\frac{q}{p-q}} \sum_{s=0}^{T-\alpha} G(s,T) < 1, \tag{3.2}$$

then

$$u(n) \le \left[A^{\frac{p-q}{p}}(T) + \frac{p-q}{p} \sum_{s=\alpha}^{n} f(s-\alpha) \right]^{\frac{1}{p-q}}, \quad n \in I_{\alpha-1}, \tag{3.3}$$

where

$$A(T) = \frac{1}{1-\lambda} \left\{ k + 2^{\frac{q}{p-q}} \sum_{s=0}^{T-\alpha} G(s,T) \left[\frac{p-q}{p} \sum_{\tau=\alpha}^{s+\alpha-1} f(\tau-\alpha) \right]^{\frac{p}{p-q}} \right\}, \tag{3.4}$$

$$G(s,n) = \frac{1}{\Gamma(\alpha)} (n-s-1)^{(\alpha-1)} g(s). \tag{3.5}$$

Proof Let $k > 0$. From (3.1) and (3.5) we have

$$\begin{aligned} u^p(n) &\le k + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n-\alpha} (n-s-1)^{(\alpha-1)} f(s) u^q(s+\alpha-1) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T-\alpha} (T-s-1)^{(\alpha-1)} g(s) u^p(s+\alpha-1) \\ &= k + \sum_{s=0}^{n-\alpha} F(s,n) u^q(s+\alpha-1) \\ &\quad + \sum_{s=0}^{T-\alpha} G(s,T) u^p(s+\alpha-1), \quad n \in I_{\alpha-1}, \end{aligned} \tag{3.6}$$

where

$$F(s,n) = \frac{1}{\Gamma(\alpha)} (n-s-1)^{(\alpha-1)} f(s).$$

Define

$$\begin{aligned} z(n) &= k + \sum_{s=0}^{n-\alpha} F(s,n) u^q(s+\alpha-1) \\ &\quad + \sum_{s=0}^{T-\alpha} G(s,T) u^p(s+\alpha-1), \quad n \in I_{\alpha-1}. \end{aligned} \tag{3.7}$$

Then $z(n) \ge 0$ is nondecreasing,

$$z(\alpha-1) = k + \sum_{s=0}^{T-\alpha} G(s,T) u^p(s+\alpha-1), \tag{3.8}$$

and

$$u^p(n) \le z(n), \quad n \in I_{\alpha-1}. \tag{3.9}$$

By the definitions of $F(s,n)$ and $t^{(\alpha)}$ we can easily get that $F(s,n)$ is decreasing in n for each $s \in \mathbf{N}_0$. So from a straightforward computation, for $n \in I_\alpha$, we obtain that

$$\begin{aligned} z(n) - z(n-1) &= F(n-\alpha, n)u^q(n-1) \\ &\quad + \sum_{s=0}^{n-\alpha-1} \bigl[F(s,n) - F(s,n-1)\bigr]u^q(s+\alpha-1) \\ &\le F(n-\alpha,n)u^q(n-1) \\ &\le F(n-\alpha,n)z^{\frac{q}{p}}(n-1) \\ &= f(n-\alpha)z^{\frac{q}{p}}(n-1). \end{aligned} \tag{3.10}$$

Using the monotonicity of z, we deduce

$$z^{\frac{q}{p}}(n-1) \ge z^{\frac{q}{p}}(\alpha-1) = \left(k + \sum_{s=0}^{T-\alpha} G(s,T)u^p(s+\alpha-1)\right)^{\frac{q}{p}} > 0, \quad n \in I_\alpha. \tag{3.11}$$

So from (3.10) and (3.11) we have

$$\frac{z(n) - z(n-1)}{z^{\frac{q}{p}}(n-1)} \le f(n-\alpha), \quad n \in I_\alpha,$$

that is,

$$\frac{\Delta z(n-1)}{z^{\frac{q}{p}}(n-1)} \le f(n-\alpha), \quad n \in I_\alpha. \tag{3.12}$$

On the other hand, by the mean value theorem we obtain

$$\begin{aligned} \Delta\left(\frac{p}{p-q}z^{\frac{p-q}{p}}(n-1)\right) &= \frac{p}{p-q}z^{\frac{p-q}{p}}(n) - \frac{p}{p-q}z^{\frac{p-q}{p}}(n-1) \\ &= \xi^{-\frac{q}{p}}\Delta z(n-1) = \frac{\Delta z(n-1)}{\xi^{\frac{q}{p}}} \\ &\le \frac{\Delta z(n-1)}{z^{\frac{q}{p}}(n-1)}, \quad \xi \in \bigl[z(n-1), z(n)\bigr]. \end{aligned} \tag{3.13}$$

