**RESEARCH**  **Open Access**

CrossMark

# On the Hausdorff measure of noncompactness for the parameterized Prokhorov metric

Ben Berckmoes[*]

[*]Correspondence:
ben.berckmoes@uantwerpen.be
Department of Mathematics and
Computer Science, University of
Antwerp, Antwerpen, Belgium

**Abstract**

We quantify the Prokhorov theorem by establishing an explicit formula for the Hausdorff measure of noncompactness (HMNC) for the parameterized Prokhorov metric on the set of Borel probability measures on a Polish space. Furthermore, we quantify the Arzelà-Ascoli theorem by obtaining upper and lower estimates for the HMNC for the uniform norm on the space of continuous maps of a compact interval into Euclidean $N$-space, using Jung's theorem on the Chebyshev radius. Finally, we combine the obtained results to quantify the stochastic Arzelà-Ascoli theorem by providing upper and lower estimates for the HMNC for the parameterized Prokhorov metric on the set of multivariate continuous stochastic processes.

**Keywords:** Arzelà-Ascoli theorem; Chebyshev radius; Hausdorff measure of noncompactness; Jung's theorem; parametrized Prokhorov metric; Prokhorov's theorem; stochastic Arzelà-Ascoli theorem

## 1 Introduction and statement of the main results

For the basic probabilistic concepts and results, we refer the reader to any standard work on probability theory such as [1].

Let $S$ be a Polish space, that is, a separable completely metrizable topological space, and $\mathcal{P}(S)$ the collection of Borel probability measures on $S$, equipped with the weak topology $\tau_w$, that is, the weakest topology for which each map

$$\mathcal{P}(S) \to \mathbb{R} : P \mapsto \int f \, dP$$

with bounded and continuous $f : S \to \mathbb{R}$ is continuous. The space $\mathcal{P}(S)$ is known to be Polish.

We call a collection $\Gamma \subset \mathcal{P}(S)$ uniformly tight iff for each $\epsilon > 0$, there exists a compact set $K \subset S$ such that $P(S \setminus K) < \epsilon$ for all $P \in \Gamma$.

The following celebrated result interrelates the $\tau_w$-relative compactness with uniform tightness.

**Theorem 1.1** (Prokhorov) *A collection $\Gamma \subset \mathcal{P}(S)$ is $\tau_w$-relatively compact if and only if it is uniformly tight.*

Fix $N \in \mathbb{N}_0$ and let $\mathcal{C}$ be the space of continuous maps $x$ of the compact interval $[0,1]$ into Euclidean $N$-space $\mathbb{R}^N$ equipped with the uniform topology $\tau_\infty$, that is, the topology derived from the uniform norm

$$\|x\|_\infty = \sup_{t \in [0,1]} |x(t)|,$$

where $|\cdot|$ stands for the Euclidean norm. The space $\mathcal{C}$ is also known to be Polish.

Recall that a set $\mathcal{X} \subset \mathcal{C}$ is said to be uniformly bounded iff there exists a constant $M > 0$ such that $|x(t)| \leq M$ for all $x \in \mathcal{X}$ and $t \in [0,1]$, and uniformly equicontinuous iff for each $\epsilon > 0$, there exists $\delta > 0$ such that $|x(s) - x(t)| < \epsilon$ for all $x \in \mathcal{X}$ and $s, t \in [0,1]$ with $|s - t| < \delta$.

In this setting, the following theorem is a classical result [2].

**Theorem 1.2** (Arzelà-Ascoli) *A collection $\mathcal{X} \subset \mathcal{C}$ is $\tau_\infty$-relatively compact if and only if it is uniformly bounded and uniformly equicontinuous.*

Let $\Omega = (\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. Throughout, a continuous stochastic process (c.s.p.) is a Borel-measurable map of $\Omega$ into $\mathcal{C}$, and we consider on the set of c.s.p.s the weak topology $\tau_w$, that is, the topology with open sets $\{\xi$ c.s.p. $\mid \mathbb{P}_\xi \in \mathcal{G}\}$, where $\mathbb{P}_\xi$ is the probability distribution of $\xi$, and $\mathcal{G}$ is a $\tau_w$-open set in $\mathcal{P}(\mathcal{C})$.

A collection $\Xi$ of c.s.p.s is said to be stochastically uniformly bounded iff for each $\epsilon > 0$, there exists $M > 0$ such that $\mathbb{P}(\|\xi\|_\infty > M) < \epsilon$ for all $\xi \in \Xi$, and stochastically uniformly equicontinuous iff for all $\epsilon, \epsilon' > 0$, there exists $\delta > 0$ such that $\mathbb{P}(\sup_{|s-t|<\delta} |\xi(s) - \xi(t)| \geq \epsilon) < \epsilon'$ for all $\xi \in \Xi$, the supremum taken over all $s, t \in [0,1]$ for which $|s - t| < \delta$.

It is not hard to see that combining Theorem 1.1 and Theorem 1.2 yields the following stochastic version of Theorem 1.2, which plays a crucial role in the development of functional central limit theory.

