Zhu et al. *Journal of Inequalities and Applications* (2016) 2016:191
DOI 10.1186/s13660-016-1130-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH
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Eigenvalue inequalities of elliptic operators in weighted divergence form on smooth metric measure spaces

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Abstract

In this paper, we study the eigenvalue problem of elliptic operators in weighted divergence form on smooth metric measure spaces. First of all, we give a general inequality for eigenvalues of the eigenvalue problem of elliptic operators in weighted divergence form on compact smooth metric measure space with boundary (possibly empty). Then applying this general inequality, we get some universal inequalities of Payne-Pólya-Weinberger-Yang type for the eigenvalues of elliptic operators in weighted divergence form on a connected bounded domain in the smooth metric measure spaces, the Gaussian shrinking solitons, and the general product solitons, respectively.

MSC: 35P15; 53C20; 53C42

Keywords: eigenvalue; universal inequalities; drifting Laplacian; elliptic operators in weighted divergence form; Gaussian shrinking soliton; general product soliton

1 Introduction

A smooth metric measure space is actually a Riemannian manifold equipped with some measure which is absolutely continuous with respect to the usual Riemannian measure. More precisely, for a given complete n-dimensional Riemannian manifold $(M, \langle,\rangle)$ with the metric $\langle,\rangle$, the triple $(M, \langle,\rangle, e^{-f}\,dv)$ is called a smooth metric measure space, where f is a smooth real-valued function on M and dv is the Riemannian volume element related to $\langle,\rangle$ (sometimes, we also call dv the volume density). Let Ω be a bounded domain in a smooth metric measure space $(M, \langle,\rangle, e^{-f}\,dv)$, and let $A : \Omega \to \mathrm{End}(T\Omega)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of $T\Omega$, we can define the elliptic operator in weighted divergence form as

$$\mathfrak{L}_f = -\operatorname{div}_f A\nabla, \tag{1.1}$$

where $\operatorname{div}_f X = e^f \operatorname{div}(e^{-f}X)$ is the weighted divergence of vector fields X, and ∇ is the gradient operator. When A is an identity map, $-\mathfrak{L}_f$ becomes the drifting Laplacian Δ_f, for the drifting Laplacian, some universal inequalities have been given in [1–5]. When f is a constant, $\mathfrak{L}_f$ becomes the elliptic operator in divergence form, for some recent developments about universal inequalities of the eigenvalue of elliptic operator in divergence form on Riemannian manifolds, we refer to [6–10] and the references therein. As briefly mentioned above, it is a natural problem how to get the universal inequalities of the eigenvalues of elliptic operator in weighted divergence form. Actually, in this paper, we first consider the eigenvalue problem as follows:

$$\begin{cases} (\mathfrak{L}_f + V)u = \lambda\rho u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in a complete smooth metric measure space $(M, \langle,\rangle, e^{-f}\,dv)$, V is a non-negative continuous function on M, and ρ is a weight function which is positive and continuous on M. For the eigenvalues of (1.2), we can give the following universal inequalities.

Theorem 1.1 *Let Ω be a connected bounded domain in an n-dimensional complete smooth metric measure space $(M, \langle,\rangle, e^{-f}\,dv)$. Assume that $\xi_1 I \le A$, $tr(A) \le n\xi_2$ throughout Ω, and $\rho_1 \le \rho(x) \le \rho_2$, $|\nabla f|(x) \le C_0$, $\forall x \in \Omega$, here I is the identity map, $\xi_1, \xi_2, \rho_1, \rho_2, C_0$ are positive constants and $tr(A)$ denotes the trace of A. Let λ_i be the ith eigenvalue of the eigenvalue problem* (1.2), *then we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4\xi_2\rho_2^2}{n\rho_1^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left\{\frac{\lambda_i-\rho_2^{-1}V_0}{\xi_1} + C_0\left(\frac{\lambda_i-\rho_2^{-1}V_0}{\xi_1}\right)^{\frac{1}{2}} + \frac{n^2H_0^2+C_0^2}{4\rho_1}\right\}, \tag{1.3}$$

where $H_0 = \sup_{x\in\Omega}|\mathbf{H}|(x)$, $V_0 = \min_{x\in\Omega}V(x)$, and $\mathbf{H}$ is the mean curvature vector field of M in a Euclidean space $\mathbb{R}^m$.

Remark 1.2 From inequality (1.3), we can get some results which are given in [5, 8], for example, if f is a constant, then $C_0 = 0$, (1.3) becomes (3.2) in [8].

For the fourth-order elliptic operator in weighted divergence, we can consider the following eigenvalue problem:

$$\begin{cases} \mathfrak{L}_f^2 u = \Lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

we also give some universal inequalities for the eigenvalues of (1.4) as follows.

Theorem 1.3 *Let Ω be a connected bounded domain in an n-dimensional complete smooth metric measure space $(M, \langle,\rangle, e^{-f}\,dv)$. Assume that $\xi_1 I \le A \le \xi_2 I$ throughout Ω, and $|\nabla f|(x) \le C_0$, $\forall x \in \Omega$, here I is the identity map, ξ_1, ξ_2, C_0 are positive constants. Let Λ_i be the ith eigenvalue of the eigenvalue problem* (1.4), *then we have*

$$\begin{aligned}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \le \frac{\xi_2}{n\xi_1}\Bigg\{&\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\Big((2n+4)\Lambda_i^{\frac{1}{2}} + 4C_0\xi_2^{\frac{1}{2}}\Lambda_i^{\frac{1}{4}} + \xi_2\big(n^2H_0^2+C_0^2\big)\Big) \\ &\times \sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Big(4\Lambda_i^{\frac{1}{2}} + 4C_0\xi_1^{\frac{1}{2}}\Lambda_i^{\frac{1}{4}} + \xi_1\big(n^2H_0^2+C_0^2\big)\Big)\Bigg\}^{\frac{1}{2}},\end{aligned} \tag{1.5}$$

where $H_0 = \sup_{x\in\Omega} |\mathbf{H}|(x)$, *and* $\mathbf{H}$ *is the mean curvature vector field of* M *in a Euclidean space* $\mathbb{R}^m$.