So from (3.12) and (3.13) we obtain

$$\Delta\left(\frac{p}{p-q}z^{\frac{p-q}{p}}(n-1)\right) \le f(n-\alpha), \quad n \in I_\alpha. \tag{3.14}$$

Setting $n = s$ in inequality (3.14) and summing with respect to s from α to $n-1$, we get

$$\sum_{s=\alpha}^{n-1} \Delta\left(\frac{p}{p-q}z^{\frac{p-q}{p}}(s-1)\right) \le \sum_{s=\alpha}^{n-1} f(s-\alpha),$$

that is,

$$z^{\frac{p-q}{p}}(n-1) \le z^{\frac{p-q}{p}}(\alpha-1) + \frac{p-q}{p}\sum_{s=\alpha}^{n-1} f(s-\alpha), \quad n \in I_\alpha. \tag{3.15}$$

Then from inequality (3.15) we conclude that

$$z(n-1) \le \left[z^{\frac{p-q}{p}}(\alpha-1) + \frac{p-q}{p}\sum_{s=\alpha}^{n-1} f(s-\alpha)\right]^{\frac{p}{p-q}}, \quad n \in I_\alpha,$$

that is,

$$z(n) \le \left[z^{\frac{p-q}{p}}(\alpha-1) + \frac{p-q}{p}\sum_{s=\alpha}^{n} f(s-\alpha)\right]^{\frac{p}{p-q}}, \quad n \in I_{\alpha-1}. \tag{3.16}$$

By (3.8), (3.9), and (3.16) we get

$$z(\alpha-1) \le k + \sum_{s=0}^{T-\alpha} G(s,T)\left[z^{\frac{p-q}{p}}(\alpha-1) + \frac{p-q}{p}\sum_{\tau=\alpha}^{s+\alpha-1} f(\tau-\alpha)\right]^{\frac{p}{p-q}}.$$

Therefore, using the inequality $(a+b)^\mu \le 2^{\mu-1}(a^\mu + b^\mu)$, $\mu \ge 1$, we have

$$z(\alpha-1) \le k + \sum_{s=0}^{T-\alpha} G(s,T)2^{\frac{q}{p-q}}\left\{z(\alpha-1) + \left[\frac{p-q}{p}\sum_{\tau=\alpha}^{s+\alpha-1} f(\tau-\alpha)\right]^{\frac{p}{p-q}}\right\}. \tag{3.17}$$

Hence, in view of (3.2), we obtain

$$z(\alpha-1) \le \frac{1}{1-\lambda}\left\{k + 2^{\frac{q}{p-q}}\sum_{s=0}^{T-\alpha} G(s,T)\left[\frac{p-q}{p}\sum_{\tau=\alpha}^{s+\alpha-1} f(\tau-\alpha)\right]^{\frac{p}{p-q}}\right\} = A(T), \tag{3.18}$$

where $A(T)$ is defined as in (3.4). From (3.16) and (3.18) we get

$$z(n) \le \left[A^{\frac{p-q}{p}}(T) + \frac{p-q}{p}\sum_{s=\alpha}^{n} f(s-\alpha)\right]^{\frac{p}{p-q}}, \quad n \in I_{\alpha-1}. \tag{3.19}$$

Using (3.9) and (3.19), we obtain

$$u(n) \le \left[A^{\frac{p-q}{p}}(T) + \frac{p-q}{p}\sum_{s=\alpha}^{n} f(s-\alpha)\right]^{\frac{1}{p-q}}, \quad n \in I_{\alpha-1}. \tag{3.20}$$

If $k = 0$, then we carry out the above procedure with $\varepsilon > 0$ instead of k and subsequently let $\varepsilon \to 0$. This completes the proof. □