**Theorem 1.3** (Stochastic Arzelà-Ascoli) *A collection $\Xi$ of c.s.p.s is $\tau_w$-relatively compact if and only if it is stochastically uniformly bounded and stochastically uniformly equicontinuous.*

In a complete metric space $(X, d)$, the Hausdorff measure of noncompactness of a set $A \subset X$ [3, 4] is given by

$$\mu_{\mathrm{H},d}(A) = \inf_F \sup_{x \in A} \inf_{y \in F} d(x, y),$$

the first infimum running through all finite sets $F \subset X$. It is well known that $A$ is $d$-bounded if and only if $\mu_{\mathrm{H},d}(A) < \infty$, and $d$-relatively compact if and only if $\mu_{\mathrm{H},d}(A) = 0$.

Fix a complete metric $d$ metrizing the topology of the Polish space $S$. The Prokhorov distance with parameter $\lambda \in \mathbb{R}_0^+$ between probability measures $P, Q \in \mathcal{P}(S)$ [5] is defined as the infimum of all numbers $\alpha \in \mathbb{R}_0^+$ such that

$$P(A) \leq Q(A^{(\lambda\alpha)}) + \alpha$$

for all Borel sets $A \subset S$, where

$$A^{(\epsilon)} = \left\{ x \in S \mid \inf_{a \in A} d(a, x) \leq \epsilon \right\}.$$

This distance is denoted by $\rho_\lambda(P, Q)$. It defines a complete metric on $\mathcal{P}(S)$ and induces the weak topology $\tau_w$. It is also known that $\rho_{\lambda_1} \leq \rho_{\lambda_2}$ if $\lambda_1 \geq \lambda_2$ and that

$$\sup_{\lambda \in \mathbb{R}_0^+} \rho_\lambda(P, Q) = \sup_A |P(A) - Q(A)|,$$

the supremum being taken over all Borel sets $A \subset S$.

For a collection $\Gamma \subset \mathcal{P}(S)$, we define the measure of nonuniform tightness as

$$\mu_{\mathrm{ut}}(\Gamma) = \sup_{\epsilon > 0} \inf_Y \sup_{P \in \Gamma} P\left(S \setminus \bigcup_{y \in Y} B(y, \epsilon)\right),$$

where the infimum runs through all finite sets $Y \subset S$, and

$$B(y, \epsilon) = \{x \in S \mid d(y, x) < \epsilon\}.$$

It is clear that $\mu_{\mathrm{ut}}(\Gamma) = 0$ if $\Gamma$ is uniformly tight. The converse holds as well. Indeed, suppose that $\mu_{\mathrm{ut}}(\Gamma) = 0$ and fix $\epsilon > 0$. Then, for each $n \in \mathbb{N}_0$, choose a finite set $Y_n \subset S$ such that

$$P\left(S \setminus \bigcup_{y \in Y_n} B(y, 1/n)\right) < \epsilon/2^n$$

for all $P \in \Gamma$. Put

$$K = \bigcap_{n \in \mathbb{N}_0} \bigcup_{y \in Y_n} B^\star(y, 1/n)$$

with $B^\star(y, 1/n)$ the closure of $B(y, 1/n)$. Then $K$ is a compact set such that $P(S \setminus K) < \epsilon$ for all $P \in \Gamma$. We conclude that $\Gamma$ is uniformly tight. The measure $\mu_{\mathrm{ut}}$ is slightly weaker than the weak measure of tightness studied in [6].

By the previous considerations we know that a set $\Gamma \subset \mathcal{P}(S)$ is $\tau_w$-relatively compact if and only if $\mu_{\mathrm{H}, \rho_\lambda}(\Gamma) = 0$ for each $\lambda \in \mathbb{R}_0^+$, and uniformly tight if and only if $\mu_{\mathrm{ut}}(\Gamma) = 0$. Therefore, Theorem 1.4, our first main result, which provides a quantitative relation between the numbers $\mu_{\mathrm{H}, \rho_\lambda}(\Gamma)$ and $\mu_{\mathrm{ut}}(\Gamma)$, is a strict generalization of Theorem 1.1. The proof is given in Section 2.

**Theorem 1.4** (Quantitative Prokhorov) *For a collection $\Gamma \subset \mathcal{P}(S)$,*

$$\sup_{\lambda \in \mathbb{R}_0^+} \mu_{\mathrm{H}, \rho_\lambda}(\Gamma) = \mu_{\mathrm{ut}}(\Gamma).$$

From now on, we consider on the space $\mathcal{C}$ the uniform metric derived from the uniform norm, and for a set $\mathcal{X} \subset \mathcal{C}$, we let $\mu_{\mathrm{H}, \infty}(\mathcal{X})$ stand for the Hausdorff measure of noncompactness; more precisely,

$$\mu_{\mathrm{H}, \infty}(\mathcal{X}) = \inf_{\mathcal{F}} \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{F}} \|x - y\|_\infty,$$

the infimum taken over all finite sets $\mathcal{F} \subset \mathcal{C}$. Clearly, $\mathcal{X}$ is $\tau_\infty$-relatively compact if and only if $\mu_{\mathrm{H},\infty}(\mathcal{X}) = 0$.

The measure of nonuniform equicontinuity of $\mathcal{X} \subset \mathcal{C}$ is defined by

$$\mu_{\mathrm{uec}}(\mathcal{X}) = \inf_{\delta > 0} \sup_{x \in \mathcal{X}} \sup_{|s-t| < \delta} |x(s) - x(t)|,$$

the second supremum running through all $s, t \in [0,1]$ with $|s - t| < \delta$. We readily see that $\mathcal{X}$ is uniformly equicontinuous if and only if $\mu_{\mathrm{uec}}(\mathcal{X}) = 0$. In [3], it was shown that $\mu_{\mathrm{uec}}$ is a measure of noncompactness on the space $\mathcal{C}$ (Theorem 11.2).

