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Bonnesen-style symmetric mixed inequalities

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Abstract
In this paper, we investigate the symmetric mixed isoperimetric deficit $\Delta_2(K_0, K_1)$ of domains K_0 and K_1 in the Euclidean plane $\mathbb{R}^2$. Via the known kinematic formulae of Poincaré and Blaschke in integral geometry, we obtain some Bonnesen-style symmetric mixed inequalities. These new Bonnesen-style symmetric mixed inequalities are known as Bonnesen-style inequalities if one of the domains is a disc. Some inequalities obtained in this paper strengthen the known Bonnesen-style inequalities.

MSC: 52A10; 52A22

Keywords: isoperimetric inequality; symmetric mixed isoperimetric deficit; symmetric mixed isoperimetric inequality; Bonnesen-style symmetric mixed inequality

1 Introductions and preliminaries

A subset of points K in the Euclidean space $\mathbb{R}^n$ is called convex if for all $x, y \in K$, the line segment $\lambda x + (1-\lambda)y$ $(0 \le \lambda \le 1)$ joining x and y is contained in K. A domain is a set with nonempty interior, and a convex body is a compact convex domain. The Minkowski sum of convex sets K and L is defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

and the scalar product of the convex set K with $\lambda \ge 0$ is defined by

$$\lambda K = \{\lambda x : x \in K, \lambda \ge 0\}.$$

A homothety of the convex set K is of the form $x + \lambda K$ $(x \in \mathbb{R}^n, \lambda > 0)$.

Let S^1 be the unit circle in $\mathbb{R}^2$, and $u \in S^1$. The support function $p_K(u) : S^1 \to \mathbb{R}$ of a convex domain $K \subseteq \mathbb{R}^2$ is defined by

$$p_K(u) = \max\{x \cdot u : x \in K, u \in S^1\} \tag{1.1}$$

and uniquely determines the convex domain K. Let K_k $(k = 0, 1)$ be two convex domains of areas A_k and perimeters P_k in $\mathbb{R}^2$. Then

$$p_{K_0}(u) \le p_{K_1}(u) \quad \text{if and only if} \quad K_0 \subseteq K_1. \tag{1.2}$$

By the Steiner formula (see [1]) the area of $sK_0 + tK_1$ is

$$A_{sK_0+tK_1} = s^2A_0 + 2stA(K_0,K_1) + t^2A_1, \tag{1.3}$$

where $A(K_0,K_1)$ is called the mixed area of K_0 and K_1. The mixed area $A(K_0,K_1)$ satisfies (see [1])

$$A(K_0,K_0) = A_0, \tag{1.4}$$

the symmetry

$$A(K_0,K_1) = A(K_1,K_0), \tag{1.5}$$

the linearity

$$A(K_0,sK_1 + tK_2) = sA(K_0,K_1) + tA(K_0,K_2), \tag{1.6}$$

and the monotonicity

$$K_1 \subseteq K_2 \quad \Rightarrow \quad A(K_0,K_1) \le A(K_0,K_2). \tag{1.7}$$

Let G_2 be the group of plane rigid motions (see [2–4]), that is, translations and rotations. Let θ be rotation angle of K_1 with respect to origin, and $g \in G_2$. Then we have (see [4])

$$\int_0^{2\pi} A(K_0,gK_1)\,d\theta = \frac{1}{2}P_0P_1. \tag{1.8}$$

The classical isoperimetric problem says that the disc encloses the maximum area among all plane domains of given perimeter. That is: Let Γ be a simple closed curve of perimeter P in the Euclidean plane $\mathbb{R}^2$, and A be the area of the domain K enclosed by Γ; then

$$P^2 - 4\pi A \ge 0, \tag{1.9}$$

where the equality holds if and only if Γ is a circle.

The classical isoperimetric problem can root back to Ancient Greece. However, a rigorous mathematical proof of the isoperimetric inequality was obtained during the 19th century (see [2, 5–11]). We can find some simplified and beautiful proofs that lead to generalizations of the discrete case, higher dimensions, the surface of constant curvature, and applications to other branches of mathematics [1, 4, 12–34].

The isoperimetric inequality (1.9) indicates that the quantity

$$\Delta_2(K) = P^2 - 4\pi A \tag{1.10}$$

measures the deficit of domain K and a disc of radius $P/2\pi$, and it is called the isoperimetric deficit of K.

During the 1920s, Bonnesen proved a series of inequalities of the form

$$\Delta_2(K) = P^2 - 4\pi A \geq B_K, \tag{1.11}$$

where B_K is a nonnegative invariant of geometric significance and vanishes if and only if K is a disc. An inequality of the form (1.11) is called the Bonnesen-style inequality, and it is stronger than the classical isoperimetric inequality. The following Bonnesen-style inequalities are known.

Proposition 1.1 *Let K be a plane domain of area A and bounded by a simple closed curve of perimeter P. Denote by R and r, respectively, the radius of the minimum circumscribed disc and radius of the maximum inscribed disc of K. Then we have*

$$\begin{aligned} &\pi t^2 - Pt + A \leq 0; \quad r \leq t \leq R, \\ &\frac{P - \sqrt{P^2 - 4\pi A}}{2\pi} \leq r \leq R \leq \frac{P + \sqrt{P^2 - 4\pi A}}{2\pi}, \\ &P^2 - 4\pi A \geq \pi^2 (R - r)^2. \end{aligned} \tag{1.12}$$

Each equality sign holds when K is a disc.