Remark 1.4 From inequality (1.5), we can get some results which are given in [3, 11], for example, if A is an identity map, then $\xi_1 = \xi_2 = 1$, (1.5) becomes (1.2) in [3].

On smooth metric measure spaces, we can also define the so-called *weighted Ricci curvature* Ric^f given by

$$\mathrm{Ric}^f = \mathrm{Ric} + \mathrm{Hess} f,$$

which is also called the ∞*-Bakry-Émery Ricci tensor*. The equation $\mathrm{Ric}^f = \kappa\langle,\rangle$ for some constant κ is just the gradient Ricci soliton equation, which plays an important role in the study of Ricci flow. We refer the reader to [12] for some recent progress about Ricci solitons. For $\kappa > 0, \kappa = 0$, or $\kappa < 0$, the gradient Ricci soliton $(M, \langle,\rangle, e^{-f}\,d\nu, \kappa)$ is called shrinking, steady, or expanding, respectively. In the following, we would like to give two examples of Ricci solitons.

Example 1 The Gaussian shrinking soliton $(\mathbb{R}^n, \langle,\rangle_{\mathrm{can}}, e^{-\frac{1}{4}|x|^2}\,d\nu, \frac{1}{2})$, where $\langle,\rangle_{\mathrm{can}}$ is the standard Euclidean metric on $\mathbb{R}^n$, $f = \frac{1}{4}|x|^2, x \in \mathbb{R}^n$, and $\mathrm{Ric}^f = \frac{1}{2}\langle,\rangle_{\mathrm{can}}$.

Example 2 More generally, consider the Riemannian product $\Sigma \times \mathbb{R}^n$, where $(\Sigma; \langle,\rangle_\Sigma)$ is an Einstein manifold satisfying $\mathrm{Ric}_\Sigma = \kappa\langle,\rangle_\Sigma$. Define the smooth function $f : \Sigma \times \mathbb{R}^n \to \mathbb{R}$ by setting $f(p;x) = \frac{\kappa}{2}|x|^2 + x \cdot a + b$, for any $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. Then $(\Sigma \times \mathbb{R}^n, \langle,\rangle_\Sigma + \langle,\rangle_{\mathrm{can}}, e^{-f}\,d\nu_\Sigma \otimes d\nu^n_{\mathbb{R}}, \kappa)$ is a non-trivial gradient Ricci soliton. Similarly, one could even construct gradient Ricci solitons with a warped product structure. More details of the product solitons can be found in the Remark 3.2 in [2].

In the following, we will give some universal inequalities for the Dirichlet eigenvalues in a connected bounded domain on the Gaussian shrinking solitons and general product solitons.

Theorem 1.5 *Let* Ω *be a connected bounded domain in the Gaussian shrinking soliton* $(\mathbb{R}^n, \langle,\rangle_{\mathrm{can}}, e^{-\frac{1}{4}|x|^2}\,d\nu, \frac{1}{2})$, *and assume that* $\xi_1 I \le A, tr(A) \le n\xi_2$ *throughout* Ω, *here* I *is the identity map,* ξ_1, ξ_2 *are positive constants and* $tr(A)$ *denotes the trace of* A. *Let* λ_i *be the ith eigenvalue of the Dirichlet problem*

$$\begin{cases} \mathfrak{L}_f u = \lambda u, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{1.6}$$

where $f = \frac{1}{4}|x|^2$, *then we have*

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \le \frac{4\xi_2}{n\xi_1}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\left\{\lambda_i + \frac{\xi_1}{16}\left(4n - \min_{x\in\overline{\Omega}}\{|x|^2\}\right)\right\}. \tag{1.7}$$

Remark 1.6 (i) For a self-shrinker, the drifting Laplacian Δ_f with $f = \frac{|x|^2}{4}$ is actually the operator $\mathfrak{L} := \Delta - \frac{1}{2}\langle x, \nabla(\cdot)\rangle$, which was introduced by Colding-Minicozzi [13] to study self-shrinker hypersurfaces. For the Dirichlet problem of the operator $\mathfrak{L}$, some universal inequalities have been obtained by Cheng and Peng [14]. In this case, our results can be regarded as conclusions for the Dirichlet problem of the elliptic operator in weighted divergence form.

(ii) Let $b = \frac{\xi_2}{\xi_1}$, using the recursive formula in Cheng and Yang [15], we can infer from (1.7) that

$$\lambda_{k+1} + \frac{\xi_1}{16}\Big(4n - \min_{x\in\overline{\Omega}}\big\{|x|^2\big\}\Big) \le C_0(n,k)k^{\frac{2b}{n}}\left(\lambda_1 + \frac{\xi_1}{16}\Big(4n - \min_{x\in\overline{\Omega}}\big\{|x|^2\big\}\Big)\right), \tag{1.8}$$

where $C_0(n,k) \le 1 + \frac{4b}{n}$ is a constant (see [15]).

Theorem 1.7 *Let Ω be a connected bounded domain in the gradient product Ricci soliton $(\Sigma \times \mathbb{R}, \langle,\rangle, e^{-\frac{\kappa t^2}{2}}\,d\nu, \kappa)$, where Σ is an Einstein manifold with constant Ricci curvature κ. Set $\overline{x} = (x,t) \in \Omega$, where $x \in \Sigma$, $t \in \mathbb{R}$, and assume that $\xi_1 I \le A \le \xi_2 I$ throughout Ω, here I is the identity map, ξ_1, ξ_2 are positive constants. Let λ_i be the ith eigenvalue of the eigenvalue problem* (1.6), *where $f = \frac{\kappa t^2}{2}$, then we have*

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \le \frac{\xi_2}{\xi_1}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\Big\{4\lambda_i + 4\xi_1 - \kappa^2\xi_1 \min_{(x,t)\in\overline{\Omega}}\big\{|t|^2\big\}\Big\}. \tag{1.9}$$

2 A general inequality

In this section, we will prove a general inequality, which will play a key role in the proof of our main results which are listed in Section 1.

