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Difference of composition operators on weighted Bergman spaces over the half-plane

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Abstract

Recently, the bounded, compact and Hilbert-Schmidt difference of composition operators on the Bergman spaces over the half-plane are characterized in (Choe *et al.* in Trans. Am. Math. Soc., 2016, in press). Motivated by this, we give a sufficient condition when two composition operators C_φ and C_ψ are in the same path component under the operator norm topology and show that there is no cancellation property for the compactness of double difference of composition operators. More precisely, we show that if C_{φ_1}, C_{φ_2}, and C_{φ_3} are distinct and bounded, then $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_1})$ is compact if and only if both $C_{\varphi_1} - C_{\varphi_2}$ and $C_{\varphi_1} - C_{\varphi_3}$ are compact on weighted Bergman spaces over the half-plane. Moreover, we prove the strong continuity of composition operators semigroup induced by a one-parameter semigroup of holomorphic self-maps of half-plane.

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Keywords: Bergman space; composition operator; Hilbert-Schmidt; semigroup

1 Introduction

Let Π^+ be the upper half of the complex plane, that is, $\Pi^+ := \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$, and let $S(\Pi^+)$ be the set of all holomorphic self-maps of Π^+. For $\varphi \in S(\Pi^+)$, the composition operator C_φ is defined by

$$C_\varphi f = f \circ \varphi$$

for functions f holomorphic on Π^+. It is clear that C_φ maps the space of holomorphic functions on Π^+ into itself. Our purpose in this paper is to study composition operators acting on the weighted Bergman spaces over Π^+. For $\alpha > -1$, let

$$dA_\alpha(z) := c_\alpha (\operatorname{Im} z)^\alpha \, dA(z),$$

where $c_\alpha = \frac{2^\alpha(\alpha+1)}{\pi}$ is a normalizing constant and A is the area measure on Π^+. The weighted Bergman space $A^2_\alpha(\Pi^+)$ consists of holomorphic functions f on Π^+ such that the norm

$$\|f\| := \left\{ \int_{\Pi^+} |f(z)|^2 \, dA_\alpha(z) \right\}^{\frac{1}{2}}$$

is finite. It is well known that $A^2_\alpha(\Pi^+)$ is a Hilbert space with the inner product

$$\langle f,g\rangle = \int_{\Pi^+} f(z)\overline{g(z)}\,dA_\alpha$$

for $f,g \in A^2_\alpha(\Pi^+)$.

An extensive study on the theory of composition operators has been established during the past four decades on various settings. We refer to [2–4] for various aspects on the theory of composition operators acting on holomorphic function spaces. With the basic questions such as boundedness and compactness settled on some symmetric regions [2], it is natural to look at the topological structure of the composition operators under the operator norm topology, and this topic is one of continuing interests in the theory of composition operators. Berkson [5] focused attention on topological structure with his isolation results on $H^p(\mathbf{D})$, where $\mathbf{D}$ is the unit disk of the complex plane, and $0 < p < \infty$. Especially, we mention that Choe-Hosokawa-Koo [6] studied the topological structure of the space of all composition operators under the Hilbert-Schmidt norm topology and gave a characterization of components and some sufficient conditions for isolation or nonisolation. In relation to the study of the topological structure, the difference or the linear sum of composition operators in various settings has been a very active topic [7–9].

Recently, composition operators on upper half-plane have received more attention; for instance, refer to [10–12]. Especially, Elliott and Wynn [13] characterized bounded composition operators and showed that there is no compact composition operator on $A^2_\alpha(\Pi^+)$. Choe-Koo-Smith [1] studied the bounded and compact difference of composition operators on $A^2_\alpha(\Pi^+)$. They also obtained conditions under which the difference of composition operators is Hilbert-Schmidt.

In this paper, we proceed along this line to give a sufficient condition when the composition operators C_φ and C_ψ are in the same path component under the operator norm topology. Moreover, we show that the cancellation of double difference cannot occur on $A^2_\alpha(\Pi^+)$. More precisely, for distinct and bounded C_{φ_1}, C_{φ_2}, and C_{φ_3}, the difference $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_1})$ is compact on $A^2_\alpha(\Pi^+)$ if and only if both $C_{\varphi_1} - C_{\varphi_2}$ and $C_{\varphi_3} - C_{\varphi_1}$ are compact. We also study the linear sum of composition operators induced by some special classes of holomorphic self-maps. In addition, we prove the strong continuity of composition operators semigroups induced by one-parameter semigroups of holomorphic self-maps of Π^+. Due to the unboundedness of the domain, some special techniques are needed.

In Section 2, we recall some basic facts to be used in later sections. In Section 3.1, we prove our sufficient condition for the path component of composition operators. In Section 3.2, we prove that there is no cancellation property for the compactness of double difference of composition operators. The continuity of composition operators semigroup is proved in Section 3.3.

In the rest of the paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \leq CY$ for some inessential constant $C > 0$. Similarly, we say that $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2 Preliminaries

In this section, we give some notation and well-known results on $A^2_\alpha(\Pi^+)$. Recall that for a Hilbert space X, a bounded linear operator $T : X \to X$ is said to be compact if T maps every bounded set into a relatively compact set. Due to the metric topology of X, T is compact if and only if the image of every bounded sequence has a convergent subsequence.

The following lemma gives a convenient compact criterion for a linear combination of composition operators acting on $A^2_\alpha(\Pi^+)$.

Lemma 2.1 *Let T be a linear combination of composition operators and assume that T is bounded on $A^2_\alpha(\Pi^+)$. Then T is compact on $A^2_\alpha(\Pi^+)$ if and only if $Tf_n \to 0$ in $A^2_\alpha(\Pi^+)$ for any bounded sequence $\{f_n\}$ in $A^2_\alpha(\Pi^+)$ satisfying $f_n \to 0$ uniformly on compact subsets of Π^+.*

A proof can be found in [2], Proposition 3.11, for composition operators on a Hardy space over the unit disk, and it can be easily modified for composition operators on $A^2_\alpha(\Pi^+)$.

The pseudo-hyperbolic distance is defined as follows:

$$\sigma(z,w) := \left|\frac{z-w}{z-\overline{w}}\right|, \quad z,w \in \Pi^+.$$

Note that σ is invariant under dilation and horizontal translation. We know from [1] that

$$\sigma(z,w) < 1, \quad z,w \in \Pi^+.$$

Given $\varphi, \psi \in S(\Pi^+)$, we put

$$\sigma(z) := \sigma\bigl(\varphi(z), \psi(z)\bigr).$$

For $z \in \Pi^+$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudo-hyperbolic disk centered at z with radius δ. We may check by an elementary calculation that $E_\delta(z)$ is actually a Euclidean disk centered at $x + i\frac{1+\delta^2}{1-\delta^2}y$, of radius $\frac{2\delta}{1-\delta^2}y$, where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

We will often use the following submean value type inequality:

$$\bigl|f(z)\bigr|^2 \le \frac{C}{(\operatorname{Im} z)^{\alpha+2}} \int_{E_\delta(z)} \bigl|f(z)\bigr|^2 \, dA_\alpha(z), \quad z \in \Pi^+ \tag{2.1}$$

for all $f \in A^2_\alpha(\Pi^+)$ and some constant $C = C(\alpha,\delta)$; see [1] for more details. In particular, we have

$$\bigl|f(z)\bigr|^2 \le \frac{C}{(\operatorname{Im} z)^{\alpha+2}} \|f\|^2, \quad z \in \Pi^+ \tag{2.2}$$

for $f \in A^2_\alpha(\Pi^+)$.