Many Bonnesen-style inequalities have been found during the past, and mathematicians are still working on unknown Bonnesen-style inequalities of geometric significance. See [1, 3, 4, 12–16, 19–29, 32, 33, 35–46] for more references.

Let K_k ($k = 0, 1$) be two domains of areas A_k and bounded by simple closed curves of perimeters P_k in $\mathbb{R}^2$. Let dg denote the kinematic density of the group G_2 of plane rigid motions [2–4]. Let K_1 be convex, and tK_1 be a dilation of K_1. Let $n\{\partial K_0 \cap \partial(t(gK_1))\}$ denote the number of points of intersection $\partial K_0 \cap \partial(t(gK_1))$. Then we have the kinematic formula of Poincaré:

$$\int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} n\big\{\partial K_0 \cap \partial\big(t(gK_1)\big)\big\}\, dg = 4tP_0P_1. \tag{1.13}$$

Let $\chi(K_0 \cap t(gK_1))$ be the Euler-Poincaré characteristic of the intersection $K_0 \cap t(gK_1)$. Then we have the fundamental kinematic formula of Blaschke:

$$\int_{\{g\in G_2:K_0\cap t(gK_1)\neq\emptyset\}} \chi\big(K_0 \cap t(gK_1)\big)\, dg = 2\pi\big(t^2A_1 + A_0\big) + tP_0P_1. \tag{1.14}$$

If μ denotes the set of all positions of K_1 in which either $t(gK_1) \subset K_0$ or $t(gK_1) \supset K_0$, then the fundamental kinematic formula of Blaschke (1.14) can be rewritten as

$$\int_\mu dg + \int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} \chi\big(K_0 \cap t(gK_1)\big)\, dg = 2\pi\big(t^2A_1 + A_0\big) + tP_0P_1. \tag{1.15}$$

When $\partial K_0 \cap \partial(t(gK_1)) \neq \emptyset$, each component of $K_0 \cap t(gK_1)$ is bounded by at least an arc of ∂K_0 and an arc of $\partial(t(gK_1))$. Therefore, $\chi(K_0 \cap t(gK_1)) \leq n\{\partial K_0 \cap \partial(t(gK_1))\}/2$. Then by the kinematic formulae of Poincaré (1.13) and Blaschke (1.15) we obtain

$$\int_\mu dg \geq 2\pi A_1 t^2 - P_0P_1 t + 2\pi A_0. \tag{1.16}$$

Inequality (1.16) immediately gives the following containment theorem [40, 47–49].

Containment theorem *Let K_k $(k = 0,1)$ be two domains of areas A_k with simple boundaries of perimeters P_k in $\mathbb{R}^2$. Let K_1 be convex, and tK_1 be a dilation of K_1. A sufficient condition for tK_1 to contain, or to be contained in K_0, is*

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 > 0. \tag{1.17}$$

Moreover, if $t^2 A_1 \geq A_0$, then tK_1 contains K_0.

Let $r_{01}^g = \max\{t : t(gK_1) \subseteq K_0; g \in G_2\}$ and $R_{01}^g = \min\{t : t(gK_1) \supseteq K_0; g \in G_2\}$ be the inradius of K_0 with respect to K_1 and the outradius of K_0 with respect to K_1, respectively. It is obvious that $r_{01}^g \leq R_{01}^g$. Since both r_{01}^g and R_{01}^g are rigid invariant, we simply call them the relative inradius and the relative outradius and denote them r_{01} and R_{01}, respectively. Note that if K_1 is the unit disc, then the relative inradius r_{01} and the relative outradius R_{01} become the inscribed radius r and the circumscribed radius R of K_0, respectively.

Note that for $t \in [r_{01}, R_{01}]$, neither tK_1 contains K_0 nor is contained in K_0. Then by inequality (1.17) we have [40, 47–49]

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 \leq 0, \quad t \in [r_{01}, R_{01}]. \tag{1.18}$$

Inequality (1.18) guarantees that the equation $B_{K_0,K_1}(t) = 2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 = 0$ has root(s). Therefore, the determinant of $B_{K_0,K_1}(t) = 0$ is nonnegative. Then we have the following symmetric mixed isoperimetric inequality [40, 47–49]:

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 0, \tag{1.19}$$

where the equality sign holds if and only if both K_0 and K_1 are discs.

When K_1 is the unit disc, then symmetric mixed isoperimetric inequality (1.19) reduces to the isoperimetric inequality (1.9).

The quantity

$$\Delta_2(K_0, K_1) = P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \tag{1.20}$$

is called the *symmetric mixed isoperimetric deficit* of K_0 and K_1.

Motivated by the Bonnesen's works in the 1920s, we consider if there is a nonnegative invariant B_{K_0,K_1} of geometric significance such that

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq B_{K_0,K_1}, \tag{1.21}$$

where B_{K_0,K_1} vanishes if and only if both K_0 and K_1 are discs. We call such inequalities *Bonnesen-style symmetric mixed inequalities* (*cf.* [40, 47–49]).

The purpose of this paper is to find some new Bonnesen-style symmetric mixed isoperimetric inequalities that strengthen the known Bonnesen-style inequalities.