Theorem 3.2 *Assume that* $0 < \alpha \le 1$ *is a constant,* $u : \mathbf{N}_{\alpha-1} \to \mathbf{R}_+$, $g, h : \mathbf{N}_0 \to \mathbf{R}_+$ *are functions,* $k \ge 0$ *is a constant,* $\varphi \in C^1(\mathbf{R}_+, \mathbf{R}_+)$ *is an increasing function with* $\varphi(\infty) = \infty$

on $\mathbf{R}_+$, and $\psi_i : \mathbf{R}_+ \to \mathbf{R}_+$ is a nondecreasing continuous function with $\psi_i(u) > 0$ for $u > 0$, $i = 1, 2$. Suppose that there is a nondecreasing continuous function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ such that both ψ_1 and ψ_2 are less than or equal to ψ, $\Psi(r) = \int_{r_0}^{r} \frac{1}{\psi(\varphi^{-1}(s))}\,\mathrm{d}s$, $r \geq r_0 > 0$, with $\lim_{r\to\infty} \Psi(r) = \infty$, and $\Omega(t) = \Psi(2t-k) - \Psi(t)$ is increasing for $t \geq k$. If u satisfies

$$\varphi\bigl(u(n)\bigr) \leq k + \Delta_0^{-\alpha}\bigl[g(n)\psi_1\bigl(u(n+\alpha-1)\bigr)\bigr] + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}h(s)\psi_2\bigl(u(s+\alpha-1)\bigr), \quad n \in I_{\alpha-1}, \tag{3.21}$$

then

$$u(n) \leq \varphi^{-1}\left\{\Psi^{-1}\left[\Psi\left(\Omega^{-1}\left(\sum_{s=\alpha}^{T} f(s-\alpha)\right)\right) + \sum_{s=\alpha}^{n} f(s-\alpha)\right]\right\}, \quad n \in I_{\alpha-1}, \tag{3.22}$$

where $f : \mathbf{N}_0 \to \mathbf{R}_+$ is a function such that both g and h are less than or equal to f, and φ^{-1}, Ψ^{-1}, Ω^{-1} are the inverse functions of φ, Ψ, Ω, respectively.

Proof Let $k > 0$. From the assumptions on g, h, ψ_i ($i = 1, 2$) and (3.21) we have

$$\begin{aligned}\varphi\bigl(u(n)\bigr) &\leq k + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}f(s)\psi\bigl(u(s+\alpha-1)\bigr) \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}f(s)\psi\bigl(u(s+\alpha-1)\bigr) \\ &= k + \sum_{s=0}^{n-\alpha}F(s,n)\psi\bigl(u(s+\alpha-1)\bigr) \\ &\quad + \sum_{s=0}^{T-\alpha}F(s,T)\psi\bigl(u(s+\alpha-1)\bigr), \quad n \in I_{\alpha-1},\end{aligned} \tag{3.23}$$

where

$$F(s,n) = \frac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}f(s).$$

Define

$$z(n) = k + \sum_{s=0}^{n-\alpha}F(s,n)\psi\bigl(u(s+\alpha-1)\bigr) + \sum_{s=0}^{T-\alpha}F(s,T)\psi\bigl(u(s+\alpha-1)\bigr), \quad n \in I_{\alpha-1}. \tag{3.24}$$

Then $z(n) \geq 0$ is nondecreasing,

$$z(\alpha-1) = k + \sum_{s=0}^{T-\alpha}F(s,T)\psi\bigl(u(s+\alpha-1)\bigr), \tag{3.25}$$

and

$$u(n) \leq \varphi^{-1}\big(z(n)\big), \quad n \in I_{\alpha-1}. \tag{3.26}$$

By the definitions of $F(s,n)$ and $t^{(\alpha)}$, we can easily get that $F(s,n)$ is decreasing in n for each $s \in \mathbf{N}_0$. So from (3.24) and a straightforward computation, for $n \in I_\alpha$, we obtain that

$$\begin{aligned} z(n) - z(n-1) &= F(n-\alpha,n)\psi\big(u(n-1)\big) \\ &\quad + \sum_{s=0}^{n-\alpha-1}\big[F(s,n) - F(s,n-1)\big]\psi\big(u(s+\alpha-1)\big) \\ &\leq F(n-\alpha,n)\psi\big(u(n-1)\big) \\ &\leq F(n-\alpha,n)\psi\big(\varphi^{-1}\big(z(n-1)\big)\big) \\ &= f(n-\alpha)\psi\big(\varphi^{-1}\big(z(n-1)\big)\big). \end{aligned} \tag{3.27}$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\varphi^{-1}\big(z(n-1)\big) > \varphi^{-1}\big(z(\alpha-1)\big) = \varphi^{-1}\left(k + \sum_{s=0}^{T-\alpha} F(s,T)\psi\big(u(s+\alpha-1)\big)\right) > 0, \quad n \in I_\alpha. \tag{3.28}$$