Given $\alpha > -1$, it follows from (2.2) that each point evaluation is a continuous linear functional on $A^2_\alpha(\Pi^+)$. Thus, for each $z \in \Pi^+$, there exists a unique reproducing kernel $K_z^{(\alpha)} \in A^2_\alpha(\Pi^+)$ that has the reproducing property

$$f(z) = \int_{\Pi^+} f(w)\overline{K_z^{(\alpha)}(w)} \, dA_\alpha(w)$$

for $f \in A^2_\alpha(\Pi^+)$. The explicit formula of $K_z^{(\alpha)}$ is given as

$$K_z^{(\alpha)}(w) = \left(\frac{i}{w-\overline{z}}\right)^{\alpha+2},$$

where $i^2 = -1$. Notice that

$$\|K_z^{(\alpha)}\| = \sqrt{K_z^{(\alpha)}(z)} = \left(\frac{1}{2\operatorname{Im} z}\right)^{\frac{\alpha+2}{2}}.$$

Thus, the normalized reproducing kernel $\frac{K_z^{(\alpha)}}{\|K_z^{(\alpha)}\|}$ converges to 0 uniformly on compact subsets of Π^+ as $z \to \partial\widehat{\Pi}^+$. Here, $\widehat{\Pi}^+ := \overline{\Pi}^+ \cup \{\infty\}$, and $\partial\widehat{\Pi}^+$ is the boundary of $\widehat{\Pi}^+$. In the sequel, we usually write $K_z = K_z^{(\alpha)}$ and $k_z = \frac{K_z^{(\alpha)}}{\|K_z^{(\alpha)}\|}$ for simplicity.

Before introducing angular derivatives in the half-plane setting, we first clarify the notion of nontangential limits at boundary points of $\widehat{\Pi}^+$. Of course, those at a finite boundary point refer to the standard notion. Meanwhile, those at $\infty \in \partial\widehat{\Pi}^+$ refer to those associated with nontangential approach regions Ω_ϵ, $\epsilon > 0$, consisting of all $z \in \mathbf{C}$ such that $\operatorname{Im} z > \epsilon|\operatorname{Re} z|$. For a function $\varphi : \Pi^+ \to \Pi^+$ and $x \in \partial\widehat{\Pi}^+$, we write $\varphi(x) = \eta$ (possibly ∞) if φ has a nontangential limit η, that is, $\angle\lim_{z\to x}\varphi(z) = \eta$. For a holomorphic self-map ψ of $\mathbf{D}$, the angular derivative of ψ exists at $\zeta \in \partial\mathbf{D}$ if there is $\eta \in \partial\mathbf{D}$ such that a nontangential limit of $\frac{\eta-\psi(z)}{\zeta-z}$ exists as a finite complex value as $z \to \zeta$. Now we introduce the notion of angular derivatives on Π^+ via the Caley transformation

$$\gamma(z) = i\frac{1+z}{1-z}, \quad z \in \mathbf{D},$$

which conformally maps $\mathbf{D}$ onto Π^+. Note that a region Γ is contained in a nontangential approach region in $\mathbf{D}$ if and only if $\gamma(\Gamma)$ is contained in some nontangential approach region in Π^+.

For $\varphi \in S(\Pi^+)$, let

$$\varphi_\gamma = \gamma^{-1} \circ \varphi \circ \gamma. \tag{2.3}$$

We say that φ has finite angular derivative at $x \in \partial\widehat{\Pi}^+$ if φ_γ has a finite angular derivative at $\widetilde{x} := \gamma^{-1}(x)$, where $\gamma^{-1}(w) = \frac{w-i}{w+i}$, $w \in \Pi^+$.

The following Julia-Carathéodory theorem for the upper half-plan is proved in [13].

Proposition 2.2 *For $\varphi \in S(\Pi^+)$, the following statements are equivalent*:

(a) $\varphi(\infty) = \infty$, *and* $\varphi'(\infty)$ *exists*;
(b) $\sup_{z\in\Pi^+} \frac{\operatorname{Im} z}{\operatorname{Im}\varphi(z)} < \infty$;
(c) $\limsup_{z\to\infty} \frac{\operatorname{Im} z}{\operatorname{Im}\varphi(z)} < \infty$.

Moreover, the quantities in (b) *and* (c) *are equal to* $\varphi'(\infty)$.

Elliott and Wynn [13] gave the following characterization of bounded composition operators by means of angular derivatives.

Theorem 2.3 *Let $\alpha > -1$ and $\varphi \in S(\Pi^+)$. Then C_φ is bounded on $A_\alpha^2(\Pi^+)$ if and only if φ has a finite angular derivative $\varphi'(\infty) = \lambda \in (0,\infty)$. Moreover, $\|C_\varphi\| = \lambda^{\frac{\alpha+2}{2}}$.*

For $z \in \Pi^+$, let τ_z be the function on Π^+ defined by

$$\tau_z(w) := \frac{1}{w - \overline{z}}.$$

In [1], the authors showed that $\tau_z^s \in A_\alpha^2(\Pi^+)$ if and only if $2s > \alpha + 2$. In this case,

$$\|\tau_z^s\|^2 = \frac{C}{(\operatorname{Im} z)^{2s-\alpha-2}}, \tag{2.4}$$

where C is a constant. Thus, $\frac{\tau_z^s}{\|\tau_z^s\|} \to 0$ uniformly on compact subsets of Π^+ as $z \to \partial\widehat{\Pi}^+$.

3 Main results

3.1 Path component of composition operators

In this subsection, we give a sufficient condition for composition operators C_φ and C_ψ to be in the same path component under the operator norm topology. To this end, we recall the definition of Hilbert-Schmidt operators on $A_\alpha^2(\Pi^+)$. A bounded linear operator T on a separable Hilbert H is Hilbert-Schmidt if

$$\|T\|_{\mathrm{HS}}^2 := \sum_{j=1}^{\infty} \|Te_j\|_{\mathrm{H}}^2 = \sum_{j,n=1}^{\infty} \left|\langle Te_j, e_n\rangle_{\mathrm{H}}\right|^2 < \infty$$

for any (or some) orthonormal basis $\{e_n\}$ of H. As is well known, the value of the sum above does not depend on the choice of orthonormal basis $\{e_n\}$ of H, and $\|T\| \le \|T\|_{\mathrm{HS}}$. We know that every Hilbert-Schmidt operator is compact; see [6] for more details. Let $\mathcal{C}(A_\alpha^2(\Pi^+))$ be the space of all bounded composition operators on $A_\alpha^2(\Pi^+)$ endowed with norm topology. The following theorem is due to Choe-Koo-Smith [1], Theorem 7.6.

Theorem 3.1 *Let $\alpha > -1$ and $\varphi, \psi \in S(\Pi^+)$. Then $C_\varphi - C_\psi$ is Hilbert-Schmidt on $A_\alpha^2(\Pi^+)$ if and only if*

$$\int_{\Pi^+} \left[\frac{1}{\operatorname{Im}\varphi(z)} + \frac{1}{\operatorname{Im}\psi(z)}\right]^{\alpha+2} \sigma^2(z)\, dA_\alpha(z) < \infty,$$

where $\sigma(z) := |\frac{\varphi(z)-\psi(z)}{\varphi(z)-\overline{\psi(z)}}|$.