Lemma 2.1 *Let $(M, \langle,\rangle, e^{-f}\,d\nu)$ be an n-dimensional compact smooth metric measure space with boundary ∂M (possibly empty), and let a, b be the random non-negative constants and $a + b \neq 0$. Let λ_i be the ith eigenvalue of the eigenvalue problem of the fourth-order elliptic operator in weighted divergence form with weight ρ such that*

$$\begin{cases} (a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)u = \lambda\rho u, & \text{in } M,\\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial M,\end{cases}$$

and u_i be the orthonormal eigenfunction corresponding to λ_i, that is,

$$\begin{cases} (a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)u_i = \lambda_i\rho u_i, & \text{in } M,\\ u_i = \frac{\partial u_i}{\partial \nu} = 0, & \text{on } \partial M,\\ \int_M \rho u_i u_j\,d\mu = \delta_{ij}, & \forall i,j = 1,2,\ldots,\end{cases}$$

where $d\mu = e^{-f}\,d\nu$. Then, for any $h \in C^4(\overline{M})$, we have

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2\int_M u_i^2|\nabla h|^2\,d\mu\\ &\quad\le \sum_{i=1}^{k}\delta(\lambda_{k+1} - \lambda_i)^2\int_M h u_i p_i\,d\mu\\ &\qquad + \sum_{i=1}^{k}\frac{\lambda_{k+1} - \lambda_i}{\delta}\int_M \frac{1}{\rho}\left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta_f h}{2}\right)^2 d\mu,\end{aligned} \tag{2.1}$$

where δ is any positive constant and

$$
\begin{aligned}
p_i &= -2a\langle\nabla h, A\nabla(\mathfrak{L}_f u_i)\rangle + a\mathfrak{L}_f h\mathfrak{L}_f u_i - 2a\mathfrak{L}_f\bigl(\langle\nabla h, A\nabla u_i\rangle\bigr) \\
&\quad + a\mathfrak{L}_f(u_i\mathfrak{L}_f h) - 2b\langle\nabla h, A\nabla u_i\rangle + bu_i\mathfrak{L}_f h.
\end{aligned}
$$

Proof Let $\varphi_i = hu_i - \sum_{j=1}^k a_{ij}u_j$, here $k \geq 1$ is any integer and $a_{ij} = \sum_{j=1}^k \int_M \rho hu_iu_j\,d\mu = a_{ji}$. Then we have

$$
\varphi_i|_{\partial M} = \frac{\partial\varphi_i}{\partial\nu}\bigg|_{\partial M} = 0, \quad \text{and} \quad \int_M \rho\varphi_i u_j\,d\mu = 0, \quad \forall i,j = 1,\ldots,k.
$$

By the Rayleigh-Ritz inequality, we get

$$
\lambda_{k+1}\int_M \rho\varphi_i^2\,d\mu \leq \int_M \varphi_i\bigl(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V\bigr)\varphi_i\,d\mu. \tag{2.2}
$$

From the definition of $\mathfrak{L}_f$, we have

$$
\begin{aligned}
\mathfrak{L}_f(hu_i) &= -\operatorname{div}_f\bigl(A\nabla(hu_i)\bigr) \\
&= -e^f\operatorname{div}\bigl(e^{-f}\bigl(A(h\nabla u_i + u_i\nabla h)\bigr)\bigr) \\
&= -\operatorname{div}\bigl(A(h\nabla u_i + u_i\nabla h)\bigr) - \bigl\langle\nabla f, A(h\nabla u_i + u_i\nabla h)\bigr\rangle \\
&= -h\operatorname{div}_f(A\nabla u_i) - \langle\nabla h, A\nabla u_i\rangle - u_i\operatorname{div}_f(A\nabla h) - \langle\nabla u_i, A\nabla h\rangle \\
&= h\mathfrak{L}_f u_i - 2\langle\nabla h, A\nabla u_i\rangle + u_i\mathfrak{L}_f h
\end{aligned} \tag{2.3}
$$

and

$$
\begin{aligned}
\mathfrak{L}_f^2(hu_i) &= \mathfrak{L}_f\bigl(h\mathfrak{L}_f u_i - 2\langle\nabla h, A\nabla u_i\rangle + u_i\mathfrak{L}_f h\bigr) \\
&= h\mathfrak{L}_f^2 u_i - 2\langle\nabla h, A\nabla(\mathfrak{L}_f u_i)\rangle + \mathfrak{L}_f h\mathfrak{L}_f u_i \\
&\quad - 2\mathfrak{L}_f\bigl(\langle\nabla h, A\nabla u_i\rangle\bigr) + \mathfrak{L}_f(u_i\mathfrak{L}_f h).
\end{aligned} \tag{2.4}
$$

It follows from (2.3) and (2.4) that

$$
\bigl(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V\bigr)(hu_i) = \lambda_i\rho hu_i + p_i, \tag{2.5}
$$

where p_i is defined by

$$
\begin{aligned}
p_i &= -2a\bigl\langle\nabla h, A\nabla\bigl(\mathfrak{L}_f(u_i)\bigr)\bigr\rangle + a\mathfrak{L}_f h\mathfrak{L}_f u_i - 2a\mathfrak{L}_f\bigl(\langle\nabla h, A\nabla u_i\rangle\bigr) \\
&\quad + a\mathfrak{L}_f\bigl(u_i L(h)\bigr) - 2b\langle\nabla h, A\nabla u_i\rangle + bu_i\mathfrak{L}_f h.
\end{aligned}
$$

Let us compute

$$
\begin{aligned}
&\int_M \varphi_i\bigl(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V\bigr)\varphi_i\,d\mu \\
&\quad = \int_M \varphi_i\bigl(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V\bigr)(hu_i)\,d\mu
\end{aligned}
$$

$$= \lambda_i \int_M \varphi_i \rho h u_i \, d\mu + \int_M \varphi_i p_i \, d\mu$$

$$= \lambda_i \int_M \rho \varphi_i^2 \, d\mu + \int_M h u_i p_i \, d\mu - \sum_{j=1}^{k} a_{ij} b_{ij}, \tag{2.6}$$

where b_{ij} is defined by $b_{ij} = \int_M p_i u_j \, d\mu$.