So from (3.27) and (3.28) we have

$$\frac{z(n) - z(n-1)}{\psi(\varphi^{-1}(z(n-1)))} \leq f(n-\alpha), \quad n \in I_\alpha,$$

that is,

$$\frac{\Delta z(n-1)}{\psi(\varphi^{-1}(z(n-1)))} \leq f(n-\alpha), \quad n \in I_\alpha. \tag{3.29}$$

On the other hand, by the mean value theorem and the monotonicity of φ^{-1} and ψ we obtain

$$\begin{aligned} \Delta\Psi\big(z(n-1)\big) &= \Psi\big(z(n)\big) - \Psi\big(z(n-1)\big) \\ &= \Psi'(\xi)\Delta z(n-1) = \frac{\Delta z(n-1)}{\psi(\varphi^{-1}(\xi))} \\ &\leq \frac{\Delta z(n-1)}{\psi(\varphi^{-1}(z(n-1)))}, \quad \xi \in \big[z(n-1), z(n)\big]. \end{aligned} \tag{3.30}$$

So from (3.29) and (3.30) we obtain

$$\Delta\Psi\big(z(n-1)\big) \leq f(n-\alpha), \quad n \in I_\alpha. \tag{3.31}$$

Setting $n = s$ in inequality (3.31) and summing with respect to s from α to $n-1$, we get

$$\sum_{s=\alpha}^{n-1} \Delta\Psi\big(z(s-1)\big) \leq \sum_{s=\alpha}^{n-1} f(s-\alpha),$$

that is,

$$\Psi\bigl(z(n-1)\bigr) \le \Psi\bigl(z(\alpha-1)\bigr) + \sum_{s=\alpha}^{n-1} f(s-\alpha), \quad n \in I_\alpha. \tag{3.32}$$

Then from inequality (3.32) we conclude that

$$z(n-1) \le \Psi^{-1}\left[\Psi\bigl(z(\alpha-1)\bigr) + \sum_{s=\alpha}^{n-1} f(s-\alpha)\right], \quad n \in I_\alpha,$$

that is,

$$z(n) \le \Psi^{-1}\left[\Psi\bigl(z(\alpha-1)\bigr) + \sum_{s=\alpha}^{n} f(s-\alpha)\right], \quad n \in I_{\alpha-1}. \tag{3.33}$$

By (3.24) we get that

$$2z(\alpha-1) - k = k + 2\sum_{s=0}^{T-\alpha} F(s,T)\psi\bigl(u(s+\alpha-1)\bigr) = z(T), \tag{3.34}$$

and then from (3.33) and (3.34) we have

$$2z(\alpha-1) - k = z(T) \le \Psi^{-1}\left[\Psi\bigl(z(\alpha-1)\bigr) + \sum_{s=\alpha}^{T} f(s-\alpha)\right],$$

that is,

$$\Psi\bigl(2z(\alpha-1)-k\bigr) - \Psi\bigl(z(\alpha-1)\bigr) \le \sum_{s=\alpha}^{T} f(s-\alpha). \tag{3.35}$$

Since $\Omega(t) = \Psi(2t-k) - \Psi(t)$ is increasing for $t \ge k$, and Ω has an inverse function Ω^{-1}, from (3.35) we get

$$z(\alpha-1) \le \Omega^{-1}\left(\sum_{s=\alpha}^{T} f(s-\alpha)\right). \tag{3.36}$$

Substituting (3.36) into (3.33), we have

$$z(n) \le \Psi^{-1}\left[\Psi\left(\Omega^{-1}\left(\sum_{s=\alpha}^{T} f(s-\alpha)\right)\right) + \sum_{s=\alpha}^{n} f(s-\alpha)\right], \quad n \in I_{\alpha-1}. \tag{3.37}$$

Combining (3.37) with (3.26), we obtain the desired inequality (3.22). If $k = 0$, then we carry out the above procedure with $\varepsilon > 0$ instead of k and subsequently let $\varepsilon \to 0$. This completes the proof. □

For the particular case $\varphi(u) = u$ and $\psi_1(u) = \psi_2(u) = u$, Theorem 3.2 gives the following discrete fractional sum inequality.

