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Monotonicity and absolute monotonicity for the two-parameter hyperbolic and trigonometric functions with applications

Zhen-Hang Yang[1,2] and Yu-Ming Chu[1*]

*Correspondence:
chuyuming2005@126.com
[1]School of Mathematics and Computation Sciences, Hunan City University, Yiyang, 413000, China
Full list of author information is available at the end of the article

Abstract
In this paper, we present the monotonicity and absolute monotonicity properties for the two-parameter hyperbolic and trigonometric functions. As applications, we find several complete monotonicity properties for the functions involving the gamma function and provide the bounds for the error function.

MSC: 33B10; 33B15; 33B20; 26A48; 26D07

Keywords: Stolarsky mean; hyperbolic function; trigonometric function; gamma function; error function; complete monotonicity; absolute monotonicity

1 Introduction

Let $p, q \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$. Then the Stolarsky mean $S_{p,q}(a,b)$ [1] is given by

$$S_{p,q}(a,b) = \begin{cases} [\frac{q(a^p-b^p)}{p(a^q-b^q)}]^{1/(p-q)}, & pq(p-q) \neq 0, \\ [\frac{a^p-b^p}{p(\log a-\log b)}]^{1/p}, & p \neq 0, q = 0, \\ [\frac{a^q-b^q}{q(\log a-\log b)}]^{1/q}, & p = 0, q \neq 0, \\ \exp[\frac{a^p\log a-b^p\log b}{a^p-b^p} - \frac{1}{p}], & p = q \neq 0, \\ \sqrt{ab}, & p = q = 0. \end{cases}$$

It is well known that $S_{p,q}(a,b)$ is continuous and symmetric on the domain $\{(p,q,a,b) : p, q \in \mathbb{R}, a > 0, b > 0\}$ and strictly increasing with respect to its parameters $p, q \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many bivariate means are particular cases of the Stolarksy mean, and many remarkable inequalities and properties for this mean can be found in the literature [2–13]. We clearly see that the value $S_{p,q}(a,b)$ in the case of $pq(p-q) = 0$ is the limit of the case of $pq(p-q) \neq 0$.

Let $b > a > 0$ and $t = \log\sqrt{b/a} \in (0, \infty)$. Then the Stolarsky mean $S_{p,q}(a,b)$ can be expressed by a hyperbolic function as follows:

$$S_{p,q}(a,b) = \sqrt{ab}H_{p,q}(t), \tag{1.1}$$

where

$$H_{p,q}(t) = \begin{cases} (\frac{q\sinh(pt)}{p\sinh(qt)})^{1/(p-q)}, & pq(p-q) \neq 0, \\ (\frac{\sinh(pt)}{pt})^{1/p}, & p \neq 0, q = 0, \\ (\frac{\sinh(qt)}{qt})^{1/q}, & p = 0, q \neq 0, \\ \exp(t\coth(pt) - \frac{1}{p}), & p = q \neq 0, \\ 1, & p = q = 0, \end{cases} \tag{1.2}$$

is the two-parameter hyperbolic sine function [14].

Let $p, q \in [-2, 2]$ and $t \in (0, \pi/2)$. Then the two-parameter trigonometric sine function $T_{p,q}(t)$ [14] is given by

$$T_{p,q}(t) = \begin{cases} (\frac{q\sin(pt)}{p\sin(qt)})^{1/(p-q)}, & pq(p-q) \neq 0, \\ (\frac{\sin(pt)}{pt})^{1/p}, & p \neq 0, q = 0, \\ (\frac{\sin(qt)}{qt})^{1/q}, & p = 0, q \neq 0, \\ \exp(t\cot(pt) - \frac{1}{p}), & p = q \neq 0, \\ 1, & p = q = 0. \end{cases} \tag{1.3}$$

The main purpose of this paper is to deal with the monotonicity of the functions $t \mapsto [\log H_{p,q}(t)]/t$ and $t \mapsto [\log H_{p,q}(t)]/t^2$ on the interval $(0, \infty)$ and with the absolute monotonicity of the functions $t \mapsto \log T_{p,q}(t)$, $t \mapsto [\log T_{p,q}(t)]/t$ and $t \mapsto [\log T_{p,q}(t)]/t^2$ on the interval $(0, \pi/2)$. As applications, we shall present several complete monotonicity properties for the functions involving the gamma function and provide bounds for the error function.