On the other hand, by (2.2) and (2.6), we have

$$(\lambda_{k+1} - \lambda_i) \int_M \rho \varphi_i^2 \, d\mu \le \int_M h u_i p_i \, d\mu - \sum_{j=1}^{k} a_{ij} b_{ij}. \tag{2.7}$$

By a similar computation to (2.8)-(2.12) in [6], we have

$$\begin{aligned} &2\int_M \mathfrak{L}_f u_j \langle A\nabla h, \nabla u_i \rangle \, d\mu - 2\int_M \mathfrak{L}_f u_i \langle A\nabla h, \nabla u_j \rangle \, d\mu \\ &\quad = -\int_M h u_i \mathfrak{L}_f^2 u_j \, d\mu + \int_M h u_j \mathfrak{L}_f^2 u_i \, d\mu + \int_M u_i \mathfrak{L}_f u_j \mathfrak{L}_f h \, d\mu - \int_M u_j \mathfrak{L}_f u_i \mathfrak{L}_f h \, d\mu, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\int_M u_j \mathfrak{L}_f \langle \nabla h, A\nabla u_i \rangle \, d\mu + \int_M u_j \big\langle \nabla h, A\nabla(\mathfrak{L}_f u_i) \big\rangle \, d\mu \\ &\quad = \int_M \mathfrak{L}_f u_j \langle \nabla h, A\nabla u_i \rangle \, d\mu - \int_M \mathfrak{L}_f u_i \langle \nabla h, A\nabla u_j \rangle \, d\mu + \int_M u_j \mathfrak{L}_f u_i \mathfrak{L}_f h \, d\mu, \end{aligned} \tag{2.9}$$

$$\int_M u_j \mathfrak{L}_f (u_i \mathfrak{L}_f h) \, d\mu = \int_M u_i \mathfrak{L}_f u_j \mathfrak{L}_f h \, d\mu, \tag{2.10}$$

and

$$\int_M u_j \big\{ -2\langle \nabla h, A\nabla u_i \rangle + u_i \mathfrak{L}_f h \big\} \, d\mu = \int_M h u_i \mathfrak{L}_f u_j \, d\mu - \int_M h u_j \mathfrak{L}_f u_i \, d\mu. \tag{2.11}$$

Combining (2.8)-(2.11) and a similar calculation to (2.13) in [6], we get

$$b_{ij} = \int_M p_i u_j \, d\mu = (\lambda_j - \lambda_i) a_{ij}. \tag{2.12}$$

We infer from (2.7) and (2.12) that

$$(\lambda_{k+1} - \lambda_i) \int_M \rho \varphi_i^2 \, d\mu \le \int_M h u_i p_i \, d\mu - \sum_{j=1}^{k} (\lambda_j - \lambda_i) a_{ij}^2. \tag{2.13}$$

Setting $t_{ij} = \int_M u_j (\langle \nabla h, \nabla u_i \rangle + \frac{u_i \Delta_f h}{2}) \, d\mu$, thus $c_{ij} = -c_{ji}$ and

$$\begin{aligned} &\int_M -2\varphi_i \left(\langle \nabla h, \nabla u_i \rangle + \frac{u_i \Delta_f h}{2} \right) d\mu \\ &\quad = \int_M \big(-2h u_i \langle \nabla h, \nabla u_i \rangle - h u_i^2 \Delta_f h \big) \, d\mu + 2\sum_{j=1}^{k} a_{ij} c_{ij} \\ &\quad = \int_M u_i^2 |\nabla h|^2 \, d\mu + 2\sum_{j=1}^{k} a_{ij} c_{ij}. \end{aligned} \tag{2.14}$$

Using (2.13), (2.14), and the Schwarz inequality, we can get

$$\begin{aligned}
&(\lambda_{k+1}-\lambda_i)^2\left(\int_M u_i^2|\nabla h|^2\,d\mu + 2\sum_{j=1}^k a_{ij}c_{ij}\right)\\
&\quad= (\lambda_{k+1}-\lambda_i)^2\int_M -2\sqrt{\rho}\varphi_i\left(\frac{1}{\sqrt{\rho}}\left(\langle\nabla h,\nabla u_i\rangle + \frac{u_i\Delta_f h}{2}\right) - \sum_{j=1}^k c_{ij}\sqrt{\rho}u_j\right)d\mu\\
&\quad\le \delta(\lambda_{k+1}-\lambda_i)^3\int_M \rho\varphi_i^2\,d\mu\\
&\qquad+ \frac{\lambda_{k+1}-\lambda_i}{\delta}\int_M\left(\frac{1}{\sqrt{\rho}}\left(\langle\nabla h,\nabla u_i\rangle + \frac{u_i\Delta_f h}{2}\right) - \sum_{j=1}^k c_{ij}\sqrt{\rho}u_j\right)^2 d\mu\\
&\quad\le \delta(\lambda_{k+1}-\lambda_i)^2\left(\int_M hu_ip_i\,d\mu - \sum_{j=1}^k(\lambda_j-\lambda_i)a_{ij}^2\right)\\
&\qquad+ \frac{\lambda_{k+1}-\lambda_i}{\delta}\left(\int_M\frac{1}{\rho}\left(\langle\nabla h,\nabla u_i\rangle + \frac{u_i\Delta_f h}{2}\right)^2 - \sum_{j=1}^k c_{ij}^2\right),
\end{aligned}\tag{2.15}$$

where δ is any positive constant. Summing over i from 1 to k in (2.15) and noticing $a_{ij} = a_{ji}, c_{ij} = -c_{ji}$, we have