Theorem 1.5, our second main result, entails that the measures $\mu_{\mathrm{H},\infty}$ and $\mu_{\mathrm{uec}}$ are Lipschitz equivalent on the collection of uniformly bounded subsets of $\mathcal{C}$, and thus it strictly generalizes Theorem 1.2. The proof, which hinges upon a classical result of Jung on the Chebyshev radius, is given in Section 3.

**Theorem 1.5** (Quantitative Arzelà-Ascoli) *For $\mathcal{X} \subset \mathcal{C}$,*

$$\frac{1}{2}\mu_{\mathrm{uec}}(\mathcal{X}) \leq \mu_{\mathrm{H},\infty}(\mathcal{X}).$$

*Suppose, in addition, that $\mathcal{X}$ is uniformly bounded. Then*

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \leq \left(\frac{N}{2N+2}\right)^{1/2} \mu_{\mathrm{uec}}(\mathcal{X}).$$

*In particular, if $N = 1$, then*

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) = \frac{1}{2}\mu_{\mathrm{uec}}(\mathcal{X}),$$

*and, regardless of $N$,*

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \leq \frac{\sqrt{2}}{2}\mu_{\mathrm{uec}}(\mathcal{X}).$$

We transport the parameterized Prokhorov metric from $\mathcal{P}(\mathcal{C})$ to the collection of c.s.p.s via their probability distributions. Thus, for c.s.p.s $\xi$ and $\eta$,

$$\rho_\lambda(\xi, \eta) = \rho_\lambda(\mathbb{P}_\xi, \mathbb{P}_\eta).$$

Note that a set of c.s.p.s $\Xi$ is $\tau_\omega$-relatively compact if and only if $\mu_{\mathrm{H},\rho_\lambda}(\Xi) = 0$ for all $\lambda \in \mathbb{R}_0^+$.

For a set of c.s.p.s $\Xi$, the measure of nonstochastic uniform boundedness is given by

$$\mu_{\mathrm{sub}}(\Xi) = \inf_{M \in \mathbb{R}_0^+} \sup_{\xi \in \Xi} \mathbb{P}(\|\xi\|_\infty > M),$$

and the measure of nonstochastic uniform equicontinuity by

$$\mu_{\mathrm{suec}}(\Xi) = \sup_{\epsilon > 0} \inf_{\delta > 0} \sup_{\xi \in \Xi} \mathbb{P}\left(\sup_{|s-t| < \delta} |\xi(s) - \xi(t)| \geq \epsilon\right),$$

where the third supremum is taken over all $s, t \in [0, 1]$ with $|s - t| < \delta$. It is easily seen that $\Xi$ is stochastically uniformly bounded if and only if $\mu_{\mathrm{sub}}(\Xi) = 0$, and stochastically uniformly equicontinuous if and only if $\mu_{\mathrm{suec}}(\Xi) = 0$. The measure $\mu_{\mathrm{suec}}$ was studied in [6].

In Section 4, we prove that combining Theorem 1.4 and Theorem 1.5 leads to Theorem 1.6, our third main result, which gives upper and lower bounds for $\sup_{\lambda \in \mathbb{R}_0^+} \mu_{\mathrm{H}, \rho_\lambda}$ in terms of $\mu_{\mathrm{sub}}$ and $\mu_{\mathrm{suec}}$. Theorem 1.6 strictly generalizes Theorem 1.3.

**Theorem 1.6** (Quantitative stochastic Arzelà-Ascoli) *Let $\Xi$ be a collection of c.s.p.s. Then*

$$\max\left\{\mu_{\mathrm{sub}}(\Xi), \mu_{\mathrm{suec}}(\Xi)\right\} \leq \sup_{\lambda \in \mathbb{R}_0^+} \mu_{\mathrm{H}, \rho_\lambda}(\Xi) \leq \mu_{\mathrm{sub}}(\Xi) + \mu_{\mathrm{suec}}(\Xi).$$

*In particular, if $\Xi$ is stochastically uniformly bounded, then*

$$\sup_{\lambda \in \mathbb{R}_0^+} \mu_{\mathrm{H}, \rho_\lambda}(\Xi) = \mu_{\mathrm{suec}}(\Xi),$$

*and, if $\Xi$ is stochastically uniformly equicontinuous, then*

$$\sup_{\lambda \in \mathbb{R}_0^+} \mu_{\mathrm{H}, \rho_\lambda}(\Xi) = \mu_{\mathrm{sub}}(\Xi).$$

## 2　Proof of Theorem 1.4

For a collection $\Gamma \subset \mathcal{P}(S)$, put

$$p_\Gamma = \sup_{\lambda \in \mathbb{R}_0^+} \mu_{\mathrm{H}, \rho_\lambda}(\Gamma)$$

and

$$t_\Gamma = \mu_{\mathrm{ut}}(\Gamma).$$

We first show that

$$p_\Gamma \leq t_\Gamma$$

with an argument that essentially refines the first part of the proof of Theorem 4.9 in [6].