Corollary 3.1 *Let α, u, and k be defined as in Theorem* 3.2, *and $f : \mathbf{N}_0 \to \mathbf{R}_+$ be a function. If u satisfies*

$$u(n) \leq k + \Delta_0^{-\alpha}\big[f(n)u(n+\alpha-1)\big] + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}f(s)u(s+\alpha-1), \quad n \in I_{\alpha-1},$$

and

$$\lambda = \exp\left(\sum_{s=\alpha}^{T} f(s-\alpha)\right) < 2,$$

then

$$u(n) \leq \frac{k}{2-\lambda}\exp\left(\sum_{s=\alpha}^{n} f(s-\alpha)\right), \quad n \in I_{\alpha-1}. \tag{3.38}$$

Proof From the definitions of Ψ and Ω, letting $\psi(u) = u$, we obtain

$$\Psi(r) = \int_{r_0}^{r}\frac{1}{s}\,\mathrm{d}s = \ln\frac{r}{r_0}, \quad r \geq r_0 > 0,$$
$$\Omega(t) = \Psi(2t-k) - \Psi(t) = \ln\frac{2t-k}{t}, \quad t \geq k,$$

and hence $\Psi^{-1}(r) = r_0\exp(r)$ and $\Omega^{-1}(t) = \frac{k}{2-\exp(t)}$. From inequality (3.22) we obtain inequality (3.38). □

Theorem 3.3 *Assume that $0 < \alpha \leq 1$ is a constant, $u : \mathbf{N}_{\alpha-1} \to \mathbf{R}_+$, $g, h : \mathbf{N}_0 \to \mathbf{R}_+$ are functions, $k \geq 0$ is a constant, $\varphi \in C^1(\mathbf{R}_+, \mathbf{R}_+)$ with $\varphi(0) = 0$, $\varphi(\infty) = \infty$, and $\varphi'(u) > 0$ for $u > 0$, the derivative φ' is increasing on $\mathbf{R}_+$, and $\psi_i : \mathbf{R}_+ \to \mathbf{R}_+$ are nondecreasing continuous functions with $\psi_i(u) > 0$ for $u > 0$, $i = 1, 2$. Suppose that there is a nondecreasing continuous function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ such that both ψ_1 and ψ_2 are less than or equal to ψ, $G(r) = \int_{r_0}^{r}\frac{1}{\psi(s)}\,\mathrm{d}s$, $r \geq r_0 > 0$, with $\lim_{r\to\infty} G(r) = \infty$, and $\Omega(t) = G(\varphi^{-1}(2t-k)) - G(\varphi^{-1}(t))$ is increasing for $t \geq k$. If u satisfies*

$$\varphi\big(u(n)\big) \leq k + \Delta_0^{-\alpha}\big[g(n)\varphi'\big(u(n+\alpha-1)\big)\psi_1\big(u(n+\alpha-1)\big)\big] + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}h(s)\varphi'\big(u(s+\alpha-1)\big)\psi_2\big(u(s+\alpha-1)\big), \quad n \in I_{\alpha-1}, \tag{3.39}$$

then

$$u(n) \leq G^{-1}\left\{G\left[\varphi^{-1}\left(\Omega^{-1}\left(\sum_{s=\alpha}^{T} f(s-\alpha)\right)\right)\right] + \sum_{s=\alpha}^{n} f(s-\alpha)\right\}, \quad n \in I_{\alpha-1}, \tag{3.40}$$

where $f : \mathbf{N}_0 \to \mathbf{R}_+$ is a function such that both g and h are less than or equal to f, and φ^{-1}, G^{-1}, and Ω^{-1} are inverse functions of φ, G, and Ω, respectively.

Proof Let $k > 0$. From the assumptions on g, h, ψ_i ($i = 1, 2$) and (3.39) we have

$$\begin{aligned} \varphi\bigl(u(n)\bigr) &\le k + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n-\alpha} (n-s-1)^{(\alpha-1)} f(s) \varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T-\alpha} (T-s-1)^{(\alpha-1)} f(s) \varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr) \\ &= k + \sum_{s=0}^{n-\alpha} F(s,n)\varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr) \\ &\quad + \sum_{s=0}^{T-\alpha} F(s,T)\varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr), \quad n \in I_{\alpha-1}, \end{aligned} \tag{3.41}$$

where

$$F(s,n) = \frac{1}{\Gamma(\alpha)} (n-s-1)^{(\alpha-1)} f(s).$$

Define

$$\begin{aligned} z(n) &= k + \sum_{s=0}^{n-\alpha} F(s,n)\varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr) \\ &\quad + \sum_{s=0}^{T-\alpha} F(s,T)\varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr), \quad n \in I_{\alpha-1}. \end{aligned} \tag{3.42}$$