$$\begin{aligned}
&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2|\nabla h|^2\,d\mu - 2\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)a_{ij}c_{ij}\\
&\quad\le \sum_{i=1}^k\delta(\lambda_{k+1}-\lambda_i)^2\int_M hu_ip_i\,d\mu + \sum_{i=1}^k\frac{\lambda_{k+1}-\lambda_i}{\delta}\int_M\frac{1}{\rho}\left(\langle\nabla h,\nabla u_i\rangle + \frac{u_i\Delta_f h}{2}\right)^2 d\mu\\
&\qquad- \sum_{i,j=1}^k\delta(\lambda_{k+1}-\lambda_i)(\lambda_j-\lambda_i)^2a_{ij}^2 - \sum_{i,j=1}^k\frac{\lambda_{k+1}-\lambda_i}{\delta}c_{ij}^2,
\end{aligned}\tag{2.16}$$

which implies that

$$\begin{aligned}
&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2|\nabla h|^2\,d\mu\\
&\quad\le \sum_{i=1}^k\delta(\lambda_{k+1}-\lambda_i)^2\int_M hu_ip_i\,d\mu + \sum_{i=1}^k\frac{\lambda_{k+1}-\lambda_i}{\delta}\int_M\frac{1}{\rho}\left(\langle\nabla h,\nabla u_i\rangle + \frac{u_i\Delta_f h}{2}\right)^2 d\mu.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

3 Proof of Theorem 1.1 and Theorem 1.3

In this section, we will give the proof of Theorem 1.1 and Theorem 1.3 by using Lemma 2.1.

Proof of Theorem 1.1 From the Nash embedding theorem, we know that there exists an isometric immersion from a complete Riemannian manifold M into a Euclidean space $\mathbb{R}^m$. Thus, M can be considered as an n-dimensional complete isometrically immersed submanifold in $\mathbb{R}^m$. Let $y_1, y_2, \dots, y_m$ be the standard coordinate functions of $\mathbb{R}^m$. Then we have

$$\sum_{\alpha=1}^{m}|\nabla y_\alpha|^2 = n, \tag{3.1}$$

$$\Delta(y_1, y_2, \dots, y_m) = (\Delta y_1, \Delta y_2, \dots, \Delta y_m) = n\mathbf{H}, \tag{3.2}$$

$$\sum_{\alpha=1}^{m}\langle\nabla y_\alpha, \nabla f\rangle^2 = \sum_{\alpha=1}^{m}\big(\nabla f(y_\alpha)\big)^2 = |\nabla f|^2, \tag{3.3}$$

$$\sum_{\alpha=1}^{m}\langle\nabla y_\alpha, \nabla u_i\rangle^2 = \sum_{\alpha=1}^{m}\big(\nabla u_i(y_\alpha)\big)^2 = |\nabla u_i|^2, \tag{3.4}$$

$$\sum_{\alpha=1}^{m}\langle\nabla y_\alpha, \nabla u_i\rangle\langle\nabla y_\alpha, \nabla f\rangle = \sum_{\alpha=1}^{m}\nabla u_i(y_\alpha)\nabla f(y_\alpha) = \langle\nabla u_i, \nabla f\rangle, \tag{3.5}$$

$$\sum_{\alpha=1}^{m}\Delta y_\alpha\langle\nabla y_\alpha, \nabla u_i\rangle = \sum_{\alpha=1}^{m}\Delta y_\alpha\nabla u_i(y_\alpha) = \langle n\mathbf{H}, \nabla u_i\rangle = 0, \tag{3.6}$$

and

$$\sum_{\alpha=1}^{m}\Delta y_\alpha\langle\nabla y_\alpha, \nabla f\rangle = \sum_{\alpha=1}^{m}\Delta y_\alpha\nabla f(y_\alpha) = \langle n\mathbf{H}, \nabla f\rangle = 0. \tag{3.7}$$

Then we infer from (3.3)-(3.7) that

$$\sum_{\alpha=1}^{m}\Delta_f y_\alpha\langle\nabla y_\alpha, \nabla u_i\rangle = \sum_{\alpha=1}^{m}\big(\Delta y_\alpha - \langle\nabla y_\alpha, \nabla f\rangle\big)\langle\nabla y_\alpha, \nabla u_i\rangle = \langle\nabla u_i, \nabla f\rangle \tag{3.8}$$

and

$$\begin{aligned}\sum_{\alpha=1}^{m}(\Delta_f y_\alpha)^2 &= \sum_{\alpha=1}^{m}\big(\Delta y_\alpha - \langle\nabla y_\alpha, \nabla f\rangle\big)^2\\ &= \sum_{\alpha=1}^{m}\big((\Delta y_\alpha)^2 - 2\Delta y_\alpha\langle\nabla y_\alpha, \nabla f\rangle + \langle\nabla y_\alpha, \nabla f\rangle^2\big)\\ &= n^2|\mathbf{H}|^2 + |\nabla f|^2.\end{aligned} \tag{3.9}$$

Let $a = 0, b = 1$ in (2.1), then taking $h = y_\alpha$ and summing over α, and noticing $\rho_2^{-1} \le \|u_i\|^2 \le \rho_1^{-1}$, we get

$$\begin{aligned}n\rho_2^{-1}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \le{}& \sum_{i=1}^{k}\delta(\lambda_{k+1} - \lambda_i)^2\int_\Omega\sum_{\alpha=1}^{m}y_\alpha u_i p_{\alpha i}\,d\mu\\ &+ \sum_{i=1}^{k}\frac{\lambda_{k+1} - \lambda_i}{\delta}\int_\Omega\frac{1}{\rho}\sum_{\alpha=1}^{m}\left(\langle\nabla y_\alpha, \nabla u_i\rangle + \frac{u_i\Delta_f y_\alpha}{2}\right)^2 d\mu,\end{aligned} \tag{3.10}$$

where $p_{\alpha i} = -2\langle\nabla y_\alpha, A\nabla u_i\rangle + u_i\mathfrak{L}_f y_\alpha$. Since