2 Bonnesen-style symmetric mixed inequality

For any two plane domains K_k $(k = 0,1)$ of areas A_k with simple boundaries of perimeters P_k, the convex hulls K_k^* of K_k increase the areas A_k^* and decrease the perimeters P_k^*, that is,

$A_k^* \geq A_k$ and $P_k^* \leq P_k$, so that $P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq P_0^{*2} P_1^{*2} - 16\pi^2 A_0^* A_1^*$, that is, $\Delta_2(K_0, K_1) \geq \Delta_2(K_0^*, K_1^*)$. Therefore, the symmetric mixed isoperimetric inequality and Bonnesen-type symmetric mixed inequality are valid for all domains with simple boundaries in $\mathbb{R}^2$ if these inequalities are valid for convex domains. Hence, from now on, we only consider convex domains when we estimate the lower bounds of the symmetric mixed isoperimetric deficit.

Lemma 2.1 *Let K_k $(k = 0, 1)$ be two convex domains of areas A_k and perimeters P_k in the Euclidean plane $\mathbb{R}^2$. Then*

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 \leq 0, \quad t \in [r_{01}, R_{01}]. \tag{2.1}$$

The inequality is strict whenever $r_{01} < t < R_{01}$. When $t = r_{01}$, equality holds if and only if K_1 is a disc and K_0 is the Minkowski sum of a disc and a line segment (which may be a point). When $t = R_{01}$, equality holds if and only if K_0 is a disc and K_1 is the Minkowski sum of a disc and a line segment (which may be a point).

Proof Let $p_{K_0}(u)$ and $p_{(gK_1)}(u)$ are the support functions of convex domains K_0 and gK_1, respectively. We can always find $g \in G_2$ such that the function $p_{K_0}(u) - tp_{(gK_1)}(u)$ about u is nonnegative for $t \in [0, r_{01}]$. Let $\widetilde{K}_t$ be given by

$$\widetilde{K}_t = \left\{x \in \mathbb{R}^2 : x \cdot u \leq p_{K_0}(u) - tp_{(gK_1)}(u); u \in S^1, g \in G_2\right\}, \quad t \in [0, r_{01}]. \tag{2.2}$$

From (2.2) we have that $\widetilde{K}_0 = K_0$ and $\widetilde{K}_{r_{01}}$ is a line segment (which may be a point); see the proof of (6.5.11) in [1]. Therefore,

$$A(\widetilde{K}_0) = A_0, \qquad A(\widetilde{K}_{r_{01}}) = 0. \tag{2.3}$$

From definitions (2.2) and (1.2) we immediately have

$$\widetilde{K}_t + t(gK_1) \subseteq K_0. \tag{2.4}$$

However, relation (2.4), together with the monotonicity (1.7), linearity (1.6), the symmetry of mixed areas (1.5), and (1.4), gives

$$A(K_0, gK_1) \geq A\big(\widetilde{K}_t + t(gK_1), gK_1\big) = A(\widetilde{K}_t, gK_1) + tA(gK_1), \tag{2.5}$$

and we have (see the proof of (2.17) in [50]) that

$$A_0 - A(\widetilde{K}_t) = 2\int_0^t A(\widetilde{K}_s, gK_1)\,ds; \quad t \in [0, r_{01}]. \tag{2.6}$$

Now (2.6), (2.5), and (1.8) give

$$\begin{aligned} \int_0^{2\pi} \big(A_0 - A(\widetilde{K}_t)\big)\,d\theta &= \int_0^{2\pi} 2\int_0^t A(\widetilde{K}_s, gK_1)\,ds\,d\theta \\ &\leq 2\int_0^{2\pi}\int_0^t \big(A(K_0, gK_1) - sA(gK_1)\big)\,ds\,d\theta \\ &= P_0 P_1 t - 2\pi A_1 t^2. \end{aligned} \tag{2.7}$$

Thus,

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 \le \int_0^{2\pi} A(\widetilde{K}_t)\,d\theta. \tag{2.8}$$

From (2.5) and (2.7) we see that equality holds in (2.8) if and only if, for all $s \in [0,t]$,

$$\begin{aligned}\int_0^{2\pi} A(\widetilde{K}_s, gK_1)\,d\theta &= \int_0^{2\pi} \bigl(A(K_0, gK_1) - sA(gK_1)\bigr)\,d\theta \\ &= \frac{1}{2}P_0P_1 - 2\pi s A_1.\end{aligned} \tag{2.9}$$

When $t = r_{01}$, by (2.3), (2.8), and (2.9) we have

$$2\pi A_1 r_{01}^2 - P_0P_1 r_{01} + 2\pi A_0 \le 0, \tag{2.10}$$

where the equality holds if and only if

$$\int_0^{2\pi} A(\widetilde{K}_{r_{01}}, gK_1)\,d\theta = \frac{1}{2}P_0P_1 - 2\pi r_{01}A_1.$$

Therefore, we have

$$2\pi A_1 r_{01}^2 - 2\left(\int_0^{2\pi} A(\widetilde{K}_{r_{01}}, gK_1)\,d\theta + 2\pi r_{01}A_1\right) r_{01} + 2\pi A_0 = 0,$$

that is,

$$r_{01}\int_0^{2\pi} A(\widetilde{K}_{r_{01}}, gK_1)\,d\theta = -\pi A_1 r_{01}^2 + \pi A_0. \tag{2.11}$$