2 Main results

Theorem 2.1 *Let $p, q \in \mathbb{R}$, $t > 0$, and $H_{p,q}(t)$ be defined by* (1.2). *Then the function $t \mapsto [\log H_{p,q}(t)]/t$ is strictly increasing* (*decreasing*) *and strictly concave* (*convex*) *from* $(0, \infty)$ *onto* $(0, (p+q)/(|p|+|q|))$ $(((p+q)/(|p|+|q|), 0))$ *if* $p+q > 0$ (< 0).

Proof We only prove the desired result in the case of $pq(p+q) \neq 0$; the other cases can be derived easily from the continuity and limit values. Let

$$\begin{aligned} f_1(t) &= t\left[\frac{p\cosh(pt)}{\sinh(pt)} - \frac{q\cosh(qt)}{\sinh(qt)}\right] - \log\sinh(|p|t) \\ &\quad + \log\sinh(|q|t) + \log|p| - \log|q|, \\ f_2(t) &= tf_1'(t) - 2f_1(t), \\ F_1(u) &= \frac{u}{\sinh(u)}, \qquad F_2(u) = \frac{u^3\cosh(u)}{\sinh^3(u)}. \end{aligned}$$

Then elaborated computations lead to

$$f_1(0^+) = f_2(0^+) = \lim_{t\to 0^+} \frac{\log H_{p,q}(t)}{t} = 0, \tag{2.1}$$

$$\log H_{p,q}(t) = \frac{1}{p-q}\log\left(\frac{q\sinh(pt)}{p\sinh(qt)}\right) = \frac{1}{p-q}\log\left(\frac{|q|\sinh(|p|t)}{|p|\sinh(|q|t)}\right)$$
$$= \frac{|p|-|q|}{p-q}t + \frac{1}{p-q}\log\left[\frac{|q|(1-e^{-2|p|t})}{|p|(1-e^{-2|q|t})}\right],$$

$$\lim_{t\to\infty}\frac{\log H_{p,q}(t)}{t} = \frac{|p|-|q|}{p-q} = \frac{p+q}{|p|+|q|}, \tag{2.2}$$

$$\left[\frac{\log H_{p,q}(t)}{t}\right]' = \frac{f_1(t)}{(p-q)t^2} = \frac{p+q}{(|p|+|q|)t^2}\times\frac{f_1(t)}{|p|-|q|}, \tag{2.3}$$

$$f_1'(t) = \frac{1}{t}\left[\frac{(qt)^2}{\sinh^2(qt)} - \frac{(pt)^2}{\sinh^2(pt)}\right],$$

$$\frac{f_1'(t)}{|p|-|q|} = \frac{F_1^2(qt)-F_1^2(pt)}{(|p|-|q|)t} = \frac{F_1^2(|qt|)-F_1^2(|pt|)}{|pt|-|qt|}$$
$$= -\left[F_1(|qt|)+F_1(|pt|)\right]\frac{F_1(|qt|)-F_1(|pt|)}{|qt|-|pt|}, \tag{2.4}$$

$$\left[\frac{\log H_{p,q}(t)}{t}\right]'' = \frac{f_2(t)}{(p-q)t^3} = \frac{p+q}{(|p|+|q|)t^3}\times\frac{f_2(t)}{|p|-|q|}, \tag{2.5}$$

$$f_2'(t) = \frac{2}{t}\left[\frac{(pt)^3\cosh(pt)}{\sinh^3(pt)} - \frac{(qt)^3\cosh(qt)}{\sinh^3(qt)}\right],$$