$$-2\int_\Omega y_\alpha u_i\langle\nabla y_\alpha, A\nabla u_i\rangle\,d\mu = \int_\Omega u_i^2\langle\nabla y_\alpha, A\nabla y_\alpha\rangle\,d\mu - \int_\Omega y_\alpha u_i^2\mathfrak{L}_f y_\alpha\,d\mu,$$

we infer from above equality and $\sum_{\alpha=1}^{m}\langle\nabla y_\alpha, A\nabla y_\alpha\rangle = tr(A) \le n\xi_2$ that

$$\begin{aligned}\int_\Omega \sum_{\alpha=1}^{m} y_\alpha u_i p_{\alpha i}\,d\mu &= \int_\Omega \sum_{\alpha=1}^{m} y_\alpha u_i\bigl(-2\langle\nabla y_\alpha, A\nabla u_i\rangle + u_i\mathfrak{L}_f y_\alpha\bigr)\,d\mu \\ &= \int_\Omega \sum_{\alpha=1}^{m} u_i^2\langle\nabla y_\alpha, A\nabla y_\alpha\rangle\,d\mu \le n\xi_2\|u_i\|^2 \le n\xi_2\rho_1^{-1}. \end{aligned} \tag{3.11}$$

From $\rho_2^{-1} \le \|u_i\|^2 \le \rho_1^{-1}$ and $A \ge \xi_1 I$, we have

$$\begin{aligned}\lambda_i &= \int_\Omega u_i(\mathfrak{L}_f + V)u_i\,d\mu = \int_\Omega -u_i\operatorname{div}_f(A\nabla u_i)\,d\mu + \int_\Omega Vu_i^2\,d\mu \\ &= \int_\Omega \langle\nabla u_i, A\nabla u_i\rangle\,d\mu + \int_\Omega Vu_i^2\,d\mu \ge \xi_1\|\nabla u_i\|^2 + \rho_2^{-1}V_0,\end{aligned}$$

which implies

$$\|\nabla u_i\|^2 \le \frac{\lambda_i - \rho_2^{-1}V_0}{\xi_1}. \tag{3.12}$$

Using the Schwarz inequality and the above inequality, we have

$$\int_\Omega \langle\nabla f, \nabla u_i\rangle\,d\mu \le \int_\Omega |\nabla f||\nabla u_i|\,d\mu \le C_0\bigl\{\|\nabla u_i\|^2\bigr\}^{\frac{1}{2}} \le C_0\left(\frac{\lambda_i - \rho_2^{-1}V_0}{\xi_1}\right)^{\frac{1}{2}}. \tag{3.13}$$

Combining (3.4), (3.8), (3.9), (3.12), and (3.13), we have

$$\begin{aligned}&\int_\Omega \frac{1}{\rho}\sum_{\alpha=1}^{m}\left(\langle\nabla y_\alpha, \nabla u_i\rangle + \frac{u_i\Delta_f y_\alpha}{2}\right)^2 d\mu \\ &\quad= \int_\Omega \frac{1}{\rho}\sum_{\alpha=1}^{m}\left(\langle\nabla y_\alpha, \nabla u_i\rangle^2 + u_i\Delta_f y_\alpha\langle\nabla y_\alpha, \nabla u_i\rangle + \frac{u_i^2(\Delta_f y_\alpha)^2}{4}\right) d\mu \\ &\quad= \int_\Omega \frac{1}{\rho}\left(|\nabla u_i|^2 + \langle\nabla f, \nabla u_i\rangle + \frac{u_i^2}{4}\bigl(n^2|\mathbf{H}|^2 + |\nabla f|^2\bigr)\right) d\mu \\ &\quad\le \frac{1}{\rho_1}\left\{\frac{\lambda_i - \rho_2^{-1}V_0}{\xi_1} + C_0\left(\frac{\lambda_i - \rho_2^{-1}V_0}{\xi_1}\right)^{\frac{1}{2}} + \frac{1}{4\rho_1}\bigl(n^2H_0^2 + C_0^2\bigr)\right\}.\end{aligned} \tag{3.14}$$

Substituting (3.11) and (3.14) into (3.10), we have

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \le{}& \sum_{i=1}^{k}\delta(\lambda_{k+1} - \lambda_i)^2\frac{\rho_2\xi_2}{\rho_1} \\ &+ \sum_{i=1}^{k}\frac{\lambda_{k+1} - \lambda_i}{\delta}\frac{\rho_2}{n\rho_1}\left\{\frac{\lambda_i - \rho_2^{-1}V_0}{\xi_1} + C_0\left(\frac{\lambda_i - \rho_2^{-1}V_0}{\xi_1}\right)^{\frac{1}{2}}\right. \\ &\left.+ \frac{n^2H_0^2 + C_0^2}{4\rho_1}\right\}.\end{aligned}$$

Taking

$$\delta = \left\{ \frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\{\frac{\lambda_i-\rho_2^{-1}V_0}{\xi_1} + C_0(\frac{\lambda_i-\rho_2^{-1}V_0}{\xi_1})^{\frac{1}{2}} + \frac{n^2H_0^2+C_0^2}{4\rho_1}\}}{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 n\xi_2} \right\}^{\frac{1}{2}},$$

we can get (3.1). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.3 Let $a = 1, b = 0, V \equiv 0, \rho \equiv 1$ in (2.1), then taking $h = y_\alpha$ and summing over α, where $\{y_\alpha\}_{\alpha=1}^m$ are defined as above, we get

$$\begin{aligned} n\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 &\le \sum_{i=1}^{k}\delta(\Lambda_{k+1}-\Lambda_i)^2\int_\Omega\sum_{\alpha=1}^{m} y_\alpha u_i p_{\alpha i}\,d\mu \\ &\quad + \sum_{i=1}^{k}\frac{\Lambda_{k+1}-\Lambda_i}{\delta}\int_\Omega\sum_{\alpha=1}^{m}\left(\langle\nabla y_\alpha,\nabla u_i\rangle + \frac{u_i\Delta_f y_\alpha}{2}\right)^2 d\mu, \end{aligned} \tag{3.15}$$