By (1.3), (2.3), and (2.11) we have

$$\begin{aligned}\int_0^{2\pi} A\bigl(\widetilde{K}_{r_{01}} + r_{01}(gK_1)\bigr)\,d\theta &= \int_0^{2\pi} \bigl(A(\widetilde{K}_{r_{01}}) + 2r_{01}A(\widetilde{K}_{r_{01}}, gK_1) + A(gK_1)r_{01}^2\bigr)\,d\theta \\ &= 2\pi A_0.\end{aligned} \tag{2.12}$$

Since $\widetilde{K}_{r_{01}} + r_{01}(gK_1) \subseteq K_0$, we have $A(\widetilde{K}_{r_{01}} + r_{01}(gK_1)) \le A_0$. Equality (2.12) forces us to conclude that $A(\widetilde{K}_{r_{01}} + r_{01}(gK_1)) = A_0$ for any $g \in G_2$, that is, $\widetilde{K}_{r_{01}} + r_{01}(gK_1) = K_0$ for any $g \in G_2$. Therefore, K_1 must be a disc, and K_0 is the Minkowski sum of a dilation of K_1 (a disc) and a line segment $\widetilde{K}_{r_{01}}$ (which may be a point).

Let $r'_{01} = \max\{t : t(gK_0) \subseteq K_1; g \in G_2\}$ be the inradius of K_1 with respect to K_0. Obviously, from the definition of r'_{01} and R_{01} it follows that

$$r'_{01} = \frac{1}{R_{01}}.$$

From inequality (2.10) we establish

$$2\pi A_0 r'^2_{01} - P_0P_1 r'_{01} + 2\pi A_1 \le 0 \tag{2.13}$$

with equality if and only if K_0 is a disc and K_1 is the Minkowski sum of a disc and a line segment (which may be a point), that is,

$$2\pi A_1 R_{01}^2 - P_0 P_1 R_{01} + 2\pi A_0 \leq 0, \tag{2.14}$$

with equality if and only if K_0 is a disc and K_1 is the Minkowski sum of a disc and a line segment (which may be a point).

Finally, inequalities (2.10) and (2.14), together with the well-known properties of quadratic functions, show that

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 < 0, \quad r_{01} < t < R_{01}. \qquad \square$$

Remark 2.1 An analogue of inequality (2.1) can already be found in Bol's work. A complete proof of the analogous inequality (2.1) with equality conditions is given by Böröczky *et al.* [50] and Luo *et al.* [51].

When K_1 is the unit disc, inequality (2.1) reduces to the following known Bonnesen inequality (see [4, 7, 20, 41]).

Corollary 2.1 *Let K be a convex domain with length P and area A in $\mathbb{R}^2$. Denote by R and r, respectively, the radius of the minimum circumscribed disc and radius of the maximum inscribed disc of K. Then*

$$\pi t^2 - Pt + A \leq 0, \quad t \in [r, R]. \tag{2.15}$$

The inequality is strict whenever $r < t < R$. When $t = r$, equality holds if and only if K is the Minkowski sum of a disc and a line segment (which may be a point). When $t = R$, equality holds if and only if K is a disc.

Lemma 2.2 *Let K_k $(k = 0, 1)$ be two convex domains with areas A_k and perimeters P_k in $\mathbb{R}^2$. Then, for $r_{01} \leq t \leq R_{01}$, we have*

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 4\pi^2 A_1^2 (R_{01} - t)^2 + \bigl[2\pi A_1 (R_{01} + t) - P_0 P_1\bigr]^2. \tag{2.16}$$

The inequality is strict whenever $r_{01} < t < R_{01}$. When $t = r_{01}$, the equality holds if and only if both K_0 and K_1 are discs. When $t = R_{01}$, the equality holds if and only if K_0 is a disc and K_1 is the Minkowski sum of a disc and a line segment (which may be a point).

Proof By inequality (2.1),

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 \leq 0, \quad t \in [r_{01}, R_{01}],$$

so that

$$2\pi A_1 R_{01}^2 - P_0 P_1 R_{01} + 2\pi A_0 \leq 0,$$

that is,

$$-8\pi^2A_0A_1 \geq 8\pi^2A_1^2t^2 - 4\pi A_1tP_0P_1,$$
$$-8\pi^2A_0A_1 \geq 8\pi^2A_1^2R_{01}^2 - 4\pi A_1R_{01}P_0P_1.$$

By adding the last inequalities side by side we have

$$-16\pi^2A_0A_1 \geq 8\pi^2A_1^2t^2 + 8\pi^2A_1^2R_{01}^2 - 4\pi A_1tP_0P_1 - 4\pi A_1R_{01}P_0P_1,$$

that is,

$$\begin{aligned} P_0^2P_1^2 - 16\pi^2A_0A_1 &\geq P_0^2P_1^2 + 8\pi^2A_1^2t^2 + 8\pi^2A_1^2R_{01}^2 - 4\pi A_1tP_0P_1 - 4\pi A_1R_{01}P_0P_1 \\ &= 4\pi^2A_1^2t^2 + 4\pi^2A_1^2R_{01}^2 - 8\pi^2A_1^2tR_{01} + P_0^2P_1^2 + 4\pi^2A_1^2t^2 \\ &\quad + 4\pi^2A_1^2R_{01}^2 + 8\pi^2A_1^2tR_{01} - 4\pi A_1tP_0P_1 - 4\pi A_1R_{01}P_0P_1 \\ &= 4\pi^2A_1^2(R_{01}-t)^2 + \big[2\pi A_1(R_{01}+t) - P_0P_1\big]^2. \end{aligned}$$