$$\frac{f_2'(t)}{|p|-|q|} = \frac{2[F_2(|pt|)-F_2(|qt|)]}{|pt|-|qt|}, \tag{2.6}$$

$$F_1'(u) = -\frac{\cosh(u)}{\sinh^2(u)}\left[u-\tanh(u)\right] < 0, \tag{2.7}$$

$$F_2'(u) = -\frac{3u^3}{\sinh^4(u)}\left[\frac{\sinh(2u)}{2u} - \frac{2+\cosh(2u)}{3}\right] < 0 \tag{2.8}$$

for $u > 0$, where the inequality in (2.8) is the Cusa-type inequality given in [15].

It follows from (2.1), (2.4), and (2.6)-(2.8) that

$$\frac{f_1(t)}{|p|-|q|} > 0 \tag{2.9}$$

and

$$\frac{f_2(t)}{|p|-|q|} < 0 \tag{2.10}$$

for $t \in (0,\infty)$.

Therefore, Theorem 2.1 follows easily from (2.1)-(2.3), (2.5), (2.9), and (2.10). □

Theorem 2.2 *Let $p,q\in\mathbb{R}$ and $t>0$, and let $H_{p,q}(t)$ be defined by* (1.2). *Then the function $t\mapsto[\log H_{p,q}(t)]/t^2$ is strictly decreasing* (*increasing*) *from* $(0,\infty)$ *onto* $(0,(p+q)/6)$ $(((p+q)/6,0))$ *if* $p+q>0$ (<0).

Proof Let $g_1(t) = [\log H_{p,q}(t)]/t$ and $g_2(t) = t$. Then we clearly see that

$$\frac{g_1'(t)}{g_2'(t)} = g_1'(t) = \left[\frac{\log H_{p,q}(t)}{t}\right]', \tag{2.11}$$

and (2.1) leads

$$\frac{\log H_{p,q}(t)}{t^2} = \frac{g_1(t)}{g_2(t)} = \frac{g_1(t) - g_1(0^+)}{g_2(t) - g_2(0^+)}. \tag{2.12}$$

From Theorem 2.1, (2.11), (2.12), and the well-known monotone form of l'Hôpital's rule [16] we know that the function $t \mapsto [\log H_{p,q}(t)]/t^2$ is strictly decreasing (increasing) on $(0, \infty)$ if $p + q > 0$ (< 0).

It follows from l'Hôpital's rule and (2.2) that

$$\lim_{t\to 0^+} \frac{\log H_{p,q}(t)}{t^2} = \frac{p+q}{6} \quad \text{and} \quad \lim_{t\to\infty} \frac{\log H_{p,q}(t)}{t^2} = 0. \qquad \square$$

From (1.1) and Theorem 2.2 we get the following corollary.

Corollary 2.1 *For $a, b > 0$ with $a \neq b$, we have the double inequality*

$$\sqrt{ab} < (>)\, S_{p,q}(a,b) < (>)\, \sqrt{ab}e^{\frac{p+q}{24}(\log b - \log a)^2}$$

if $p + q > 0$ (< 0).

Letting $b > a > 0$, $t = \log \sqrt{b/a} > 0$, and $(p,q) = (1,0), (1,1), (3/2,1/2)$ in Corollary 2.1, we get the following corollary.

Corollary 2.2 *We have the inequalities*

$$\frac{\sinh(t)}{t} < e^{t^2/6}, \qquad e^{t\cosh(t)-1} < e^{t^2/3}, \qquad \frac{2\cosh(t)+1}{3} < e^{t^2/3}$$

for all $t > 0$.

Next, we recall the definition of absolutely monotonic function [17]. A real-valued function f is said to be absolutely monotonic on the interval I if f has derivatives of all orders on I and

$$f^{(n)}(x) > 0$$

for all $x \in I$ and $n \geq 0$.