Write $C_\varphi \sim C_\psi$ if C_φ and C_ψ are in the same path component of $\mathcal{C}(A_\alpha^2(\Pi^+))$. For $t \in [0,1]$, we put $\varphi_t = (1-t)\varphi + t\psi$. It is easy to see that $\varphi_t \in S(\Pi^+)$. In order to give a sufficient condition of path connected of two compositions, we need the following lemma.

Lemma 3.2 *Let $\alpha > -1$ and $\varphi, \psi \in S(\Pi^+)$. Assume that $C_\varphi - C_\psi$ is Hilbert-Schmidt on $A_\alpha^2(\Pi^+)$. Let $\varphi_t = (1-t)\varphi + t\psi$ for $t \in [0,1]$. Then $C_{\varphi_s} - C_{\varphi_t}$ is Hilbert-Schmidt on $A_\alpha^2(\Pi^+)$ for any $s, t \in [0,1]$.*

Proof Since $\|C_{\varphi_s} - C_{\varphi_t}\|_{\mathrm{HS}} \le \|C_\varphi - C_{\varphi_s}\|_{\mathrm{HS}} + \|C_\varphi - C_{\varphi_t}\|_{\mathrm{HS}}$, it is enough to prove that $C_\varphi - C_{\varphi_t}$ is Hilbert-Schmidt on $A_\alpha^2(\Pi^+)$ for every $t \in [0,1]$.

Fix $t \in [0,1]$ and put $\sigma_t(z) := \sigma(\varphi(z), \varphi_t(z))$. Then

$$\begin{aligned}
\sigma_t(z) &= \left|\frac{\varphi(z) - \varphi_t(z)}{\varphi(z) - \overline{\varphi_t(z)}}\right| \\
&= \left|\frac{\varphi(z) - [(1-t)\varphi(z) + t\psi(z)]}{\varphi(z) - \overline{[(1-t)\varphi(z) + t\psi(z)]}}\right|
\end{aligned}$$

$$\begin{aligned}
&= \left| \frac{t(\varphi(z) - \psi(z))}{\varphi(z) - \overline{\psi(z)} - \overline{[(1-t)\varphi(z) - (1-t)\psi(z)]}} \right| \\
&\le \frac{t\sigma(z)}{1-(1-t)\sigma(z)} < \sigma(z) \qquad (3.1)
\end{aligned}$$

for all $z \in \Pi^+$.

Now, note that $\frac{1}{\operatorname{Im}\xi} \le \frac{1}{\operatorname{Im} z} + \frac{1}{\operatorname{Im} w}$ for any ξ on the line segment connecting z and w. Thus, we have

$$\frac{1}{\operatorname{Im}\varphi_t(z)} \le \frac{1}{\operatorname{Im}\varphi(z)} + \frac{1}{\operatorname{Im}\psi(z)} \qquad (3.2)$$

for all $0 \le t \le 1$. By (3.1) and (3.2) we have

$$\int_{\Pi^+} \frac{\sigma_t^2(z)}{(\operatorname{Im}\varphi_t(z))^{\alpha+2}}\, dA_\alpha(z) \lesssim \int_{\Pi^+} \left(\frac{1}{\operatorname{Im}\varphi(z)} + \frac{1}{\operatorname{Im}\psi(z)} \right)^{\alpha+2} \sigma^2(z)\, dA_\alpha(z).$$

Similarly,

$$\int_{\Pi^+} \frac{\sigma_t^2(z)}{(\operatorname{Im}\varphi(z))^{\alpha+2}}\, dA_\alpha(z) \lesssim \int_{\Pi^+} \left(\frac{1}{\operatorname{Im}\varphi(z)} + \frac{1}{\operatorname{Im}\psi(z)} \right)^{\alpha+2} \sigma^2(z)\, dA_\alpha(z).$$

So, we conclude by Theorem 3.1 that $C_\varphi - C_{\varphi_t}$ is Hilbert-Schmidt on $A_\alpha^2(\Pi^+)$. □

Now, we give our first main result on sufficient conditions of path connected.

Theorem 3.3 *Let $\alpha > -1$ and $\varphi, \psi \in S(\Pi^+)$. Assume that both C_φ and C_ψ are bounded on $A_\alpha^2(\Pi^+)$ and $C_\varphi - C_\psi$ is Hilbert-Schmidt on $A_\alpha^2(\Pi^+)$. Then $C_\varphi \sim C_\psi$.*

Proof Suppose that $C_\varphi - C_\psi$ is Hilbert-Schmidt on $A_\alpha^2(\Pi^+)$. Since

$$\|C_{\varphi_t}\| \le \|C_{\varphi_t} - C_\varphi\| + \|C_\varphi\| \le \|C_{\varphi_t} - C_\varphi\|_{\mathrm{HS}} + \|C_\varphi\|, \quad t \in [0,1],$$

by Lemma 3.2 we obtain that $C_{\varphi_t} \in \mathcal{C}(A_\alpha^2(\Pi^+))$. We will show that $t \in [0,1] \mapsto C_{\varphi_t}$ is a continuous path in $\mathcal{C}(A_\alpha^2(\Pi^+))$. Since $\|C_{\varphi_t} - C_{\varphi_s}\| \le \|C_{\varphi_t} - C_{\varphi_s}\|_{\mathrm{HS}}$, it is sufficient to prove that

$$\lim_{t\to s} \|C_{\varphi_t} - C_{\varphi_s}\|_{\mathrm{HS}} = 0.$$

Given $t, s \in [0,1]$, $t \ne s$, put $\sigma_{t,s}(z) = \sigma(\varphi_t(z), \varphi_s(z))$, $t, s \in [0,1]$ and $z \in \Pi^+$. From (3.1) we have

$$\sigma_{t,s}(z) \le \sigma_{t,0}(z) + \sigma_{s,0}(z) \le 2\sigma(z). \qquad (3.3)$$

From [1], Theorem 7.5, we know that

$$\|C_{\varphi_t} - C_{\varphi_s}\|_{\mathrm{HS}}^2 \approx \int_{\Pi^+} \left[\frac{1}{\operatorname{Im}\varphi_t(z)} + \frac{1}{\operatorname{Im}\varphi_s(z)} \right]^{\alpha+2} \sigma_{t,s}^2(z)\, dA_\alpha(z).$$

Put

$$\Phi_{t,s}(z) = \left(\frac{1}{\operatorname{Im}\varphi_t(z)} + \frac{1}{\operatorname{Im}\varphi_s(z)}\right)^{\alpha+2} \sigma_{t,s}^2(z), \quad z \in \Pi^+$$

for short. By (3.2) and (3.3) we have

$$\Phi_{t,s}(z) \lesssim \left(\frac{1}{\operatorname{Im}\varphi(z)} + \frac{1}{\operatorname{Im}\psi(z)}\right)^{\alpha+2} \sigma^2(z) = \Phi_{0,1}(z), \quad z \in \Pi^+.$$

Since $C_\varphi - C_\psi$ is a Hilbert-Schmidt operator, $\Phi_{0,1}$ is integrable by Theorem 3.1. Again $\sigma_{t,s}(z) \to 0$ in Π^+ as $t \to s$, and we conclude by the dominated convergence theorem that

$$\lim_{t\to s} \|C_{\varphi_t} - C_{\varphi_s}\|_{\mathrm{HS}} = 0,$$

which completes the proof. □

Remark It follows immediately from the theorem above that

- Given $C_\varphi \in \mathcal{C}(A_\alpha^2(\Pi^+))$, the set $N(\varphi) := \{C_\psi : \|C_\varphi - C_\psi\|_{\mathrm{HS}} < \infty\}$ is the a path-connected set in $\mathcal{C}(A_\alpha^2(\Pi^+))$ containing C_φ.
- The set $N(\varphi)$ is 'convex' in the sense that if $C_\psi \in N(\varphi)$, then $\{C_{\varphi_t}\}_{t\in[0,1]} \in N(\varphi)$.