Then $z(n) \ge 0$ is nondecreasing,

$$z(\alpha-1) = k + \sum_{s=0}^{T-\alpha} F(s,T)\varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr), \tag{3.43}$$

and

$$u(n) \le \varphi^{-1}\bigl(z(n)\bigr), \quad n \in I_{\alpha-1}. \tag{3.44}$$

By the definition of $F(s,n)$ and $t^{(\alpha)}$, we can easily get that $F(s,n)$ is decreasing in n for each $s \in \mathbf{N}_0$. So from (3.42) and a straightforward computation, for $n \in I_\alpha$, we obtain that

$$\begin{aligned} &z(n) - z(n-1) \\ &\quad = F(n-\alpha,n)\varphi'\bigl(u(n-1)\bigr)\psi\bigl(u(n-1)\bigr) \\ &\qquad + \sum_{s=0}^{n-\alpha-1} \bigl[F(s,n) - F(s,n-1)\bigr]\varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr) \\ &\quad \le F(n-\alpha,n)\varphi'\bigl(u(n-1)\bigr)\psi\bigl(u(n-1)\bigr) \\ &\quad \le F(n-\alpha,n)\varphi'\bigl(\varphi^{-1}\bigl(z(n-1)\bigr)\bigr)\psi\bigl(\varphi^{-1}\bigl(z(n-1)\bigr)\bigr) \\ &\quad = f(n-\alpha)\varphi'\bigl(\varphi^{-1}\bigl(z(n-1)\bigr)\bigr)\psi\bigl(\varphi^{-1}\bigl(z(n-1)\bigr)\bigr). \end{aligned} \tag{3.45}$$

Using the monotonicity of φ', φ^{-1}, and z, we deduce

$$\begin{aligned}\varphi'\bigl(\varphi^{-1}\bigl(z(n-1)\bigr)\bigr) &\geq \varphi'\bigl(\varphi^{-1}\bigl(z(\alpha-1)\bigr)\bigr)\\ &= \varphi'\left(\varphi^{-1}\left(k+\sum_{s=0}^{T-\alpha}F(s,T)\varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr)\right)\right)\\ &> 0,\quad n\in I_\alpha.\end{aligned}\tag{3.46}$$

So from (3.45) and (3.46) we have

$$\frac{z(n)-z(n-1)}{\varphi'[\varphi^{-1}(z(n-1))]}\leq f(n-\alpha)\psi\bigl(\varphi^{-1}\bigl(z(n-1)\bigr)\bigr),\quad n\in I_\alpha,$$

that is,

$$\frac{\Delta z(n-1)}{\varphi'[\varphi^{-1}(z(n-1))]}\leq f(n-\alpha)\psi\bigl(\varphi^{-1}\bigl(z(n-1)\bigr)\bigr),\quad n\in I_\alpha.\tag{3.47}$$

On the other hand, by the mean value theorem and the monotonicity of φ' and φ^{-1} we have, for $n\in I_\alpha$,

$$\begin{aligned}\Delta\varphi^{-1}\bigl(z(n-1)\bigr) &= \varphi^{-1}\bigl(z(n)\bigr)-\varphi^{-1}\bigl(z(n-1)\bigr)\\ &= \frac{1}{\varphi'(\varphi^{-1}(\xi))}\Delta z(n-1)\\ &\leq \frac{\Delta z(n-1)}{\varphi'[\varphi^{-1}(z(n-1))]},\quad \xi\in\bigl[z(n-1),z(n)\bigr].\end{aligned}\tag{3.48}$$

So from (3.47) and (3.48) we obtain

$$\Delta\varphi^{-1}\bigl(z(n-1)\bigr)\leq f(n-\alpha)\psi\bigl(\varphi^{-1}\bigl(z(n-1)\bigr)\bigr).\tag{3.49}$$

Setting $n=s$ in inequality (3.49) and summing with respect to s from α to $n-1$, we get

$$\varphi^{-1}\bigl(z(n-1)\bigr)\leq\varphi^{-1}\bigl(z(\alpha-1)\bigr)+\sum_{s=\alpha}^{n-1}f(s-\alpha)\psi\bigl(\varphi^{-1}\bigl(z(s-1)\bigr)\bigr).$$