Theorem 2.3 *Let $p, q \in [-2, 2]$ and $t \in (0, \pi/2)$, and let $T_{p,q}(t)$ be defined by* (1.3). *Then the functions $t \to \log T_{p,q}(t)$, $t \to [\log T_{p,q}(t)]/t$, and $t \to [\log T_{p,q}(t)]/t^2$ are absolutely monotonic on $(0, \pi/2)$ if $p + q < 0$. Moreover, the functions $t \to -\log T_{p,q}(t)$, $t \to -[\log T_{p,q}(t)]/t$, and $t \to -[\log T_{p,q}(t)]/t^2$ are absolutely monotonic on $(0, \pi/2)$ if $p + q > 0$.*

Proof We only prove the desired result in the case of $pq(p+q) \neq 0$; the other cases can be derived easily from the continuity and limit values.

Let $i = 0, 1, 2$. Then from (1.3) and the power series formula

$$\log \frac{\sin(t)}{t} = -\sum_{n=1}^{\infty} \frac{2^{2n-1}|B_{2n}|}{n(2n)!} t^{2n}, \quad |t| < \pi,$$

listed in [18], 4.3.71, we get

$$\log T_{p,q}(t) = \frac{1}{p-q}\log\left(\frac{q\sin(pt)}{p\sin(qt)}\right) = \frac{1}{p-q}\log\left(\frac{|qt|\sin(|pt|)}{|pt|\sin(|qt|)}\right)$$
$$= -(p+q)t^2\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|(p^{2n}-q^{2n})}{2n(2n)!(p^2-q^2)}t^{2n-2},$$
$$\frac{\log T_{p,q}(t)}{t^i} = -(p+q)t^{2-i}\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|(p^{2n}-q^{2n})}{2n(2n)!(p^2-q^2)}t^{2n-2}, \tag{2.13}$$

where B_n are the Bernoulli numbers.

Therefore, Theorem 2.3 follows easily from (2.13). □

Let $(p,q) = (1,0), (1,1), (3/2,1/2)$ in Theorem 2.3. Then we immediately get the following corollary.

Corollary 2.3 *We have the inequalities*

$$\left(\frac{2}{\pi}\right)^{4t^2/\pi^2} < \frac{\sin(t)}{t} < e^{-t^2/6}, \tag{2.14}$$
$$1-\frac{4t^2}{\pi^2} < \frac{t}{\tan(t)} < 1-\frac{t^2}{3}, \tag{2.15}$$
$$3^{-4t^2/\pi^2} < \frac{2\cos(t)+1}{3} < e^{-t^2/3}$$

for all $t \in (0, \pi/2)$.

Remark 2.1 The second inequality in (2.14) was first proved by Yang [19], and the double inequality (2.15) can be found in [20], which is better than the Redheffer-type inequality in Theorem 3 of [21].

Remark 2.2 Bhayo and Sándor [22], equation (3.3), presented the double inequality

$$1-\frac{4t^2}{\pi^2} < \frac{t}{\tan(t)} < \frac{\pi^2}{8}-\frac{t^2}{2} \tag{2.16}$$

for all $t \in (0, \pi/2)$. The second inequality in (2.16) is better than the second inequality in (2.15) for $t \in (\sqrt{3\pi^2/4-6}, \pi/2)$.

3 Applications

Recall that a real-valued function f is said to be completely monotonic [23] on the interval I if f has derivatives of all order on I and

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $n \geq 0$ and $x \in I$. The set of all completely monotonic functions on I is denoted by $\mathrm{CM}[I]$. A positive function f is said to be logarithmically completely monotonic on the interval I if its logarithm $\log f$ is completely monotonic on I. The class of all logarithmically completely monotonic functions on I is denoted by LCM[I]. The famous Bernstein theorem [17] implies that the function

$$f(x) = \int_0^\infty e^{-xt} g(t)\,dt$$

is completely monotonic on $(0,\infty)$ if and only if $g(t) \geq 0$ for all $t \in (0,\infty)$ if $g(t)$ is continuous on $(0,\infty)$.