3.2 Cancellation properties of composition operators

In this subsection, we study cancellation properties of composition operators on $A_\alpha^2(\Pi^+)$. The following lemma is cited from [1], Corollary 4.7.

Lemma 3.4 *Let $\alpha > -1$ and $\varphi, \psi \in S(\Pi^+)$. Assume that both C_φ and C_ψ are bounded on $A_\alpha^2(\Pi^+)$. Then $C_\varphi - C_\psi$ is compact on $A_\alpha^2(\Pi^+)$ if and only if*

$$\lim_{z\to\partial\widehat{\Pi}^+} \left[\frac{\operatorname{Im} z}{\operatorname{Im}\varphi(z)} + \frac{\operatorname{Im} z}{\operatorname{Im}\psi(z)}\right]\sigma(z) = 0. \tag{3.4}$$

Here $\lim_{z\to\partial\widehat{\Pi}^+} g(z) = 0$ means that $\sup_{\Pi^+\setminus K} |g| \to 0$ as the compact set $K \subset \Pi^+$ expands to the whole of Π^+ or, equivalently, that $g(z) \to 0$ as $\operatorname{Im} z \to 0^+$ and $g(z) \to 0$ as $|z| \to \infty$.

The following theorem shows that there is no cancellation property for the compactness of double difference of composition operators.

Theorem 3.5 *Let $\alpha > -1$, $a, b \in \mathbf{C}\setminus\{0\}$, and $a + b \neq 0$. Assume that $\varphi_j \in S(\Pi^+)$ and φ_j are distinct for $j = 1, 2, 3$ and each C_{φ_j} is bounded on $A_\alpha^2(\Pi^+)$. Then $T := a(C_{\varphi_2} - C_{\varphi_1}) + b(C_{\varphi_3} - C_{\varphi_1})$ is compact on $A_\alpha^2(\Pi^+)$ if and only if both $C_{\varphi_2} - C_{\varphi_1}$ and $C_{\varphi_3} - C_{\varphi_1}$ are compact on $A_\alpha^2(\Pi^+)$.*

Proof The sufficiency is trivial, and we only prove the necessity. Assume that $T = a(C_{\varphi_2} - C_{\varphi_1}) + b(C_{\varphi_3} - C_{\varphi_1})$ is compact. We will get a contradiction if either $C_{\varphi_2} - C_{\varphi_1}$ or $C_{\varphi_3} - C_{\varphi_1}$ is not compact. Without loss of generality, we assume that $C_{\varphi_2} - C_{\varphi_1}$ is not compact. Let $\sigma_{j,s}(z) := |\frac{\varphi_j(z)-\varphi_s(z)}{\varphi_j(z)-\overline{\varphi_s(z)}}|$ for $j, s = 1, 2, 3$ and $s \neq j$.

Since $C_{\varphi_2} - C_{\varphi_1}$ is not compact, by (3.4) there are $\epsilon > 0$ and a sequence $\{z_n\} \subset \Pi^+$ such that $z_n \to \partial\widehat{\Pi}^+$ $(n \to \infty)$ and

$$\left[\frac{\operatorname{Im} z_n}{\operatorname{Im}\varphi_1(z_n)} + \frac{\operatorname{Im} z_n}{\operatorname{Im}\varphi_2(z_n)}\right]\sigma_{12}(z_n) \geq \epsilon. \tag{3.5}$$

For each $j = 1, 2, 3$, since C_{φ_j} is bounded on $A^2_\alpha(\Pi^+)$, we have $\|C^*_{\varphi_j}K_z\| \leq \|C_{\varphi_j}\|\|K_z\|$ for any $z \in \Pi^+$, where $C^*_{\varphi_j}$ is the adjoint of C_{φ_j}. Due to $C^*_{\varphi_j}K_z = K_{\varphi_j(z)}$, this is equivalent to

$$\frac{\operatorname{Im} z}{\operatorname{Im}\varphi_j(z)} \leq C \quad \text{for all } z \in \Pi^+,$$

where C is some positive constant. Duo to this and the fact $\sigma_{12} < 1$, taking a smaller ϵ if necessary, formula (3.5) gives that

$$\sigma_{1,2}(z_n) \gtrsim \epsilon \tag{3.6}$$

and

$$M_1(z_n) := \frac{\operatorname{Im} z_n}{\operatorname{Im}\varphi_1(z_n)} \gtrsim \epsilon$$

or

$$M_2(z_n) := \frac{\operatorname{Im} z_n}{\operatorname{Im}\varphi_2(z_n)} \gtrsim \epsilon.$$

Without loss of generality, we assume that $M_2(z_n) \gtrsim \epsilon$ (the proof for the case $M_1(z_n) \gtrsim \epsilon$ is similar); thus, $\operatorname{Im} z_n \approx \operatorname{Im}\varphi_2(z_n)$. For $j = 1, 2, 3$, we define $g_{j,n}(z) := \tau^k_{\varphi_j(z_n)}(z) = \frac{1}{(z-\overline{\varphi_j(z_n)})^k}$ with $2k > \alpha + 2$. For $m, j = 1, 2, 3$, $m \neq j$, let $x^m_{j,n} = \frac{\varphi_m(z_n)-\overline{\varphi_m(z_n)}}{\varphi_j(z_n)-\overline{\varphi_m(z_n)}}$. Notice that

$$\begin{aligned} \left|x^m_{j,n}\right| &= \left|\frac{\varphi_m(z_n) - \overline{\varphi_m(z_n)}}{\varphi_j(z_n) - \overline{\varphi_m(z_n)}}\right| \\ &\leq \left|\frac{\varphi_m(z_n) - \varphi_j(z_n)}{\varphi_j(z_n) - \overline{\varphi_m(z_n)}}\right| + \left|\frac{\varphi_j(z_n) - \overline{\varphi_m(z_n)}}{\varphi_j(z_n) - \overline{\varphi_m(z_n)}}\right| \\ &= \sigma_{m,j}(z_n) + 1 < 2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \left|x^m_{j,n}\right| &= \left|\frac{\varphi_m(z_n) - \overline{\varphi_m(z_n)}}{\varphi_j(z_n) - \overline{\varphi_m(z_n)}}\right| \\ &= \left|\frac{\varphi_j(z_n) - \overline{\varphi_j(z_n)}}{\varphi_j(z_n) - \overline{\varphi_m(z_n)}}\right| \left[\frac{\varphi_m(z_n) - \overline{\varphi_m(z_n)}}{\varphi_j(z_n) - \overline{\varphi_j(z_n)}}\right] \\ &< 2\frac{\varphi_m(z_n) - \overline{\varphi_m(z_n)}}{\varphi_j(z_n) - \overline{\varphi_j(z_n)}} \\ &:= 2y^m_{j,n}. \end{aligned} \tag{3.8}$$