When $t = r_{01}$, the equality holds in (2.16) if and only if the equalities hold in (2.1) when $t = r_{01}$ and $t = R_{01}$, that is, K_0 and K_1 are discs. When $t = R_{01}$, the equality holds in (2.16) if and only if the equalities hold in (2.1) when $t = R_{01}$, that is, K_0 is a disc, and K_1 is the Minkowski sum of a disc and a line segment (which may be a point). From the equality conditions in (2.1) we know that inequality (2.16) is strict whenever $r_{01} < t < R_{01}$. □

Let

$$f(t) = 4\pi^2A_1^2(R_{01}-t)^2 + \big[2\pi A_1(R_{01}+t) - P_0P_1\big]^2, \quad t \in [r_{01}, R_{01}].$$

Then

$$f'(t) = 16\pi^2A_1^2\left(t - \frac{P_0P_1}{4\pi A_1}\right)$$

and

$$f''(t) = 16\pi^2A_1^2 > 0.$$

Therefore, $f(t)$ is concave and reaches the minimum at $t = \frac{P_0P_1}{4\pi A_1}$ and the maximum at $t = r_{01}$ or $t = R_{01}$. Then we obtain the following Bonnesen-style symmetric mixed inequality.

Corollary 2.2 *Let K_k $(k = 0, 1)$ be two convex domains with areas A_k and perimeters P_k in $\mathbb{R}^2$. Then we have*

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \geq 4\pi^2A_1^2\left(R_{01} - \frac{P_0P_1}{4\pi A_1}\right)^2 + \left[2\pi A_1\left(R_{01} + \frac{P_0P_1}{4\pi A_1}\right) - P_0P_1\right]^2, \quad (2.17)$$

where the equality holds if and only if both K_0 and K_1 are discs.

Proof When $t = \frac{P_0P_1}{4\pi A_1}$ in (2.16), we immediately obtain (2.17). From the proof of Lemma 2.2 we see that the equality holds in (2.17) if and only if the equalities hold in (2.1) when $t = R_{01}$ and $t = \frac{P_0P_1}{4\pi A_1}$, that is, R_{01} and $\frac{P_0P_1}{4\pi A_1}$ are roots of the equation $B_{K_0,K_1}(t) = 2\pi A_1t^2 - P_0P_1t + 2\pi A_0 = 0$. It is obvious that $B_{K_0,K_1}(t)$ reaches the minimum at $t = \frac{P_0P_1}{4\pi A_1}$, therefore, there is only one root $R_{01} = \frac{P_0P_1}{4\pi A_1}$ for the equation $B_{K_0,K_1}(t) = 0$, that is, the determinant $P_0^2P_1^2 - 16\pi^2A_0A_1 = 0$. By the symmetric mixed isoperimetric inequality (1.19), K_0 and K_1 are discs. □

Letting $t = r_{01}$ in inequality (2.16), we immediately obtain the following:

Theorem 2.1 *Let K_k $(k = 0,1)$ be two convex domains with areas A_k and perimeters P_k in $\mathbb{R}^2$. Then we have*

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \geq 4\pi^2A_1^2(R_{01} - r_{01})^2 + \left[2\pi A_1(R_{01} + r_{01}) - P_0P_1\right]^2, \tag{2.18}$$

where the equality holds if and only if both K_0 and K_1 are discs.

The following Kotlyar inequality (see [17, 40, 47–49]) is an immediate consequence of Theorem 2.1.

Corollary 2.3 *Let K_k $(k = 0,1)$ be two convex domains with areas A_k and perimeters P_k in $\mathbb{R}^2$. Then we have*

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \geq 4\pi^2A_1^2(R_{01} - r_{01})^2, \tag{2.19}$$

where the equality holds if and only if both K_0 and K_1 are discs.

When $t = R_{01}$ in inequality (2.16), we immediately have the following:

Theorem 2.2 *Let K_k $(k = 0,1)$ be two convex domains with areas A_k and perimeters P_k in $\mathbb{R}^2$. Then we have*

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \geq 16\pi^2A_1^2\left(R_{01} - \frac{P_0P_1}{4\pi A_1}\right)^2, \tag{2.20}$$

where the equality holds if and only if K_0 is a disc and K_1 is the Minkowski sum of a disc and a line segment (which may be a point).

We also have the following:

Lemma 2.3 *Let K_k $(k = 0,1)$ be two convex domains with areas A_k and perimeters P_k in $\mathbb{R}^2$. Then we have*

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \geq 4\pi^2A_1^2(t - r_{01})^2 + \left[2\pi A_1(t + r_{01}) - P_0P_1\right]^2, \quad t \in [r_{01}, R_{01}]. \tag{2.21}$$

The inequality is strict whenever $r_{01} < t < R_{01}$. When $t = r_{01}$, the equality holds if and only if K_1 is a disc and K_0 is the Minkowski sum of a disc and a line segment (which may be a point). When $t = R_{01}$, the equality holds if and only if both K_0 and K_1 are discs.