Fix $\lambda \in \mathbb{R}_0^+$, $\epsilon > 0$, and choose pairwise disjoint Borel sets

$$A_1, \ldots, A_n \subset S$$

with diameters less than $\lambda \epsilon$ such that

$$\forall P \in \Gamma: \quad P\left(S \setminus \bigcup_{i=1}^n A_i\right) \leq t_\Gamma + \epsilon/2.$$

Then, for each $i \in \{1, \ldots, n\}$, pick $x_i \in A_i$, and, assuming without loss of generality that $S \setminus \bigcup_{i=1}^n A_i$ is nonempty, $x_{n+1} \in S \setminus \bigcup_{i=1}^n A_i$. Finally, fix $m \in \mathbb{N}_0$ such that

$$n/m \leq \epsilon/2,$$

and let $\Phi$ be a finite collection of Borel probability measures on $S$ of the form

$$Q = \sum_{i=1}^{n+1} (k_i/m)\delta_{x_i},$$

where the $k_i$ range in $\{0, \ldots, m\}$ so that

$$\sum_{i=1}^{n+1} k_i = m,$$

and $\delta_{x_i}$ stands for the Dirac probability measure putting all its mass on $x_i$.

We now claim that

$$\forall P \in \Gamma, \exists Q \in \Phi: \quad \rho_\lambda(P, Q) \le t_\Gamma + \epsilon,$$

which finishes the proof of the desired inequality.

To prove the claim, take $P \in \Gamma$ and construct

$$Q = \sum_{i=1}^{n+1} (k_i/m)\delta_{x_i}$$

in $\Phi$ such that

$$P(A_i) \le k_i/m + 1/m$$

for all $i \in \{1, \ldots, n\}$. For a Borel set $A \subset S$, let $I$ stand for the set of those $i \in \{1, \ldots, n\}$ for which $A_i \cap A$ is nonempty. Then we derive from the calculation

$$
\begin{aligned}
P(A) &\le P\left(\bigcup_{i \in I} A_i\right) + P\left(S \setminus \bigcup_{i=1}^{n} A_i\right) \\
&\le \sum_{i \in I} P(A_i) + t_\Gamma + \epsilon/2 \\
&\le \sum_{i \in I} (k_i/m + 1/m) + t_\Gamma + \epsilon/2 \\
&\le Q\left(\bigcup_{i \in I} A_i\right) + n/m + t_\Gamma + \epsilon/2 \\
&\le Q\left(A^{(\lambda(t_\Gamma + \epsilon))}\right) + t_\Gamma + \epsilon
\end{aligned}
$$

that

$$\rho_\lambda(P, Q) \le t_\Gamma + \epsilon,$$

establishing the claim.

We now show that

$$t_\Gamma \le p_\Gamma.$$

Fix $\epsilon, \epsilon' > 0$. Choose $\lambda \in \mathbb{R}_0^+$ such that

$$\lambda\left(p_\Gamma + \epsilon'/2\right) \leq \epsilon/2$$

and take a finite collection $\Phi \subset \mathcal{P}(S)$ such that, for each $P \in \Gamma$, there exists $Q \in \Phi$ for which

$$\rho_\lambda(P, Q) \leq \mu_{\mathrm{H}, \rho_\lambda}(\Gamma) + \epsilon'/2 \leq p_\Gamma + \epsilon'/2.$$

The collection $\Phi$ being finite, we can pick a finite set $Y \subset S$ such that

$$\forall Q \in \Phi: \quad Q\left(S \setminus \bigcup_{y \in Y} B(y, \epsilon/2)\right) \leq \epsilon'/2.$$

We claim that

$$\forall P \in \Gamma: \quad P\left(S \setminus \bigcup_{y \in Y} B(y, \epsilon)\right) \leq p_\Gamma + \epsilon',$$

proving the desired inequality.

To establish the claim, take $P \in \Gamma$, and let $Q$ be a probability measure in $\Phi$ such that

$$\rho_\lambda(P, Q) \leq p_\Gamma + \epsilon'/2.$$

Then

$$P\left(S \setminus \bigcup_{y \in Y} B(y, \epsilon)\right)$$

$$\leq Q\left(\left(S \setminus \bigcup_{y \in Y} B(y, \epsilon)\right)^{(\lambda(p_\Gamma + \epsilon'/2))}\right) + p_\Gamma + \epsilon'/2$$

$$\leq Q\left(\left(S \setminus \bigcup_{y \in Y} B(y, \epsilon)\right)^{(\epsilon/2)}\right) + p_\Gamma + \epsilon'/2$$

$$\leq Q\left(S \setminus \bigcup_{y \in Y} B(y, \epsilon/2)\right) + p_\Gamma + \epsilon'/2$$

$$\leq p_\Gamma + \epsilon',$$

which finishes the proof of the claim.

## 3 Proof of Theorem 1.5

Before writing down the proof of Theorem 1.5, we give the required preparation.

For a bounded set $A \subset \mathbb{R}^N$, its diameter is given by

$$\mathrm{diam}(A) = \sup_{x, y \in A} |x - y|,$$

and the Chebyshev radius by

$$r(A) = \inf_{x \in \mathbb{R}^N} \sup_{y \in A} |x - y|.$$

It is well known that, for each bounded set $A \subset \mathbb{R}^N$, there exists a unique $x_A \in \mathbb{R}^N$ such that

$$\sup_{y \in A} |x_A - y| = r(A).$$

The point $x_A$ is called the Chebyshev center of $A$. A good exposition of the previous notions in a general normed vector space can be found in [7], Section 33.