Theorem 3.1 *Let $s,t,r \in \mathbb{R}$, $\rho = \min\{s,t,r\}$, $x \in (-\rho,\infty)$, let $\Gamma(u) = \int_0^\infty e^{-t}t^{u-1}\,dt$ $(u > 0)$ be the gamma function, $\psi(u) = \Gamma'(u)/\Gamma(u)$ be the psi function, and the function $x \to \nu(s,t,r;x)$ be defined by*

$$\nu(s,t,r;x) = \begin{cases} e^{-\psi(x+r)}[\frac{\Gamma(x+t)}{\Gamma(x+s)}]^{1/(t-s)}, & t \neq s, \\ e^{-\psi(x+r)} \lim_{t\to s}[\frac{\Gamma(x+t)}{\Gamma(x+s)}]^{1/(t-s)} = e^{\psi(x+s)-\psi(x+r)}, & t = s. \end{cases} \tag{3.1}$$

Then $\nu(s,t,r;x) \in$ LCM$[(-\rho,\infty)]$ if and only if $r \leq \min\{s,t\}$, and $1/\nu(s,t,r;x) \in$ LCM$[(-\rho,\infty)]$ if and only if $r \geq (s+t)/2$.

Proof We only prove the desired result in the case of $t \neq s$ because the case of $t = s$ can be derived easily from the continuity and limit values.

Let $L(a,b) = (b-a)/(\log b - \log a)$ be the logarithmic mean of two distinct positive real numbers a and b, $u > 0$, $y = |(t-s)u/2|$, and $p(s,t,r;u)$ and $q(s,t,r;u)$ be respectively defined by

$$p(s,t,r;u) = \frac{\log e^{(\rho-r)u} - \log \frac{e^{(\rho-s)u}-e^{(\rho-t)u}}{(t-s)u}}{u},$$

$$q(s,t,r;u) = \frac{e^{(\rho-r)u} - \frac{e^{(\rho-s)u}-e^{(\rho-t)u}}{(t-s)u}}{1-e^{-u}}.$$

Then we clearly see that

$$p(s,t,r;u) = -r - \frac{1}{u}\log\frac{e^{-su}-e^{-tu}}{(t-s)u} = -r + \frac{t+s}{2} - \frac{|t-s|}{2}\left(\frac{1}{y}\log\frac{\sinh(y)}{y}\right), \tag{3.2}$$

$$q(s,t,r;u) = \frac{u}{1-e^{-u}} L\left(e^{(\rho-r)u}, \frac{e^{(\rho-s)u}-e^{(\rho-t)u}}{(t-s)u}\right) p(s,t,r;u). \tag{3.3}$$

It follows from (1.2) and Theorem 2.1 that the function $y \to [\log(\sinh(y)/y)]/y$ is strictly increasing from $(0,\infty)$ onto $(0,1)$. Then (3.2) leads to the conclusion that

$$\min\{s,t\} - r = -r + \frac{t+s}{2} - \frac{|t-s|}{2} < p(s,t,r;u) < -r + \frac{t+s}{2}.$$

Therefore,

$$p(s,t,r;u) \geq 0 \tag{3.4}$$

for all $u > 0$ if and only if $r \leq \min\{s, t\}$, and

$$p(s,t,r;u) \leq 0 \tag{3.5}$$

for all $u > 0$ if and only if $r \geq (s+t)/2$.

From (3.1) and the integral formulas

$$\log\Gamma(x) = \int_0^\infty \frac{1}{u}\left((x-1)e^{-u} - \frac{e^{-u} - e^{-xu}}{1-e^{-u}}\right) du,$$
$$\psi(x) = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-xu}}{1-e^{-u}}\right) du,$$

given in [18], 6.1.50, 6.3.21, we get

$$\begin{aligned}\log v(s,t,r;x) &= \frac{\log\Gamma(x+t) - \log\Gamma(x+x)}{t-s} - \psi(x+r) \\ &= \int_0^\infty \frac{e^{-xu}}{1-e^{-u}}\left[\frac{e^{-tu} - e^{-su}}{(t-s)u} + e^{-ru}\right] du \\ &= \int_0^\infty e^{-(x+\rho)u} q(s,t,r;u)\, du. \end{aligned} \tag{3.6}$$