By (2.4) and $M_2(z_n) \gtrsim \epsilon$ we have

$$\frac{|g_{2,n}(z)|}{\|g_{2,n}\|} = \frac{C[\operatorname{Im}\varphi_2(z_n)]^{2k-\alpha-2}}{|z-\overline{\varphi_2(z_n)}|^k} \lesssim \frac{(\operatorname{Im} z_n)^{2k-\alpha-2}}{|z-\overline{\varphi_2(z_n)}|^k}. \tag{3.9}$$

Thus,

$$\frac{g_{2,n}}{\|g_{2,n}\|} \to 0 \text{ uniformly on compact subsets of } \Pi^+ \text{ as } n \to \infty.$$

By (2.1) we have

$$\left|Tg_{j,n}(z_n)\right|^2 \lesssim \frac{\|Tg_{j,n}\|^2}{(\operatorname{Im} z_n)^{\alpha+2}}, \quad j = 1, 2, 3. \tag{3.10}$$

Thus, using the fact that $M_2(z_n) \geq \epsilon$, we obtain

$$\begin{aligned}
\frac{\|Tg_{2,n}\|^2}{\|g_{2,n}\|^2} &\gtrsim \left(\operatorname{Im}\varphi_2(z_n)\right)^{2k-\alpha-2}(\operatorname{Im} z_n)^{\alpha+2}\left|Tg_{2,n}(z_n)\right|^2 \\
&\gtrsim |a|^2\left|1 - \frac{a+b}{a}\left(\frac{\varphi_2(z_n)-\overline{\varphi_2(z_n)}}{\varphi_1(z_n)-\overline{\varphi_2(z_n)}}\right)^k + \frac{b}{a}\left(\frac{\varphi_2(z_n)-\overline{\varphi_2(z_n)}}{\varphi_3(z_n)-\overline{\varphi_2(z_n)}}\right)^k\right|^2 \\
&\gtrsim \left[1 - \left|\frac{a+b}{a}\right|\left|x_{1,n}^2\right|^k - \left|\frac{b}{a}\right|\left|x_{3,n}^2\right|^k\right]^2 \\
&\gtrsim \left[1 - 2\left|\frac{a+b}{a}\right|\left(y_{1,n}^2\right)^k - 2\left|\frac{b}{a}\right|\left(y_{3,n}^2\right)^k\right]^2.
\end{aligned}$$

Since T is compact, we have $\frac{\|Tg_{2,n}\|}{\|g_{2,n}\|} \to 0$ ($n \to \infty$). Therefore, at least one of $y_{1,n}^2$ and $y_{3,n}^2$ does not converge to 0.

Suppose $y_{1,n}^2 \to 0$ but $y_{3,n}^2 \nrightarrow 0$. Then $\frac{\operatorname{Im}\varphi_2(z_n)}{\operatorname{Im}\varphi_3(z_n)} \geq C > 0$ for some subsequence, which we still denote by $\{z_n\}$. Since $\operatorname{Im} z_n \approx \operatorname{Im}\varphi_2(z_n)$, $M_3(z_n) := \frac{\operatorname{Im} z_n}{\operatorname{Im}\varphi_3(z_n)} \geq C > 0$. Therefore, $\operatorname{Im}\varphi_3(z_n) \lesssim \operatorname{Im} z_n$. Similarly to (3.9), we have

$$\frac{g_{3,n}}{\|g_{3,n}\|} \to 0 \text{ uniformly on compact subsets of } \Pi^+ \text{ as } n \to \infty.$$

From (3.8) we have

$$\left|x_{1,n}^3\right| \lesssim y_{1,n}^3 \lesssim \frac{\operatorname{Im} z_n}{\operatorname{Im}\varphi_1(z_n)} \approx y_{1,n}^2,$$

which implies $x_{1,n}^3 \to 0$ as $n \to \infty$. By (3.10) and $M_3(z_n) \geq C$ we obtain

$$\begin{aligned}
\frac{\|Tg_{3,n}\|^2}{\|g_{3,n}\|^2} &\gtrsim \left(\operatorname{Im}\varphi_3(z_n)\right)^{2k-\alpha-2}(\operatorname{Im} z_n)^{\alpha+2}\left|Tg_{3,n}(z_n)\right|^2 \\
&\gtrsim |b|^2\left|1 - \frac{a+b}{b}\left(\frac{\varphi_3(z_n)-\overline{\varphi_3(z_n)}}{\varphi_1(z_n)-\overline{\varphi_3(z_n)}}\right)^k + \frac{a}{b}\left(\frac{\varphi_3(z_n)-\overline{\varphi_3(z_n)}}{\varphi_2(z_n)-\overline{\varphi_3(z_n)}}\right)^k\right|^2.
\end{aligned}$$

Since T is compact, we have $\frac{\|Tg_{3,n}\|}{\|g_{3,n}\|} \to 0$ as $n \to \infty$. Since $\frac{\varphi_3(z_n)-\overline{\varphi_3(z_n)}}{\varphi_1(z_n)-\overline{\varphi_3(z_n)}} = x_{1,n}^3 \to 0$, we have

$$1 + \frac{a}{b}\left(\frac{\varphi_3(z_n)-\overline{\varphi_3(z_n)}}{\varphi_2(z_n)-\overline{\varphi_3(z_n)}}\right)^k \to 0.$$

Note that this holds for any $2k > \alpha + 2$, which implies $a + b = 0$. This is a contradiction to the assumption $a + b \neq 0$. Therefore, $y_{1,n}^2$ does not converge to 0.

Taking a subsequence of $\{z_n\}$ if necessary, we have $M_1(z_n) \geq C > 0$, which implies $\operatorname{Im} \varphi_1(z_n) \lesssim \operatorname{Im} z_n$. Similarly to (3.9), we have

$$\frac{g_{1,n}}{\|g_{1,n}\|} \to 0 \text{ uniformly on compact subsets of } \Pi^+ \text{ as } n \to \infty.$$

Notice that

$$\begin{aligned} x_{j,n}^1 &= \frac{\varphi_1(z_n) - \overline{\varphi_1(z_n)}}{\varphi_j(z_n) - \overline{\varphi_1(z_n)}} \\ &= \frac{\varphi_1(z_n) - \overline{\varphi_1(z_n)}}{\varphi_1(z_n) - \overline{\varphi_1(z_n)} + \varphi_j(z_n) - \varphi_1(z_n)} \\ &= \frac{1}{1 + \frac{\varphi_j(z_n) - \varphi_1(z_n)}{\varphi_1(z_n) - \overline{\varphi_1(z_n)}}} \end{aligned}$$

and

$$\begin{aligned} \sigma_{1,2}(z_n) &= \left| \frac{\varphi_1(z_n) - \varphi_2(z_n)}{\varphi_1(z_n) - \overline{\varphi_2(z_n)}} \right| \\ &= \left| \frac{\varphi_1(z_n) - \varphi_2(z_n)}{\varphi_1(z_n) - \overline{\varphi_1(z_n)}} \right| \left| \frac{\varphi_1(z_n) - \overline{\varphi_1(z_n)}}{\varphi_1(z_n) - \overline{\varphi_2(z_n)}} \right| \\ &= \left| \frac{\varphi_1(z_n) - \varphi_2(z_n)}{\varphi_1(z_n) - \overline{\varphi_1(z_n)}} \right| \left| x_{2,n}^1 \right|. \end{aligned}$$