where $p_{\alpha i} = -2\langle\nabla y_\alpha, A\nabla(\mathfrak{L}_f u_i)\rangle + \mathfrak{L}_f y_\alpha \mathfrak{L}_f u_i - 2\mathfrak{L}_f(\langle\nabla y_\alpha, A\nabla u_i\rangle) + \mathfrak{L}_f(u_i\mathfrak{L}_f y_\alpha)$. By a direct computation, we have

$$\begin{aligned} &\int_\Omega y_\alpha u_i p_{\alpha i}\,d\mu \\ &\quad = \int_\Omega y_\alpha u_i\Big\{-2\big\langle\nabla y_\alpha, A\nabla(\mathfrak{L}_f u_i)\big\rangle + \mathfrak{L}_f y_\alpha\mathfrak{L}_f u_i - 2\mathfrak{L}_f\big(\langle\nabla y_\alpha, A\nabla u_i\rangle\big) \\ &\qquad + \mathfrak{L}_f(u_i\mathfrak{L}_f y_\alpha)\Big\}\,d\mu \\ &\quad = \int_\Omega 2\Big\{u_i\mathfrak{L}_f u_i\langle\nabla y_\alpha, A\nabla y_\alpha\rangle + y_\alpha\mathfrak{L}_f u_i\langle\nabla u_i, A\nabla y_\alpha\rangle - y_\alpha u_i\mathfrak{L}_f y_\alpha\mathfrak{L}_f u_i\Big\}\,d\mu \\ &\qquad + \int_\Omega y_\alpha u_i\mathfrak{L}_f y_\alpha\mathfrak{L}_f u_i\,d\mu \\ &\qquad + \int_\Omega\Big\{\mathfrak{L}_f y_\alpha u_i + y_\alpha\mathfrak{L}_f u_i - 2\langle\nabla y_\alpha, A\nabla u_i\rangle\Big\}\Big\{-2\langle\nabla y_\alpha, A\nabla u_i\rangle + u_i\mathfrak{L}_f y_\alpha\Big\}\,d\mu \\ &\quad = \int_\Omega 2u_i\mathfrak{L}_f u_i\langle\nabla y_\alpha, A\nabla y_\alpha\rangle\,d\mu + \int_\Omega 4\langle\nabla y_\alpha, A\nabla u_i\rangle^2\,d\mu \\ &\qquad - \int_\Omega 4u_i\mathfrak{L}_f y_\alpha\langle\nabla y_\alpha, A\nabla u_i\rangle\,d\mu + \int_\Omega(u_i\mathfrak{L}_f y_\alpha)^2\,d\mu. \end{aligned} \tag{3.16}$$

Since $\xi_1 I \le A \le \xi_2 I$, we can infer from (3.1)-(3.9) that

$$\begin{aligned} \sum_{\alpha=1}^{m}\int_\Omega 2u_i\mathfrak{L}_f u_i\langle\nabla y_\alpha, A\nabla y_\alpha\rangle\,d\mu &\le 2n\xi_2\int_\Omega u_i\mathfrak{L}_f u_i\,d\mu \\ &\le 2n\xi_2\big\{\|u_i\|^2\|\mathfrak{L}_f u_i\|^2\big\}^{\frac{1}{2}} = 2n\xi_2\Lambda_i^{\frac{1}{2}}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \sum_{\alpha=1}^{m}\int_\Omega 4\langle\nabla y_\alpha, A\nabla u_i\rangle^2\,d\mu = 4\|A\nabla u_i\|^2 &\le 4\xi_2\int_\Omega\langle\nabla u_i, A\nabla u_i\rangle \\ &= 4\xi_2\int_\Omega u_i\mathfrak{L}_f u_i\,d\mu \le 4\xi_2\Lambda_i^{\frac{1}{2}}, \end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\sum_{\alpha=1}^{m} -\int_\Omega 4u_i \mathfrak{L}_f y_\alpha \langle \nabla y_\alpha, A\nabla u_i\rangle \, d\mu \\
&\quad \le \left| 4\xi_2 \int_\Omega u_i \Delta_f y_\alpha \langle \nabla y_\alpha, A\nabla u_i\rangle \, d\mu \right| \\
&\quad = \left| 4\xi_2 \int_\Omega u_i \langle \nabla f, A\nabla u_i\rangle \, d\mu \right| \\
&\quad \le 4\xi_2 \int_\Omega u_i |\nabla f||\nabla u_i| \, d\mu \\
&\quad \le 4C_0\xi_2 \left\{ \|u_i\|^2 \|A\nabla u_i\|^2 \right\}^{\frac{1}{2}} \\
&\quad = 4C_0\xi_2^{\frac{3}{2}} \left\{ \int_\Omega \langle \nabla u_i, A\nabla u_i\rangle \, d\mu \right\}^{\frac{1}{2}} \le 4C_0\xi_2^{\frac{3}{2}} \Lambda_i^{\frac{1}{4}},
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
\sum_{\alpha=1}^{m} \int_\Omega (u_i \mathfrak{L}_f y_\alpha)^2 \, d\mu &\le \xi_2^2 \sum_{\alpha=1}^{m} \int_\Omega u_i^2 (\Delta_f y_\alpha)^2 \, d\mu \\
&= \xi_2^2 \int_\Omega u_i^2 \left(n^2 |\mathbf{H}|^2 + |\nabla f|^2 \right) d\mu \\
&\le \xi_2^2 \left(n^2 H_0^2 + C_0^2 \right).
\end{aligned} \tag{3.20}$$

Combining (3.16)-(3.20), we have

$$\sum_{\alpha=1}^{m} \int_\Omega y_\alpha u_i p_{\alpha i} \, d\mu \le \xi_2 \left((2n+4)\Lambda_i^{\frac{1}{2}} + 4C_0 \xi_2^{\frac{1}{2}} \Lambda_i^{\frac{1}{4}} + \xi_2 \left(n^2 H_0^2 + C_0^2 \right) \right). \tag{3.21}$$