Now by applying Lemma 2.1 to the function $\varphi^{-1}(z(n-1))$ we have

$$\varphi^{-1}\bigl(z(n-1)\bigr)\leq G^{-1}\left(G\bigl(\varphi^{-1}\bigl(z(\alpha-1)\bigr)\bigr)+\sum_{s=\alpha}^{n-1}f(s-\alpha)\right),\quad n\in I_\alpha,$$

that is,

$$\varphi^{-1}\bigl(z(n)\bigr)\leq G^{-1}\left(G\bigl(\varphi^{-1}\bigl(z(\alpha-1)\bigr)\bigr)+\sum_{s=\alpha}^{n}f(s-\alpha)\right),\quad n\in I_{\alpha-1},\tag{3.50}$$

where $G(r)=\int_{r_0}^{r}\frac{1}{\psi(s)}\,\mathrm{d}s$. By (3.42) we get

$$2z(\alpha-1)-k=k+2\sum_{s=0}^{T-\alpha}F(s,T)\varphi'\bigl(u(s+\alpha-1)\bigr)\psi\bigl(u(s+\alpha-1)\bigr)=z(T),\tag{3.51}$$

and then from (3.50) and (3.51) we have

$$\begin{aligned}\varphi^{-1}\bigl(2z(\alpha-1)-k\bigr) &= \varphi^{-1}\bigl(z(T)\bigr) \\ &\le G^{-1}\left(G\bigl(\varphi^{-1}\bigl(z(\alpha-1)\bigr)\bigr)+\sum_{s=\alpha}^{T} f(s-\alpha)\right),\end{aligned}$$

that is,

$$G\bigl(\varphi^{-1}\bigl(2z(\alpha-1)-k\bigr)\bigr)-G\bigl(\varphi^{-1}\bigl(z(\alpha-1)\bigr)\bigr)\le\sum_{s=\alpha}^{T} f(s-\alpha). \tag{3.52}$$

Since $\Omega(t)=G(\varphi^{-1}(2t-k))-G(\varphi^{-1}(t))$ is increasing for $t\ge k$ and Ω has an inverse function Ω^{-1}, from (3.52) we get

$$z(\alpha-1)\le\Omega^{-1}\left(\sum_{s=\alpha}^{T} f(s-\alpha)\right). \tag{3.53}$$

Substituting (3.53) into (3.50) we have

$$\varphi^{-1}\bigl(z(n)\bigr)\le G^{-1}\left\{G\left[\varphi^{-1}\left(\Omega^{-1}\left(\sum_{s=\alpha}^{T} f(s-\alpha)\right)\right)\right]+\sum_{s=\alpha}^{n} f(s-\alpha)\right\},\quad n\in I_{\alpha-1}. \tag{3.54}$$

Combining (3.44) with (3.54), we obtain the desired inequality (3.40). If $k=0$, then we carry out the above procedure with $\varepsilon>0$ instead of k and subsequently let $\varepsilon\to0$. This completes the proof. □

For the particular case $\varphi(u)=u^p$ ($p\ge1$ is a constant), Theorem 3.3 gives the following discrete fractional sum inequality.

Corollary 3.2 *Let α, u, k, g, h, f, ψ_1, ψ_2, ψ, and G be defined as in Theorem* 3.3, *and $p\ge1$ be a constant. If u satisfies*

$$\begin{aligned}u^p(n) &\le k+\Delta_0^{-\alpha}\bigl[pg(n)u^{p-1}(n+\alpha-1)\psi_1\bigl(u(n+\alpha-1)\bigr)\bigr] \\ &\quad+\frac{p}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}h(s)u^{p-1}(s+\alpha-1)\psi_2\bigl(u(s+\alpha-1)\bigr), \\ &\quad n\in I_{\alpha-1},\end{aligned}$$

and $\Omega(t)=G((2t-k)^{\frac{1}{p}})-G(t^{\frac{1}{p}})$ is increasing for $t\ge k$, then

$$u(n)\le G^{-1}\left\{G\left[\left(\Omega^{-1}\left(\sum_{s=\alpha}^{T} f(s-\alpha)\right)\right)^{\frac{1}{p}}\right]+\sum_{s=\alpha}^{n} f(s-\alpha)\right\},\quad n\in I_{\alpha-1},$$

where G^{-1} and Ω^{-1} are inverse functions of G and Ω, respectively.

4 Applications

In this section, we apply our results to study the boundedness and uniqueness of the solutions of Volterra-Fredholm fractional sum-difference equations of the form

$$u^p(n) = k + \Delta_0^{-\alpha}\big[F\big(n, u(n+\alpha-1)\big)\big] + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}G\big(s, u(s+\alpha-1)\big), \quad n \in I_{\alpha-1}, \tag{4.1}$$

where $0 < \alpha < 1$ and $p > 0$ are constants, u is an unknown function defined on $\mathbf{N}_{\alpha-1}$, and $F, G : \mathbf{N}_0 \times \mathbf{R} \to \mathbf{R}$ are functions.