Proof By inequality (2.1),

$$2\pi A_1 t^2 - P_0P_1 t + 2\pi A_0 \le 0, \quad t \in [r_{01}, R_{01}],$$

and thus

$$2\pi A_1 r_{01}^2 - P_0P_1 r_{01} + 2\pi A_0 \le 0,$$

so that

$$-8\pi^2 A_0A_1 \ge 8\pi^2 A_1^2 t^2 - 4\pi A_1 t P_0P_1,$$
$$-8\pi^2 A_0A_1 \ge 8\pi^2 A_1^2 r_{01}^2 - 4\pi A_1 r_{01} P_0P_1.$$

By adding the last inequalities side by side we have

$$-16\pi^2 A_0A_1 \ge 8\pi^2 A_1^2 t^2 + 8\pi^2 A_1^2 r_{01}^2 - 4\pi A_1 t P_0P_1 - 4\pi A_1 r_{01} P_0P_1.$$

Then,

$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \ge 4\pi^2 A_1^2 (t - r_{01})^2 + \big[2\pi A_1(t + r_{01}) - P_0P_1\big]^2.$$

Similarly, following the equality conditions in Lemma 2.2, we have the equality conditions for (2.21). □

The function

$$g(t) = 4\pi^2 A_1^2 (t - r_{01})^2 + \big[2\pi A_1(t + r_{01}) - P_0P_1\big]^2, \quad t \in [r_{01}, R_{01}],$$

is concave and reaches the minimum at $t = \frac{P_0P_1}{4\pi A_1}$. Then we immediately obtain the following Bonnesen-style symmetric mixed inequality.

Corollary 2.4 *Let K_k $(k = 0, 1)$ be two convex domains with areas A_k and perimeters P_k in $\mathbb{R}^2$. Then we have*

$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \ge 4\pi^2 A_1^2 \left(\frac{P_0P_1}{4\pi A_1} - r_{01}\right)^2 + \left[2\pi A_1\left(\frac{P_0P_1}{4\pi A_1} + r_{01}\right) - P_0P_1\right]^2, \quad (2.22)$$

where the equality holds if and only if K_0 and K_1 are discs.

Letting $t = R_{01}$ in inequality (2.21), we obtain Theorem 2.1.

When $t = r_{01}$ in inequality (2.21), we have the following Bonnesen-style symmetric mixed inequality.

Theorem 2.3 *Let K_k $(k = 0, 1)$ be two convex domains with areas A_k and perimeters P_k in $\mathbb{R}^2$. Then we have*

$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \ge 16\pi^2 A_1^2 \left(\frac{P_0P_1}{4\pi A_1} - r_{01}\right)^2, \quad (2.23)$$

where the equality holds if and only if K_1 is a disc and K_0 is the Minkowski sum of a disc and a line segment (*which may be a point*).

The lower bound of symmetric mixed isoperimetric deficit in inequality (2.18) or (2.20) is the maximum of the function $f(t)$. The lower bound of symmetric mixed isoperimetric deficit in inequality (2.18) or (2.23) is the maximum of the function $g(t)$. Which one is the best lower bound of symmetric mixed isoperimetric deficit in inequalities (2.18), (2.20), and (2.23)?

Since

$$\begin{aligned} &16\pi^2 A_1^2\left(R_{01}-\frac{P_0P_1}{4\pi A_1}\right)^2 \\ &\quad-\left\{4\pi^2A_1^2(R_{01}-r_{01})^2+\left[2\pi A_1(R_{01}+r_{01})-P_0P_1\right]^2\right\} \\ &= 8\pi^2A_1^2(R_{01}-r_{01})\left(R_{01}+r_{01}-\frac{P_0P_1}{2\pi A_1}\right), \\ &16\pi^2 A_1^2\left(\frac{P_0P_1}{4\pi A_1}-r_{01}\right)^2 \\ &\quad-\left\{4\pi^2A_1^2(R_{01}-r_{01})^2+\left[2\pi A_1(R_{01}+r_{01})-P_0P_1\right]^2\right\} \\ &= -8\pi^2A_1^2(R_{01}-r_{01})\left(R_{01}+r_{01}-\frac{P_0P_1}{2\pi A_1}\right), \end{aligned}$$

and

$$\begin{aligned} &16\pi^2 A_1^2\left(R_{01}-\frac{P_0P_1}{4\pi A_1}\right)^2-16\pi^2 A_1^2\left(\frac{P_0P_1}{4\pi A_1}-r_{01}\right)^2 \\ &= 16\pi^2A_1^2(R_{01}-r_{01})\left(R_{01}+r_{01}-\frac{P_0P_1}{2\pi A_1}\right), \end{aligned}$$

when $R_{01}+r_{01}-\frac{P_0P_1}{2\pi A_1}\geq 0$, these lower bounds in inequalities (2.18), (2.20), and (2.23) satisfy

$$\begin{aligned} 16\pi^2 A_1^2\left(R_{01}-\frac{P_0P_1}{4\pi A_1}\right)^2 &\geq 4\pi^2A_1^2(R_{01}-r_{01})^2+\left[2\pi A_1(R_{01}+r_{01})-P_0P_1\right]^2 \\ &\geq 16\pi^2 A_1^2\left(\frac{P_0P_1}{4\pi A_1}-r_{01}\right)^2. \end{aligned}$$

Therefore, the lower bound $16\pi^2A_1^2(R_{01}-\frac{P_0P_1}{4\pi A_1})^2$ in inequality (2.20) is the best among (2.18), (2.20), and (2.23), that is, inequality (2.20) is the strongest Bonnesen-style symmetric mixed inequality among inequalities (2.18), (2.20), and (2.23).