By formula (3.7) we have

$$\left| x_{2,n}^1 \right| < 2.$$

Therefore, if $\limsup_{n\to\infty} |x_{2,n}^1| = 1$, then there exists some subsequence $\{z_{n_l}\}$ such that

$$\frac{\varphi_2(z_{n_l}) - \varphi_1(z_{n_l})}{\varphi_1(z_{n_l}) - \overline{\varphi_1(z_{n_l})}} \to 0.$$

Then $\sigma_{1,2}(z_{n_l}) \to 0$, which contradicts (3.6). Hence, we have $\limsup_{n\to\infty} |x_{2,n}^1| \lneqq 1$. From (3.10) and the fact that $M_1(z_n) \geq \epsilon$ we have

$$\left(\operatorname{Im} \varphi_1(z_n)\right)^{2k} \left| T g_{1,n}(z_n) \right|^2 \lesssim \frac{\| T g_{1,n} \|^2}{\| g_{1,n} \|^2}.$$

Thus, we get

$$\begin{aligned} 0 \leftarrow \frac{\| T g_{1,n} \|^2}{\| g_{1,n} \|^2} &\gtrsim \left| \frac{a}{a+b} \left(\frac{\varphi_1(z_n) - \overline{\varphi_1(z_n)}}{\varphi_2(z_n) - \overline{\varphi_1(z_n)}} \right)^k + \frac{b}{a+b} \left(\frac{\varphi_1(z_n) - \overline{\varphi_1(z_n)}}{\varphi_3(z_n) - \overline{\varphi_1(z_n)}} \right)^k - 1 \right|^2 \\ &= \left| \frac{a}{a+b} \left(x_{2,n}^1 \right)^k + \frac{b}{a+b} \left(x_{3,n}^1 \right)^k - 1 \right|^2 \end{aligned}$$

for all $2k > \alpha + 2$. Then

$$\frac{a}{a+b}\left(x_{2,n}^1\right)^k + \frac{b}{a+b}\left(x_{3,n}^1\right)^k - 1 \to 0.$$

Since $\limsup_{n\to\infty} |x_{2,n}^1| \lneqq 1$, we obtain $\frac{b}{a+b} - 1 = 0$. Namely, $a = 0$, which contradicts our assumption. Therefore, the compactness of $T = a(C_{\varphi_2} - C_{\varphi_1}) + b(C_{\varphi_3} - C_{\varphi_1})$ implies that both $C_{\varphi_2} - C_{\varphi_1}$ and $C_{\varphi_3} - C_{\varphi_1}$ are compact. The proof is complete. □

The following theorem involves the lower estimate of the essential norm for a linear sum of some special composition operators on $A_\alpha^2(\Pi^+)$. Here the essential norm of an operator means the distance to the space of compact operators. To state the result, we need to introduce some notation. For $\varphi \in S(\Pi^+)$, if C_φ is bounded on $A_\alpha^2(\Pi^+)$, then $\varphi(\infty) = \infty$ and $0 < \varphi'(\infty) < \infty$ by Theorem 2.3. Let

$$\mathcal{D}[\varphi, \infty] := \left(\varphi(\infty), \varphi'(\infty)\right) \in \{\infty\} \times (0, \infty).$$

For $\varepsilon > 0$, let

$$R_{\varepsilon,\infty} := \left\{z \in \Pi^+ : \operatorname{Im} z = -\varepsilon \operatorname{Re} z\right\}.$$

Note that $R_{\varepsilon,\infty}$ is a nontangential curve having ∞ as the end point. Let

$$S\left(\Pi^+\right)_c := \left\{\varphi \in S\left(\Pi^+\right) : \operatorname{Im} z \geq c \operatorname{Im} \varphi(z) \text{ for each } z \in \Pi^+\right\},$$

where c is a constant.

The following lemma can be found in [1], Lemma 5.2.

Lemma 3.6 *Let $\varphi, \psi \in S(\Pi^+)$. Assume that C_φ, C_ψ are bounded on $A_\alpha^2(\Pi^+)$. Then*

$$\lim_{\varepsilon\to 0^+} \lim_{z\to\infty, z\in R_{\varepsilon,\infty}} \frac{\varphi(z) - \overline{\varphi(z)}}{\varphi(z) - \overline{\psi(z)}} = \begin{cases} 1, & \text{if } \mathcal{D}[\varphi,\infty] = \mathcal{D}[\psi,\infty], \\ 0, & \text{otherwise.} \end{cases}$$

Now we give a lower estimate of the essential norm of a linear sum of composition operators induced by symbols in $S(\Pi^+)_c$.

Theorem 3.7 *Let $\alpha > -1$ and $\varphi_j \in S(\Pi^+)_c$, $j = 1, 2, \ldots, N$. For each $j = 1, 2, \ldots, N$, assume that C_{φ_j} is bounded on $A_\alpha^2(\Pi^+)$. Then we have the inequality*

$$\left\|\sum_{j=1}^N a_j C_{\varphi_j}\right\|_e^2 \geq \sum_{(u,v)\in\{\infty\}\times(0,\infty)} \left|\sum_{\mathcal{D}[\varphi_j,\infty]=(u,v)} a_j\right|^2$$

for any $a_1, a_2, \ldots, a_N \in \mathbf{C}$.

Proof Let $a_1, a_2, \ldots, a_N \in \mathbf{C}$. Since $C_{\varphi_j}^* K_z = K_{\varphi_j(z)}$, we have

$$\left\|\sum_{j=1}^N a_j C_{\varphi_j}\right\|_e^2 = \left\|\sum_{j=1}^N \overline{a}_j C_{\varphi_j}^*\right\|_e^2 \geq \lim_{\varepsilon\to 0^+} \lim_{z\to\infty, z\in R_{\varepsilon,\infty}} \left\|\sum_{j=1}^N \overline{a}_j C_{\varphi_j}^* k_z\right\|^2.$$

Meanwhile, we have

$$\left\| \sum_{j=1}^{N} \overline{a}_j C_{\varphi_j}^* k_z \right\|^2 = \sum_{j,k} \overline{a}_j a_k \left(\frac{2 \operatorname{Im} z}{\varphi_j(z) - \overline{\varphi_k(z)}} \right)^{\alpha+2} \gtrsim \sum_{j,k} \overline{a}_j a_k \left(\frac{\varphi_j(z) - \overline{\varphi_j(z)}}{\varphi_j(z) - \overline{\varphi_k(z)}} \right)^{\alpha+2}.$$

Using Lemma 3.6, we have

$$\lim_{\varepsilon \to 0^+} \lim_{z \to \infty, z \in R_{\varepsilon,\infty}} \frac{\varphi_j(z) - \overline{\varphi_j(z)}}{\varphi_j(z) - \overline{\varphi_k(z)}} = \begin{cases} 1, & \mathcal{D}[\varphi_j, \infty] = \mathcal{D}[\varphi_k, \infty], \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we obtain

$$\left\| \sum_{j=1}^{N} a_j C_{\varphi_j} \right\|_e^2 \geq \sum_{\mathcal{D}[\varphi_j,\infty] = \mathcal{D}[\varphi_k,\infty]} \overline{a}_j a_k,$$

which is the same as the desired inequality. The proof is complete. □

As an immediate consequence, we have a necessary coefficient relation for the compactness of linear combination of composition operators.