Theorem 3.1 provides a relation between the diameter and the Chebyshev radius of a bounded set in $\mathbb{R}^N$. A beautiful proof can be found in [8]. For extensions of the result, we refer to [9–11], and [12].

**Theorem 3.1** (Jung) *Let $A \subset \mathbb{R}^N$ be a bounded set. Then*

$$\frac{1}{2} \operatorname{diam}(A) \leq r(A) \leq \left( \frac{N}{2N + 2} \right)^{1/2} \operatorname{diam}(A).$$

We need two additional simple lemmas on linear interpolation.

For $c_0 \in \mathbb{R}^N$ and $r \in \mathbb{R}_0^+$, we denote by $B^\star(c_0, r)$ the closed ball with center $c_0$ and radius $r$.

**Lemma 3.2** *Consider $c_1, c_2 \in \mathbb{R}^N$ and $r \in \mathbb{R}_0^+$, and assume that*

$$B^\star(c_1, r) \cap B^\star(c_2, r) \neq \emptyset.$$

*Let $L$ be the $\mathbb{R}^N$-valued map on the compact interval $[\alpha, \beta]$ defined by*

$$L(t) = \frac{\beta - t}{\beta - \alpha} c_1 + \frac{t - \alpha}{\beta - \alpha} c_2.$$

*Then, for all $t \in [\alpha, \beta]$ and $y \in B^\star(c_1, r) \cap B^\star(c_2, r)$,*

$$\left| L(t) - y \right| \leq r.$$

*Proof* The calculation

$$
\begin{aligned}
\left| L(t) - y \right| &= \left| \frac{\beta - t}{\beta - \alpha}(c_1 - y) + \frac{t - \alpha}{\beta - \alpha}(c_2 - y) \right| \\
&\leq \frac{\beta - t}{\beta - \alpha}|c_1 - y| + \frac{t - \alpha}{\beta - \alpha}|c_2 - y| \\
&\leq \frac{\beta - t}{\beta - \alpha}r + \frac{t - \alpha}{\beta - \alpha}r \\
&= r
\end{aligned}
$$

proves the lemma. □

**Lemma 3.3** *Consider $c_1, c_2, y_1, y_2 \in \mathbb{R}^N$ and $\epsilon > 0$, and suppose that*

$$|c_1 - y_1| \leq \epsilon$$

*and*

$$|c_2 - y_2| \leq \epsilon.$$

*Let L and M be the $\mathbb{R}^N$-valued maps on the compact interval $[\alpha, \beta]$ defined by*

$$L(t) = \frac{\beta - t}{\beta - \alpha} c_1 + \frac{t - \alpha}{\beta - \alpha} c_2$$

*and*

$$M(t) = \frac{\beta - t}{\beta - \alpha} y_1 + \frac{t - \alpha}{\beta - \alpha} y_2.$$

*Then*

$$\|L - M\|_\infty \leq \epsilon.$$

*Proof* It is analogous to the proof of Lemma 3.2. □

*Proof of Theorem* 1.5  We first prove that

$$\frac{1}{2} \mu_{\mathrm{uec}}(\mathcal{X}) \leq \mu_{\mathrm{H},\infty}(\mathcal{X}).$$

Let $\alpha > 0$ be such that $\mu_{\mathrm{H},\infty}(\mathcal{X}) < \alpha$. Then there exists a finite set $\mathcal{F} \subset \mathcal{C}$ such that, for all $x \in \mathcal{X}$, there exists $y \in \mathcal{F}$ for which $\|y - x\|_\infty \leq \alpha$. Take $\epsilon > 0$. Since $\mathcal{F}$ is uniformly equicontinuous, there exists $\delta > 0$ such that

$$\forall y \in \mathcal{F}, \forall s, t \in [0,1]: \quad |s - t| < \delta \quad \Rightarrow \quad |y(s) - y(t)| \leq \epsilon. \tag{1}$$

Now, for $x \in \mathcal{X}$, choose $y \in \mathcal{F}$ such that

$$\|y - x\|_\infty \leq \alpha. \tag{2}$$

Then, for $s, t \in [0,1]$ with $|s - t| < \delta$, we have, by (1) and (2),

$$|x(s) - x(t)| \leq |x(s) - y(s)| + |y(s) - y(t)| + |y(t) - x(t)| \leq 2\alpha + \epsilon,$$

which, by the arbitrariness of $\epsilon$, reveals that $\mu_{\mathrm{uec}}(\mathcal{X}) \leq 2\alpha$, and thus, by the arbitrariness of $\alpha$, we have the inequality

$$\frac{1}{2} \mu_{\mathrm{uec}}(\mathcal{X}) \leq \mu_{\mathrm{H},\infty}(\mathcal{X}).$$

Next, assume that $\mathcal{X} \subset \mathcal{C}$ is uniformly bounded. We show that

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \leq \left( \frac{N}{2N+2} \right)^{1/2} \mu_{\mathrm{uec}}(\mathcal{X}).$$

Fix $\epsilon > 0$. Then, $\mathcal{X}$ being uniformly bounded, we can take a constant $M > 0$ such that

$$\forall x \in \mathcal{X}, \forall t \in [0,1]: \quad \left| x(t) \right| \leq M. \tag{3}$$