Therefore, Theorem 3.1 follows easily from (3.3)-(3.6) and the Bernstein theorem. □

Remark 3.1 Qi and Guo [24] gave a sufficient condition for $v(s,t,r;x) \in \mathrm{LCM}[(-\rho,\infty)]$ and a necessary and sufficient condition for $1/v(s,t,r;x) \in \mathrm{LCM}[(-\rho,\infty)]$ by using different methods.

Theorem 3.2 *Let $a, b, c \in \mathbb{R}$, $\rho = \min\{a, b, c\}$, $x \in (-\rho, \infty)$, and let the function $x \to U(a,b,c;x)$ be defined by*

$$U(a,b,c;x) = \begin{cases} \frac{1}{x+c}\left(\frac{\Gamma(x+a)}{\Gamma(x+b)}\right)^{1/(a-b)}, & b \neq a, \\ \lim_{b\to a} \frac{1}{x+c}\left(\frac{\Gamma(x+a)}{\Gamma(x+b)}\right)^{1/(a-b)} = \frac{1}{x+c} e^{\psi(x+a)}, & b = a. \end{cases} \tag{3.7}$$

Then $U(a,b,c;x) \in \mathrm{LCM}[(-\rho,\infty)]$ if and only if $c \leq (a + b - \max\{|a-b|, 1\})/2$, and $1/U(a,b,c;x) \in \mathrm{LCM}[(-\rho,\infty)]$ if and only if $c \geq (a + b - \min\{|a-b|, 1\})/2$.

Proof We only prove the desired result in the case of $b \neq a$ because the case of $b = a$ can be derived easily from the continuity and limit values.

We clearly see that $U(a,b,c;x) \in \mathrm{LCM}[(-\rho,\infty)]$ if and only if $-[\log U(a,b,c;x)]' \in \mathrm{CM}[(-\rho,\infty)]$ and that $1/U(a,b,c;x) \in \mathrm{LCM}[(-\rho,\infty)]$ if and only if $[\log U(a,b,c;x)]' \in \mathrm{CM}[(-\rho,\infty)]$.

Let $t > 0$, $H_{p,q}(t)$ be defined by (1.2), and $p(a,b,c;t)$ and $q(a,b,c;t)$ be respectively defined by

$$p(a,b,c;t) = \frac{\log e^{(\rho-c)t} - \log \frac{e^{(\rho-a)t} - e^{(\rho-b)t}}{(b-a)(1-e^{-t})}}{t}$$

and

$$q(a,b,c;t)=e^{(\rho-c)t}-\frac{e^{(\rho-a)t}-e^{(\rho-b)t}}{(b-a)(1-e^{-t})}.$$

Then we clearly see that

$$\begin{aligned}p(a,b,c;t)&=-c-\frac{1}{t}\log\frac{e^{-at}-e^{-bt}}{(b-a)(1-e^{-t})}\\&=\frac{a+b-1}{2}-c-\frac{1}{t}\log\left[\frac{\sinh(|\frac{(b-a)t}{2}|)}{|b-a|\sinh(\frac{t}{2})}\right]\\&=\frac{a+b-2c-1}{2}-\frac{|b-a|-1}{2}\frac{\log H_{|b-a|,1}(t/2)}{t/2}\end{aligned}\tag{3.8}$$

and

$$q(a,b,c;t)=tL\left(e^{(\rho-c)t},\frac{e^{(\rho-a)t}-e^{(\rho-b)t}}{(b-a)(1-e^{-t})}\right)p(a,b,c;t).\tag{3.9}$$