Corollary 3.8 *Let $\alpha > -1$ and $\varphi_j \in S(\Pi^+)_c$, $j = 1, 2, \ldots, N$. Assume that each C_{φ_j} is bounded on $A_\alpha^2(\Pi^+)$ and $a_j \in \mathbf{C}$. If $\sum_{j=1}^N a_j C_{\varphi_j}$ is compact on $A_\alpha^2(\Pi^+)$, then*

$$\sum_{\mathcal{D}[\varphi_j,\infty] = (u,v)} a_j = 0,$$

$(u, v) \in \{\infty\} \times (0, \infty)$.

Especially, we have the following useful corollary.

Corollary 3.9 *Let $\alpha > -1$ and $\varphi, \psi \in S(\Pi^+)_c$. Assume that C_φ, C_ψ are bounded on $A_\alpha^2(\Pi^+)$ and $a, b \in \mathbf{C} \backslash \{0\}$. If $aC_\varphi + bC_\psi$ is compact on $A_\alpha^2(\Pi^+)$, then the following statements hold:*

(a) $a + b = 0$;

(b) $\varphi'(\infty) = \psi'(\infty)$.

3.3 Composition operators induced by a one-parameter semigroup

In the last subsection, we consider the strong continuity of the composition operator semigroup induced by a one-parameter semigroup of holomorphic self-maps of Π^+. We first recall some definitions and notation.

A one-parameter semigroup of holomorphic self-maps of Π^+ is a family $\{\varphi_t\}_{t \geq 0} \subset S(\Pi^+)$ satisfying

(1) $\varphi_0(z) = z$ for all $z \in \Pi^+$;

(2) $\varphi_{t+s}(z) = \varphi_t \circ \varphi_s(z)$ for all $s, t \geq 0$ and $z \in \Pi^+$;

(3) $(t, z) \mapsto \varphi_t(z)$ is jointly continuous on $[0, \infty) \times \Pi^+$.

We know that the continuity of $(t, z) \mapsto \varphi_t(z)$ on $[0, \infty) \times \Pi^+$ is equivalent to the continuity of $t \mapsto \varphi_t(z)$ for each $z \in \Pi^+$. By [14] the holomorphic function $G : \Pi^+ \to \mathbf{C}$ given by

$$G(z) = \lim_{t \to 0} \frac{\partial \varphi_t(z)}{\partial t}$$

is the infinitesimal generator of $\{\varphi_t\}$, which characterizes $\{\varphi_t\}$ uniquely and satisfies

$$\frac{\partial \varphi_t(z)}{\partial t} = G\big(\varphi_t(z)\big), \quad z \in \Pi^+, t \geq 0.$$

Let $\widetilde{G}$ be the infinitesimal generator of the one-parameter semigroup $\{(\varphi_t)_\gamma\}_{t\geq 0}$, where $(\varphi_t)_\gamma = \gamma^{-1} \circ \varphi_t \circ \gamma$, and $\gamma(\zeta) = i\frac{1+\zeta}{1-\zeta}$, $\gamma^{-1}(w) = \frac{w-i}{w+i}$, $\zeta \in \mathbf{D}$, $w \in \Pi^+$. From [14] we also have

$$\widetilde{G}\big((\varphi_t)_\gamma\big(\gamma^{-1}(z)\big)\big) = \frac{\partial(\varphi_t)_\gamma(\gamma^{-1}(z))}{\partial t}, \quad z \in \Pi^+, t \geq 0.$$

From [15] we obtain

$$\widetilde{G}\big(\gamma^{-1}(z)\big) = \frac{2i}{(z+i)^2}G(z), \quad z \in \Pi^+. \tag{3.11}$$

Assume that $\{C_{\varphi_t}\}_{t\geq 0}$ is the bounded composition operator semigroup on $A^2_\alpha(\Pi^+)$ induced by the one-parameter semigroup $\{\varphi_t\}_{t\geq 0} \subset S(\Pi^+)$. The linear operator A defined by

$$D(A) = \left\{f \in A^2_\alpha\big(\Pi^+\big) : \lim_{t\to 0} \frac{C_{\varphi_t} f - f}{t} \text{ exists}\right\},$$

and

$$Af = \lim_{t\to 0} \frac{C_{\varphi_t} f - f}{t} = \frac{\partial C_{\varphi_t} f}{\partial t}\bigg|_{t=0}, \quad f \in D(A),$$

is the infinitesimal generator of the semigroup $\{C_{\varphi_t}\}_{t\geq 0}$, where $D(A)$ is the domain of A. If $\{C_{\varphi_t}\}_{t\geq 0}$ satisfies

$$\lim_{t\to 0} \|C_{\varphi_t} f - f\| = 0, \quad f \in A^2_\alpha\big(\Pi^+\big),$$

then we say that $\{C_{\varphi_t}\}_{t\geq 0}$ is strongly continuous.

The following theorem gives some characterizations about the boundedness of $\{C_{\varphi_t}\}$, $t \geq 0$.

Theorem 3.10 *Let $\alpha > -1$, and let $\{\varphi_t\}_{t\geq 0}$ be a one-parameter semigroup with infinitesimal generator G, where $\{\varphi_t\}_{t\geq 0} \subset S(\Pi^+)$. Then the following are equivalent*:

(a) *For each $t > 0$, C_{φ_t} is bounded on $A^2_\alpha(\Pi^+)$.*

(b) *For some $t > 0$, C_{φ_t} is bounded on $A^2_\alpha(\Pi^+)$.*

(c) *The nontangential limit $\delta := \angle \lim_{z\to\infty} \frac{G(z)}{z}$ exists finitely.*

Moreover, if one of these assertions holds, then $\|C_{\varphi_t}\| = e^{\frac{(\alpha+2)\delta t}{2}}$ for each $t > 0$.