When $R_{01}+r_{01}-\frac{P_0P_1}{2\pi A_1}\leq 0$, the lower bounds in inequalities (2.18), (2.20), and (2.23) satisfy

$$\begin{aligned} 16\pi^2 A_1^2\left(\frac{P_0P_1}{4\pi A_1}-r_{01}\right)^2 &\geq 4\pi^2A_1^2(R_{01}-r_{01})^2+\left[2\pi A_1(R_{01}+r_{01})-P_0P_1\right]^2 \\ &\geq 16\pi^2 A_1^2\left(R_{01}-\frac{P_0P_1}{4\pi A_1}\right)^2. \end{aligned}$$

The Bonnesen-style symmetric mixed inequality (2.23), that is,

$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 16\pi^2 A_1^2\left(\frac{P_0P_1}{4\pi A_1} - r_{01}\right)^2,$$

is the best one among inequalities (2.18), (2.20), and (2.23).

When K_1 is the unit disc, these Bonnesen-style symmetric mixed inequalities immediately lead to the following known Bonnesen-style inequalities of Burago, Grinberg, Hsiung, Hadwiger, Osserman, Zhou, and Ren (see [9, 12, 16, 20, 21, 41, 44]).

Corollary 2.5 *Let K be a convex domain with area A and perimeter P in $\mathbb{R}^2$. Denote by R and r, respectively, the radius of the minimum circumscribed disc and radius of the maximum inscribed disc of K. Then*

$$P^2 - 4\pi A \geq (P - 2\pi r)^2,$$

$$P^2 - 4\pi A \geq (2\pi R - P)^2,$$

$$P^2 - 4\pi A \geq \pi^2(R - r)^2 + \left[\pi(R + r) - P\right]^2,$$

$$P^2 - 4\pi A \geq \pi^2\left(R - \frac{P}{2\pi}\right)^2 + \left[\pi\left(R + \frac{P}{2\pi}\right) - P\right]^2,$$

$$P^2 - 4\pi A \geq \pi^2\left(\frac{P}{2\pi} - r\right)^2 + \left[\pi\left(\frac{P}{2\pi} + r\right) - P\right]^2.$$

The equality in the first inequality holds if and only if K is the Minkowski sum of a disc and a line segment (*which may be a point*). *The equalities of the other inequalities hold if and only if K is a disc*.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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References

1. Schneider, R: Convex Bodies: The Brunn-Minkowski Theory. Cambridge University Press, Cambridge (1993)
2. Blaschké, W: Vorlesungen über Intergral Geometrie, 3rd edn. Deutsch. Verlag Wiss., Berlin (1955)
3. Hadwiger, H: Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. Springer, Berlin (1957)
4. Santaló, LA: Integral Geometry and Geometric Probability. Addison-Wesley, Reading (1976)
5. Banchoff, T, Pohl, W: A generalization of the isoperimetric inequality. J. Differ. Geom. **6**, 175-213 (1971)
6. Bokowski, J, Heil, E: Integral representation of quermassintegrals and Bonnesen-style inequalities. Arch. Math. **4**, 79-89 (1986)
7. Bonnesen, T: Les problèmes des isopérimètres et des isépiphanes. Gauthier-Villars, Paris (1929)
8. Bottema, O: Eine obere Grenze für das isoperimetrische Defizit ebener Kurven. Ned. Akad. Wet. Proc. **66**, 442-446 (1933)