The following theorem gives the bound on the solution of Eq. (4.1).

Theorem 4.1 *For Eq.* (4.1), *assume that there exist functions $f, g : \mathbf{N}_0 \to \mathbf{R}_+$ and a constant q satisfying $p > q > 0$ such that*

$$\big|F(n,u)\big| \le f(n)|u|^q, \qquad \big|G(n,u)\big| \le g(n)|u|^p \quad \text{for any } n \in \mathbf{N}_0, u \in \mathbf{R}, \tag{4.2}$$

and

$$\lambda = 2^{\frac{q}{p-q}}\sum_{s=0}^{T-\alpha}\frac{1}{\Gamma(\alpha)}(T-s-1)^{(\alpha-1)}g(s) < 1.$$

If u is any solution of Eq. (4.1), *then*

$$u(n) \le \left[A^{\frac{p-q}{p}}(T) + \frac{p-q}{p}\sum_{s=\alpha}^{n}f(s-\alpha)\right]^{\frac{1}{p-q}}, \quad n \in I_{\alpha-1}, \tag{4.3}$$

where $A(T)$ is as in Theorem 3.1.

Proof From (4.1) and (4.2) we get

$$\begin{aligned} \big|u(n)\big|^p &\le |k| + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\big|F\big(s, u(s+\alpha-1)\big)\big| \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}\big|G\big(s, u(s+\alpha-1)\big)\big| \\ &\le |k| + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}f(s)\big|u(s+\alpha-1)\big|^q \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}g(s)\big|u(s+\alpha-1)\big|^p. \end{aligned} \tag{4.4}$$

Now, a suitable application of the inequality given in Theorem 3.1 to (4.4) yields the desired result. This completes the proof. □

Secondly, we consider the Volterra-Fredholm fractional sum-difference equations of the form

$$\begin{aligned} u(n) &= k + \Delta_0^{-\alpha}\big[F\big(n, u(n+\alpha-1)\big)\big] \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}F\big(s, u(s+\alpha-1)\big), \quad n \in I_{\alpha-1}. \end{aligned} \tag{4.5}$$

The next result deals with the uniqueness of solutions of Eq. (4.5).

Theorem 4.2 *For Eq.* (4.5), *assume that there exists a function* $f : \mathbf{N}_0 \to \mathbf{R}_+$ *satisfying* $\exp(\sum_{s=\alpha}^{T} f(s-\alpha)) < 2$ *and*

$$\big|F(n,u) - F(n,v)\big| \le f(n)|u-v| \quad \textit{for any } n \in \mathbf{N}_0, u, v \in \mathbf{R}. \tag{4.6}$$

Then Eq. (4.5) *has at most one solution.*

Proof Suppose that Eq. (4.5) has two solutions $u_1(n)$ and $u_2(n)$. Then we have

$$\begin{aligned} u_1(n) &= k + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}F\big(s, u_1(s+\alpha-1)\big) \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}F\big(s, u_1(s+\alpha-1)\big), \\ u_2(n) &= k + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}F\big(s, u_2(s+\alpha-1)\big) \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}F\big(s, u_2(s+\alpha-1)\big). \end{aligned}$$

Furthermore,

$$\begin{aligned} &u_1(n) - u_2(n) \\ &\quad = \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\big[F\big(s, u_1(s+\alpha-1)\big) - F\big(s, u_2(s+\alpha-1)\big)\big] \\ &\qquad + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}\big[F\big(s, u_(s+\alpha-1)\big) - F\big(s, u_2(s+\alpha-1)\big)\big]. \end{aligned}$$

From (4.6) we have

$$\begin{aligned} &\big|u_1(n) - u_2(n)\big| \\ &\quad \le \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}f(s)\big|u_1(s+\alpha-1) - u_2(s+\alpha-1)\big| \\ &\qquad + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}f(s)\big|u_1(s+\alpha-1) - u_2(s+\alpha-1)\big| \end{aligned}$$

$$= \Delta_0^{-\alpha}\big[f(n)\big|u_1(n+\alpha-1)-u_2(n+\alpha-1)\big|\big]$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-\alpha}(T-s-1)^{(\alpha-1)}f(s)\big|u_1(s+\alpha-1)-u_2(s+\alpha-1)\big|. \quad (4.7)$$

With respect to the function $|u_1(n)-u_2(n)|$, by a suitable application of Corollary 3.1 to (4.7) we can deduce that $|u_1(n)-u_2(n)|\leq 0$, which implies $u_1(n)\equiv u_2(n)$. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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