It follows from Theorem 2.1 and (3.8) that the function $t\to p(a,b,c;t)$ is strictly monotonic on $(0,\infty)$ and

$$p(a,b,c;0^+)=\frac{a+b-2c}{2},\qquad p(a,b,c;\infty)=\frac{a+b-2c}{2}-\frac{|b-a|-1}{2}.\tag{3.10}$$

The monotonicity of the function $t\to p(a,b,c;t)$ on the interval $(0,\infty)$ and (3.10) lead to the conclusion that

$$p(a,b,c;t)\ge(\le)\,0\tag{3.11}$$

for all $t\in(0,\infty)$ if and only if $\min(\max)\{p(a,b,c;0^+),p(a,b,c;\infty)\}\ge(\le)\,0$, that is, $c\le(\ge)$ $(a+b-\max(\min)\{|a-b|,1\})/2$.

From (3.7) and the formulas

$$\psi(x)=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-xt}}{1-e^{-t}}\right)dt,\qquad \frac{1}{x}=\int_0^\infty e^{-xt}\,dt$$

we have

$$\begin{aligned}-\big(\log U(a,b,c;x)\big)'&=\frac{1}{x+c}-\frac{\psi(x+b)-\psi(x+a)}{b-a}\\&=\int_0^\infty e^{-(x+c)t}\,dt-\int_0^\infty\frac{e^{-(x+a)t}-e^{-(x+b)t}}{(b-a)(1-e^{-t})}\,dt\\&=\int_0^\infty e^{-(x+\rho)t}q(a,b,c;t)\,dt.\end{aligned}\tag{3.12}$$

Therefore, Theorem 3.2 follows from (3.9), (3.11), (3.12), and the Bernstein theorem. □

Remark 3.2 Qi [25] presented a sufficient condition for $U(a,b,c;x)\in\mathrm{LCM}[(-\rho,\infty)]$ or $1/U(a,b,c;x)\in\mathrm{LCM}[(-\rho,\infty)]$.

Theorem 3.3 *Let* $\operatorname{erf}(x) = 2\int_0^x e^{-t^2}\,dt/\sqrt{\pi}$ *be the error function. Then we have the double inequality*

$$\frac{4}{\sqrt{\pi}}\arctan\frac{2e^{\sqrt{3}x}+1}{\sqrt{3}} + 1 - 2\sqrt{\pi} < \operatorname{erf}(x) < \frac{4}{\sqrt{\pi}}\arctan\frac{2e^{\sqrt{3}x}+1}{\sqrt{3}} - \frac{4\sqrt{\pi}}{3}$$

for all $x > 0$.

Proof It follows from the third inequality in Corollary 2.2 that

$$e^{-u^2} - \frac{3}{2\cosh(\sqrt{3}u)+1} < 0 \tag{3.13}$$

for $u > 0$.

Let

$$\begin{aligned} F(x) &= \frac{2}{\sqrt{\pi}}\int_0^x \left(e^{-u^2} - \frac{3}{2\cosh(\sqrt{3}u)+1}\right)du \\ &= \operatorname{erf}(x) - \frac{6}{\sqrt{\pi}}\int_0^x \frac{1}{2\cosh(\sqrt{3}u)+1}\,du. \end{aligned} \tag{3.14}$$

Then

$$F(0) = 0, \qquad F(\infty) = 1 - \frac{2\sqrt{\pi}}{3}. \tag{3.15}$$

It follows from (3.13)-(3.15) that

$$\frac{6}{\sqrt{\pi}}\int_0^x \frac{1}{2\cosh(\sqrt{3}u)+1}\,du + 1 - \frac{2\sqrt{\pi}}{3} < \operatorname{erf}(x) < \frac{6}{\sqrt{\pi}}\int_0^x \frac{1}{2\cosh(\sqrt{3}u)+1}\,du \tag{3.16}$$

for $x > 0$.

Therefore, Theorem 3.3 follows easily from (3.16). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Computation Sciences, Hunan City University, Yiyang, 413000, China. [2]Customer Service Center, State Grid Zhejiang Electric Power Research Institute, Hangzhou, 310009, China.

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