9. Burago, YD, Zalgaller, VA: Geometric Inequalities. Springer, Berlin (1988)
10. Diskant, V: A generalization of Bonnesen's inequalities. Sov. Math. Dokl. **14**, 1728-1731 (1973)
11. Enomoto, K: A generalization of the isoperimetric inequality on S^2 and flat tori in S^3. Proc. Am. Math. Soc. **120**, 553-558 (1994)
12. Grinberg, E, Li, S, Zhang, G, Zhou, J: Integral Geometry and Convexity. Proceedings of the International Conference. World Scientific, Wuhan (2006)
13. Gysin, L: The isoperimetric inequality for nonsimple closed curves. Proc. Am. Math. Soc. **118**, 197-203 (1993)
14. Hadwiger, H: Die isoperimetrische Ungleichung in Raum. Elem. Math. **3**, 25-38 (1948)
15. Howard, R: The sharp Sobolev inequality and the Banchoff-Pohl inequality on surfaces. Proc. Am. Math. Soc. **126**, 2779-2787 (1998)
16. Hsiung, CC: Isoperimetric inequalities for two-dimensional Riemannian manifolds with boundary. Ann. Math. **73**, 213-220 (1961)
17. Kotlyar, BD: On a geometric inequality. Ukr. Geom. Sb. **30**, 49-52 (1987)
18. Ku, H, Ku, M, Zhang, X: Isoperimetric inequalities on surfaces of constant curvature. Can. J. Math. **49**, 1162-1187 (1997)
19. Li, M, Zhou, J: An isoperimetric deficit upper bound of the convex domain in a surface of constant curvature. Sci. China Math. **53**, 1941-1946 (2010)
20. Osserman, R: The isoperimetric inequality. Bull. Am. Math. Soc. **84**, 1182-1238 (1978)
21. Osserman, R: Bonnesen-style isoperimetric inequality. Am. Math. Mon. **86**, 1-29 (1979)
22. Pleijel, A: On konvexa kurvor. Nord. Math. Tidskr. **3**, 57-64 (1955)
23. Polya, G, Szegö, G: Isoperimetric Inequalities in Mathematical Physics. Ann. Math. Studies. Princeton University Press, Princeton (1951)
24. Stone, A: On the isoperimetric inequality on a minimal surface. Calc. Var. Partial Differ. Equ. **17**, 369-391 (2003)
25. Tang, D: Discrete Wirtinger and isoperimetric type inequalities. Bull. Aust. Math. Soc. **43**, 467-474 (1991)
26. Teufel, E: A generalization of the isoperimetric inequality in the hyperbolic plane. Arch. Math. **57**, 508-513 (1991)
27. Teufel, E: Isoperimetric inequalities for closed curves in spaces of constant curvature. Results Math. **22**, 622-630 (1992)
28. Wei, S, Zhu, M: Sharp isoperimetric inequalities and sphere theorems. Pac. J. Math. **220**, 183-195 (2005)
29. Weiner, J: A generalization of the isoperimetric inequality on the 2-sphere. Indiana Univ. Math. J. **24**, 243-248 (1974)
30. Weiner, J: Isoperimetric inequalities for immersed closed spherical curves. Proc. Am. Math. Soc. **120**, 501-506 (1994)
31. Yau, ST: Isoperimetric constants and the first eigenvalue of a compact manifold. Ann. Sci. Éc. Norm. Supér. **8**, 487-507 (1975)
32. Zhang, X-M: Bonnesen-style inequalities and pseudo-perimeters for polygons. J. Geom. **60**, 188-201 (1997)
33. Zhang, X-M: Schur-convex functions and isoperimetric inequalities. Proc. Am. Math. Soc. **126**, 461-470 (1998)
34. Zhang, G, Zhou, J: Containment measures in integral geometry. In: Integral Geometry and Convexity, pp. 153-168. World Scientific, Singapore (2006)
35. Gao, X: A note on the reverse isoperimetric inequality. Results Math. **59**, 83-90 (2011)
36. Gao, X: A new reverse isoperimetric inequality and its stability. Math. Inequal. Appl. **15**, 733-743 (2012)
37. Pan, S, Tang, X, Wang, X: A refined reverse isoperimetric inequality in the plane. Math. Inequal. Appl. **13**, 329-338 (2010)
38. Pan, S, Zhang, H: A reverse isoperimetric inequality for closed strictly convex plane curves. Beitr. Algebra Geom. **48**, 303-308 (2007)
39. Xia, Y, Xu, W, Zhou, J, Zhu, B: Reverse Bonnesen style inequalities in a surface X_ϵ^2 of constant curvature. Sci. China Math. **56**, 1145-1154 (2013)
40. Zeng, C, Zhou, J, Yue, S: The symmetric mixed isoperimetric inequality of two planar convex domains. Acta Math. Sin. **55**, 355-362 (2012)
41. Zhou, J: Plan Bonnesen-type inequalities. Acta Math. Sin. **50**, 1397-1402 (2007)
42. Zhou, J, Du, Y, Cheng, F: Some Bonnesen-style inequalities for higher dimensions. Acta Math. Sin. **28**, 2561-2568 (2012)
43. Zhou, J, Ma, L, Xu, W: On the isoperimetric deficit upper limit. Bull. Korean Math. Soc. **50**, 175-184 (2013)
44. Zhou, J, Ren, D: Geometric inequalities - from integral geometry point of view. Acta Math. Sci. **30**, 1322-1339 (2010)
45. Zhou, J, Xia, Y, Zeng, C: Some new Bonnesen-style inequalities. J. Korean Math. Soc. **48**, 421-430 (2011)
46. Zhou, J, Zhou, C, Ma, F: Isoperimetric deficit upper limit of a planar convex set. Rend. Circ. Mat. Palermo (2) Suppl. **81**, 363-367 (2009)
47. Wang, P, Xu, W, Zhou, J, Zhu, B: On Bonnesen-style symmetric mixed inequality of two planar convex domains. Sci. China Math. **45**, 245-254 (2015)
48. Xu, W, Zhou, J, Zhu, B: Bonnesen-style symmetric mixed isoperimetric inequality. In: Real and Complex Submanifolds. Springer Proceedings in Mathematics and Statistics, vol. 106, pp. 97-107 (2014)
49. Xu, W, Zhou, J, Zhu, B: On containment measure and the mixed isoperimetric inequality. J. Inequal. Appl. **2013**, 540 (2013)
50. Böröczky, K, Lutwak, E, Yang, D, Zhang, G: The log-Brunn-Minkowski inequality. Adv. Math. **231**, 1974-1997 (2012)
51. Luo, M, Xu, W, Zhou, J: Translative containment measure and symmetric mixed isohomothetic inequalities. Sci. China Math. **58**, 2593-2610 (2015)