Pick a finite set $Y \subset \mathbb{R}^N$ for which

$$\forall z \in B^{\star}(0,3M), \exists y \in Y: \quad |y - z| \leq \epsilon. \tag{4}$$

Now let

$$0 < \alpha \leq 2M \tag{5}$$

be such that $\mu_{\mathrm{uec}}(\mathcal{X}) < \alpha$, that is, there exists $\delta > 0$ such that

$$\forall x \in \mathcal{X}, \forall s, t \in [0,1]: \quad |s - t| < \delta \quad \Rightarrow \quad \left| x(s) - x(t) \right| \leq \alpha. \tag{6}$$

Then choose points

$$0 = t_0 < t_1 < \cdots < t_{2n} < t_{2n+1} = 1,$$

put

$$I_0 = [0, t_2[,$$
$$I_k = ]t_{2k-1}, t_{2k+2}[ \quad \text{if } k \in \{1, \ldots, n-1\},$$
$$I_n = ]t_{2n-1}, 1],$$

and assume that we have made this choice such that

$$\forall k \in \{0, \ldots, n\}: \quad \mathrm{diam}(I_k) < \delta. \tag{7}$$

Furthermore, for each $(y_0, \ldots, y_{2n+1}) \in Y^{2n+2}$, let $L_{(y_0,\ldots,y_{2n+1})}$ be the $\mathbb{R}^N$-valued map on $[0,1]$ defined by

$$L_{(y_0,\ldots,y_{2n+1})}(t) = \begin{cases} \frac{t_1-t}{t_1-t_0} y_0 + \frac{t-t_0}{t_1-t_0} y_1 & \text{if } t \in [t_0, t_1], \\ \ldots, \\ \frac{t_{k+1}-t}{t_{k+1}-t_k} y_k + \frac{t-t_k}{t_{k+1}-t_k} y_{k+1} & \text{if } t \in [t_k, t_{k+1}], \\ \ldots, \\ \frac{t_{2n+1}-t}{t_{2n+1}-t_{2n}} y_{2n} + \frac{t-t_{2n}}{t_{2n+1}-t_{2n}} y_{2n+1} & \text{if } t \in [t_{2n}, t_{2n+1}], \end{cases}$$

and put

$$\mathcal{F} = \left\{ L_{(y_0,\ldots,y_{2n+1})} \mid (y_0, \ldots, y_{2n+1}) \in Y^{2n+2} \right\}.$$

Then $\mathcal{F}$ is a finite subset of $\mathcal{C}$. Now fix $x \in \mathcal{X}$ and let $c_{x,k}$ stand for the Chebyshev center of $x(I_k)$ for each $k \in \{0,\dots,n\}$. It follows from (6) and (7) that $\operatorname{diam} f(I_k) \le \alpha$, and thus, by Theorem 3.1,

$$\forall k \in \{0,\dots,n\}: \quad \sup_{t \in I_k} \left| c_{x,k} - x(t) \right| \le \left( \frac{N}{2N+2} \right)^{1/2} \alpha. \tag{8}$$

Let $\tilde{x}$ be the $\mathbb{R}^N$-valued map on $[0,1]$ defined by

$$\tilde{x}(t) = \begin{cases} c_{x,0} & \text{if } t \in [t_0, t_1], \\ \frac{t_2-t}{t_2-t_1} c_{x,0} + \frac{t-t_1}{t_2-t_1} c_{x,1} & \text{if } t \in [t_1, t_2], \\ c_{x,1} & \text{if } t \in [t_2, t_3[, \\ \frac{t_4-t}{t_4-t_3} c_{x,1} + \frac{t-t_3}{t_4-t_3} c_{x,2} & \text{if } t \in [t_3, t_4], \\ \dots, \\ \frac{t_{2k}-t}{t_{2k}-t_{2k-1}} c_{x,k-1} + \frac{t-t_{2k-1}}{t_{2k}-t_{2k-1}} c_{x,k} & \text{if } t \in [t_{2k-1}, t_{2k}], \\ c_{x,k} & \text{if } t \in [t_{2k}, t_{2k+1}], \\ \frac{t_{2k+2}-t}{t_{2k+2}-t_{2k+1}} c_{x,k} + \frac{t-t_{2k+1}}{t_{2k+2}-t_{2k+1}} c_{x,k+1} & \text{if } t \in [t_{2k+1}, t_{2k+2}], \\ \dots, \\ \frac{t_{2n-2}-t}{t_{2n-2}-t_{2n-3}} c_{x,n-2} + \frac{t-t_{2n-3}}{t_{2n-2}-t_{2n-3}} c_{x,n-1} & \text{if } t \in [t_{2n-3}, t_{2n-2}], \\ c_{x,n-1} & \text{if } t \in [t_{2n-2}, t_{2n-1}], \\ \frac{t_{2n}-t}{t_{2n}-t_{2n-1}} c_{x,n-1} + \frac{t-t_{2n-1}}{t_{2n}-t_{2n-1}} c_{x,n} & \text{if } t \in [t_{2n-1}, t_{2n}], \\ c_{x,n} & \text{if } t \in [t_{2n}, t_{2n+1}]. \end{cases}$$

Then (8) and Lemma 3.2 yield that

$$\|\tilde{x} - x\|_\infty \le \left( \frac{N}{2N+2} \right)^{1/2} \alpha. \tag{9}$$