Proof (a) $\Leftrightarrow$ (b). Since (a) implies (b), we only prove the converse. We now assume that there exists $t_0 > 0$ such that $C_{\varphi_{t_0}}$ is bounded on $A^2_\alpha(\Pi^+)$. Then $\varphi_{t_0}(\infty) = \infty$, and $\varphi'_{t_0}(\infty)$ exists finitely. From [15], Lemma 2.1, we know that $\varphi_{t_0}(\infty) = \infty$ if and only if $(\varphi_{t_0})_\gamma(1) = 1$. From [16], Theorem 5, we know that all members of the semigroup $\{(\varphi_t)_\gamma\}_{t\geq 0}$ have common boundary fixed points, that is, $(\varphi_t)_\gamma(1) = 1$ for each $t \geq 0$. Then, by [17], Theorem 1, we have $(\varphi_t)'_\gamma(1) = ((\varphi_{t_0})'_\gamma(1))^{\frac{t}{t_0}}$. For each $t > 0$, since

$$\angle\lim_{w\to 1}\frac{1-(\varphi_t)_\gamma(w)}{1-w} = \angle\lim_{w\to 1}\frac{[1-(\varphi_t)_\gamma(w)][1+w]}{[1+(\varphi_t)_\gamma(w)][1-w]}$$
$$= \angle\lim_{w\to 1}\frac{\gamma(w)}{\gamma((\varphi_t)_\gamma)(w)} = \angle\lim_{z\to\infty}\frac{z}{\varphi_t(z)},$$

we have

$$(\varphi_t)'_\gamma(1) = \varphi'_t(\infty). \tag{3.12}$$

Then we obtain

$$\varphi'_t(\infty) = \bigl((\varphi_{t_0})'_\gamma(1)\bigr)^{\frac{t}{t_0}} = \bigl(\varphi'_{t_0}(\infty)\bigr)^{\frac{t}{t_0}},$$

which implies that $\varphi'_t(\infty)$ exists finitely. So C_{φ_t} is bounded on $A^2_\alpha(\Pi^+)$ for each $t > 0$ by Theorem 2.3.

(a) $\Leftrightarrow$ (c). For each $t > 0$, C_{φ_t} is bounded on $A^2_\alpha(\Pi^+)$ if and only if $\varphi_t(\infty) = \infty$, $\varphi'_t(\infty)$ exists finitely by Proposition 2.2 and Theorem 2.3. From [15], Lemma 2.1, we have $(\varphi_t)_\gamma(1) = 1$ if and only if $\varphi_t(\infty) = \infty$. Also, by (3.12), C_{φ_t} is bounded on $A^2_\alpha(\Pi^+)$ if and only if $(\varphi_t)_\gamma(1) = 1$, $(\varphi_t)'_\gamma(1)$ exists finitely, which is equivalent to the finite existence of $\angle\lim_{w\to 1}\frac{\widetilde{G}(w)}{1-w}$, $w \in \mathbf{D}$, by [17], Theorem 1, where $\widetilde{G}$ is the infinitesimal generator of the one-parameter semigroup $\{(\varphi_t)_\gamma\}_{t\geq 0}$. By (3.11) we have

$$\angle\lim_{w\to 1}\frac{\widetilde{G}(w)}{1-w} = \angle\lim_{w\to 1}\frac{2iG(\gamma(w))}{(\gamma(w)+i)^2(1-w)} = \angle\lim_{w\to 1}\frac{G(\gamma(w))}{\gamma(w)+i}$$
$$= \angle\lim_{z\to\infty}\frac{G(z)}{z+i} = \angle\lim_{z\to\infty}\frac{G(z)}{z}, \tag{3.13}$$

which implies (a) $\Leftrightarrow$ (c).

For each $t > 0$, if one of the conditions holds, we obtain $\varphi'_t(\infty) = e^{\delta t}$ from (3.12), (3.13), and [17], Theorem 1, where $\delta = \angle\lim_{w\to\infty}\frac{G(w)}{w}$, $w \in \Pi^+$. By Theorem 2.3 we have

$$\|C_{\varphi_t}\| = e^{\frac{(\alpha+2)\delta t}{2}},$$

which completes the proof. □

Next, we prove the strong continuity of composition operator semigroups induced by one-parameter semigroups of holomorphic self-maps of the upper half-plane.

Theorem 3.11 *Let $\alpha > -1$, and let $\{\varphi_t\}_{t\geq 0} \subset S(\Pi^+)$ be a one-parameter semigroup on Π^+. For each $t \geq 0$, assume that C_{φ_t} is bounded on $A^2_\alpha(\Pi^+)$. Then $\{C_{\varphi_t}\}_{t\geq 0}$ is strongly continuous on $A^2_\alpha(\Pi^+)$.*

Proof Due to the denseness of *Span*$\{k_z : z \in \Pi^+\}$ in $A^2_\alpha(\Pi^+)$, it is sufficient to prove

$$\lim_{t\to 0}\|C_{\varphi_t}k_z - k_z\| = 0, \quad z \in \Pi^+.$$

By the property of reproducing kernel of $A^2_\alpha(\Pi^+)$ and the Cauchy-Schwarz inequality we have

$$\left|f(z)\right|^2 = \left|\langle f, K_z\rangle\right|^2 \le \|f\|^2\|K_z\|^2. \tag{3.14}$$

Because

$$\|C_{\varphi_t}k_z - k_z\|^2 + \|C_{\varphi_t}k_z + k_z\|^2 = 2\left(\|C_{\varphi_t}k_z\|^2 + \|k_z\|^2\right),$$

we have

$$\|C_{\varphi_t}k_z - k_z\|^2 \le 2\left(1 + \|C_{\varphi_t}\|^2\right) - \|C_{\varphi_t}k_z + k_z\|^2.$$

By Theorem 2.3 we have $\|C_{\varphi_t}\|^2 = \varphi_t'(\infty)^{\alpha+2}$. Thus, we obtain

$$\|C_{\varphi_t}k_z - k_z\|^2 \le 2\left(1 + \left(\varphi_t'(\infty)\right)^{\alpha+2}\right) - \|C_{\varphi_t}k_z + k_z\|^2. \tag{3.15}$$

Taking $f = k_z \circ \varphi_t + k_z$ in (3.14), we obtain

$$\|C_{\varphi_t}k_z + k_z\|^2 \ge \frac{|K_z \circ \varphi_t(z) + \|K_z\|^2|^2}{\|K_z\|^4}. \tag{3.16}$$

Combining (3.15) and (3.16), we obtain

$$\|C_{\varphi_t}k_z - k_z\|^2 \le 2\left(1 + \left(\varphi_t'(\infty)\right)^{\alpha+2}\right) - \frac{|K_z \circ \varphi_t(z) + \|K_z\|^2|^2}{\|K_z\|^4}.$$

Since $2(1 + (\varphi_t'(\infty))^{\alpha+2}) \to 4$ and $\frac{|K_z\circ\varphi_t(z)+\|K_z\|^2|^2}{\|K_z\|^4} \to 4$ as $t \to 0$, we obtain

$$\lim_{t\to 0}\|C_{\varphi_t}k_z - k_z\| = 0, \quad z \in \Pi^+.$$

The proof is complete. □

As an application, we have the following corollary by using standard arguments as in [15], Theorem 3.3. We omit its proof to the reader.

Corollary 3.12 *Let $\alpha > -1$, and let $\{\varphi_t\}_{t\ge 0} \subset S(\Pi^+)$ be a one-parameter semigroup on Π^+. Assume that each C_{φ_t} is bounded on $A^2_\alpha(\Pi^+)$. If G is the infinitesimal generator of $\{\varphi_t\}$, then the infinitesimal generator A of $\{C_{\varphi_t}\}$ has the domain of definition*

$$D(A) = \left\{f \in A^2_\alpha\left(\Pi^+\right) : Gf' \in A^2_\alpha\left(\Pi^+\right)\right\}$$

and is given by

$$Af = Gf'.$$

4 Conclusion

This paper studied the path component of composition operators spaces and the continuity of composition operator semigroups. In addition, the paper showed that the cancellation of double difference cannot occur in our settings. The results obtained extend some classical results on the unit disk to the upper half-plane. Due to the unboundedness of the half-plane, some special new techniques are used to overcome obstacles.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final version of this paper.

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