Also, it easily follows from (3), (5), and (9) that $\|\tilde{x}\|_\infty \le 3M$, and thus (4) allows us to choose $(y_0, \dots, y_{2n+1}) \in Y^{2n+2}$ such that

$$\forall k \in \{0,\dots,2n+1\}: \quad \left| y_k - \tilde{x}(t_k) \right| \le \epsilon. \tag{10}$$

Combining (10) and Lemma 3.3 reveals that

$$\|L_{(y_0,\dots,y_{2n+1})} - \tilde{x}\|_\infty \le \epsilon. \tag{11}$$

Thus, we have found $L_{(y_0,\dots,y_{2n+1})}$ in $\mathcal{F}$ for which, by (9) and (11),

$$\|L_{(y_0,\dots,y_{2n+1})} - x\|_\infty \le \left( \frac{N}{2N+2} \right)^{1/2} \alpha + \epsilon,$$

which, by the arbitrariness of $\epsilon$, entails that $\mu_{\mathrm{H},\infty}(\mathcal{F}) \le (\frac{N}{2N+2})^{1/2}\alpha$, and thus, by the arbitrariness of $\alpha$, the inequality

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \le \left( \frac{N}{2N+2} \right)^{1/2} \mu_{\mathrm{uec}}(\mathcal{X})$$

is established. □

## 4 Proof of Theorem 1.6

We transport the measure of nonuniform tightness from $\mathcal{P}(\mathcal{C})$ to the collection of c.s.p.s via their probability distributions. Thus, for a set $\Xi$ of c.s.p.s,

$$\mu_{\mathrm{ut}}(\Xi) = \sup_{\epsilon > 0} \inf_{\mathcal{F}} \sup_{\xi \in \Xi} \mathbb{P}\left( \xi \notin \bigcup_{y \in \mathcal{F}} B_\infty(y, \epsilon) \right),$$

where the infimum is taken over all finite sets $\mathcal{F} \subset \mathcal{C}$, and

$$B_\infty(y, \epsilon) = \left\{ x \in \mathcal{C} \mid \|y - x\|_\infty < \epsilon \right\}.$$

Before giving the proof of Theorem 1.6, we state three lemmas, which are easily seen to follow from the definitions.

**Lemma 4.1** *Let $\Xi$ be a collection of c.s.p.s, and $\alpha \in \mathbb{R}_0^+$. Then the following assertions are equivalent.*

(1) $\mu_{\mathrm{ut}}(\Xi) < \alpha$.

(2) *For each $\epsilon > 0$, there exists a uniformly bounded set $\mathcal{X} \subset \mathcal{C}$ such that*

  (a) $\mu_{\mathrm{H},\infty}(\mathcal{X}) < \epsilon$,

  (b) $\forall \xi \in \Xi \colon \mathbb{P}(\xi \notin \mathcal{X}) < \alpha$.

**Lemma 4.2** *Let $\Xi$ be a collection of c.s.p.s, and $\alpha \in \mathbb{R}_0^+$. Then the following assertions are equivalent.*

(1) $\mu_{\mathrm{sub}}(\Xi) < \alpha$.

(2) *There exists a uniformly bounded set $\mathcal{X} \subset \mathcal{C}$ such that*

$$\forall \xi \in \Xi \colon \quad \mathbb{P}(\xi \notin \mathcal{X}) < \alpha.$$

**Lemma 4.3** *Let $\Xi$ be a collection of c.s.p.s, and $\alpha \in \mathbb{R}_0^+$. Then the following assertions are equivalent.*

(1) $\mu_{\mathrm{suec}}(\Xi) < \alpha$.

(2) *For each $\epsilon > 0$, there exists a set $\mathcal{X} \subset \mathcal{C}$ such that*

  (a) $\mu_{\mathrm{uec}}(\mathcal{X}) < \epsilon$,

  (b) $\forall \xi \in \Xi \colon \mathbb{P}(\xi \notin \mathcal{X}) < \alpha$.

*Proof of Theorem* 1.6  Let $\Xi$ be a collection of c.s.p.s. By Theorem 1.4,

$$\sup_{\lambda \in \mathbb{R}_0^+} \mu_{\mathrm{H},\lambda}(\Xi) = \mu_{\mathrm{ut}}(\Xi),$$

whence it suffices to show that

$$\max\left\{ \mu_{\mathrm{sub}}(\Xi), \mu_{\mathrm{suec}}(\Xi) \right\} \leq \mu_{\mathrm{ut}}(\Xi) \leq \mu_{\mathrm{sub}}(\Xi) + \mu_{\mathrm{suec}}(\Xi).$$

We first establish that

$$\mu_{\mathrm{ut}}(\Xi) \leq \mu_{\mathrm{sub}}(\Xi) + \mu_{\mathrm{suec}}(\Xi).$$

Fix $\epsilon > 0$ and $\alpha, \beta \in \mathbb{R}_0^+$ such that

$$\mu_{\mathrm{sub}}(\Xi) < \alpha$$

and

$$\mu_{\mathrm{suec}}(\Xi) < \beta.$$

By Lemma 4.2 there exists a uniformly bounded set $\mathcal{Y} \subset \mathcal{C}$ such that

$$\forall \xi \in \Xi: \quad \mathbb{P}(\xi \notin \mathcal{Y}) < \alpha,$$

and by Lemma 4.3 there exists a set $\mathcal{Z} \subset \mathcal{C}$ such that

$$\mu_{\mathrm{uec}}(\mathcal{Z}) < \left(\frac{N}{2N+2}\right)^{-1/2} \epsilon \tag{12}$$

and

$$\forall \xi \in \Xi: \quad \mathbb{P}(\xi \notin \mathcal{Z}) < \beta.$$

Put

$$\mathcal{X} = \mathcal{Y} \cap \mathcal{Z}.$$

Then $\mathcal{X}$ is uniformly bounded. Also, by Theorem 1.5 and (12),

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \leq \left(\frac{N}{2N+2}\right)^{1/2} \mu_{\mathrm{uec}}(\mathcal{X}) \leq \left(\frac{N}{2N+2}\right)^{1/2} \mu_{\mathrm{uec}}(\mathcal{Z}) < \epsilon,$$

and, for $\xi \in \Xi$,

$$\mathbb{P}(\xi \notin \mathcal{X}) \leq \mathbb{P}(\xi \notin \mathcal{Y}) + \mathbb{P}(\xi \notin \mathcal{Z}) < \alpha + \beta.$$

We conclude from Lemma 4.1 that

$$\mu_{\mathrm{ut}}(\Xi) < \alpha + \beta,$$

from which the desired inequality follows.

Next, we prove that

$$\max\{\mu_{\mathrm{sub}}(\Xi), \mu_{\mathrm{suec}}(\Xi)\} \leq \mu_{\mathrm{ut}}(\Xi).$$

Fix $\epsilon > 0$ and $\alpha \in \mathbb{R}_0^+$ such that

$$\mu_{\mathrm{ut}}(\Xi) < \alpha.$$

By Lemma 4.1 there exists a uniformly bounded set $\mathcal{X} \subset \mathcal{C}$ such that

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) < \epsilon/2 \tag{13}$$

and

$$\forall \xi \in \Xi: \quad \mathbb{P}(\xi \notin \mathcal{X}) < \alpha.$$

We conclude from Lemma 4.2 that

$$\mu_{\mathrm{sub}}(\Xi) < \alpha.$$

Moreover, by Theorem 1.5 and (13),

$$\mu_{\mathrm{uec}}(\mathcal{X}) \leq 2\mu_{\mathrm{H},\infty}(\mathcal{X}) < \epsilon,$$

and Lemma 4.3 allows us to infer that

$$\mu_{\mathrm{suec}}(\Xi) < \alpha,$$

which finishes the proof of the desired inequality.                                                             □

## 5 Conclusions

In this work, we have quantified the Prokhorov theorem by establishing an explicit formula for the Hausdorff measure of noncompactness (HMNC) for the parameterized Prokhorov metric on the set of Borel probability measures on a Polish space (Theorem 1.4). Furthermore, we have quantified the Arzelà-Ascoli theorem by obtaining upper and lower estimates for the HMNC for the uniform norm on the space of continuous maps of a compact interval into Euclidean $N$-space, using the Jung theorem on the Chebyshev radius (Theorem 1.5). Finally, we have combined the obtained results to quantify the stochastic Arzelà-Ascoli theorem by providing upper and lower estimates for the HMNC for the parameterized Prokhorov metric on the set of multivariate continuous stochastic processes (Theorem 1.6). This work fits nicely in the research initiated in [6], the aim of which is to systematically study quantitative measures, such as the HMNC, in the realm of probability theory.

## References

1. Kallenberg, O: Foundations of Modern Probability, 2nd edn. Probability and Its Applications. Springer, New York (2002)
2. Lang, S: Real and Functional Analysis, 3rd edn. Graduate Texts in Mathematics, vol. 142. Springer, New York (1993)
3. Banaś, J, Goebel, K: Measures of Noncompactness in Banach Spaces. Lecture Notes in Pure and Applied Mathematics, vol. 60. Dekker, New York (1980)
4. Wiśnicki, A, Wośko, J: On relative Hausdorff measures of noncompactness and relative Chebyshev radii in Banach spaces. Proc. Am. Math. Soc. **124**(8), 2465-2474 (1996)
5. Rachev, ST: Probability Metrics and the Stability of Stochastic Models. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. Wiley, Chichester (1991)
6. Berckmoes, B, Lowen, R, van Casteren, J: Distances on probability measures and random variables. J. Math. Anal. Appl. **374**(2), 412-428 (2011)
7. Holmes, RB: A Course on Optimization and Best Approximation. Lecture Notes in Mathematics, vol. 257. Springer, Berlin (1972)
8. Blumenthal, LM, Wahlin, GE: On the spherical surface of smallest radius enclosing a bounded subset of $n$-dimensional Euclidean space. Bull. Am. Math. Soc. **47**, 771-777 (1941)
9. Amir, D: On Jung's constant and related constants in normed linear spaces. Pac. J. Math. **118**(1), 1-15 (1985)
10. Appell, J, Franchetti, C, Semenov, EM: Estimates for the Jung constant in Banach lattices. Isr. J. Math. **116**, 171-187 (2000)
11. Rao, TSSRK: Chebyshev centres and centrable sets. Proc. Am. Math. Soc. **130**(9), 2593-2598 (2002)
12. Nguen-Kkhak, V, Nguen-Van, K: An infinite-dimensional generalization of Jung's theorem. Mat. Zametki **80**(2), 231-239 (2006) (in Russian); translation in Math. Notes **80**(1-2), 224-232 (2006)