Since $\|\nabla u_i\|^2 \le \frac{1}{\xi_1} \int_\Omega \langle \nabla u_i, A\nabla u_i\rangle \, d\mu = \frac{1}{\xi_1} \int_\Omega u_i \mathfrak{L}_f u_i \, d\mu \le \frac{\Lambda_i^{\frac{1}{2}}}{\xi_1}$, then from (3.14), we have

$$\begin{aligned}
&\int_\Omega \sum_{\alpha=1}^{m} \left(\langle \nabla y_\alpha, \nabla u_i\rangle + \frac{u_i \Delta_f y_\alpha}{2} \right)^2 d\mu \\
&\quad = \int_\Omega \left(|\nabla u_i|^2 + \langle \nabla f, \nabla u_i\rangle + \frac{u_i^2}{4} \left(n^2 |\mathbf{H}|^2 + |\nabla f|^2 \right) \right) d\mu \\
&\quad \le \frac{\Lambda_i^{\frac{1}{2}}}{\xi_1} + \frac{C_0 \Lambda_i^{\frac{1}{4}}}{\xi_1^{\frac{1}{2}}} + \frac{n^2 H_0^2 + C_0^2}{4}.
\end{aligned} \tag{3.22}$$

Taking (3.21) and (3.22) into (3.15), we have

$$\begin{aligned}
n \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 &\le \sum_{i=1}^{k} \delta (\Lambda_{k+1} - \Lambda_i)^2 \xi_2 \left((2n+4)\Lambda_i^{\frac{1}{2}} + 4C_0 \xi_2^{\frac{1}{2}} \Lambda_i^{\frac{1}{4}} + \xi_2 \left(n^2 H_0^2 + C_0^2 \right) \right) \\
&\quad + \sum_{i=1}^{k} \frac{\Lambda_{k+1} - \Lambda_i}{\delta} \left(\frac{\Lambda_i^{\frac{1}{2}}}{\xi_1} + \frac{C_0 \Lambda_i^{\frac{1}{4}}}{\xi_1^{\frac{1}{2}}} + \frac{n^2 H_0^2 + C_0^2}{4} \right).
\end{aligned} \tag{3.23}$$

Let

$$\delta = \left\{ \frac{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\{\frac{\Lambda_i^{\frac{1}{2}}}{\xi_1} + \frac{C_0\Lambda_i^{\frac{1}{4}}}{\xi_1^{\frac{1}{2}}} + \frac{n^2H_0^2+C_0^2}{4}\}}{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\xi_2((2n+4)\Lambda_i^{\frac{1}{2}} + 4C_0\xi_2^{\frac{1}{2}}\Lambda_i^{\frac{1}{4}} + \xi_2(n^2H_0^2+C_0^2))} \right\}^{\frac{1}{2}},$$

we can infer from (3.23) that

$$\begin{aligned} \sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \le \frac{\xi_2}{n\xi_1}\Bigg\{ & \sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\big((2n+4)\Lambda_i^{\frac{1}{2}} + 4C_0\xi_2^{\frac{1}{2}}\Lambda_i^{\frac{1}{4}} + \xi_2\big(n^2H_0^2+C_0^2\big)\big) \\ & \times \sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\big(4\Lambda_i^{\frac{1}{2}} + 4C_0\xi_1^{\frac{1}{2}}\Lambda_i^{\frac{1}{4}} + \xi_1\big(n^2H_0^2+C_0^2\big)\big)\Bigg\}^{\frac{1}{2}}, \end{aligned} \tag{3.24}$$

this completes the proof of Theorem 1.3. □

4 Proof of Theorem 1.5 and Theorem 1.7

In this section, applying Lemma 2.1, we will give the proof of Theorem 1.5 and Theorem 1.7.

Proof of Theorem 1.5 Let $a = 0, b = 1, V \equiv 0, \rho \equiv 1$ in (2.1), then taking $h = x_\alpha$ and summing over α, where $\{x_\alpha\}_{\alpha=1}^n$ are the coordinate functions of $\mathbb{R}^n$, we have

$$\begin{aligned} n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le {} & \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\int_\Omega\sum_{\alpha=1}^{n}x_\alpha u_i p_{\alpha i}\,d\mu \\ & + \sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\delta}\sum_{\alpha=1}^{n}\left(\langle\nabla x_\alpha,\nabla u_i\rangle + \frac{u_i\Delta_f x_\alpha}{2}\right)^2 d\mu, \end{aligned} \tag{4.1}$$

where $p_{\alpha i} = -2\langle\nabla x_\alpha, A\nabla u_i\rangle + u_i\mathfrak{L}_f x_\alpha$.

By a similar computation to (3.11), we have

$$\int_\Omega\sum_{\alpha=1}^{n}x_\alpha u_i p_{\alpha i}\,d\mu \le n\xi_2. \tag{4.2}$$

Since $f = \frac{|x|^2}{4}$, we have

$$\Delta_f x_\alpha = \Delta x_\alpha - \left\langle\nabla\left(\frac{|x|^2}{4}\right),\nabla x_\alpha\right\rangle = -\frac{1}{2}x_\alpha,$$

hence, we infer from the above equality that

$$\begin{aligned} & \int_\Omega\sum_{\alpha=1}^{n}\left(\langle\nabla x_\alpha,\nabla u_i\rangle + \frac{u_i\Delta_f x_\alpha}{2}\right)^2 d\mu \\ & = \int_\Omega\sum_{\alpha=1}^{n}\left(\langle\nabla x_\alpha,\nabla u_i\rangle^2 + u_i\Delta_f x_\alpha\langle\nabla x_\alpha,\nabla u_i\rangle + \frac{u_i^2(\Delta_f x_\alpha)^2}{4}\right) d\mu \\ & = \int_\Omega\left(|\nabla u_i|^2 + \sum_{\alpha=1}^{n}x_\alpha u_i\frac{\partial u_i}{\partial x_\alpha} + \sum_{\alpha=1}^{n}\frac{x_\alpha^2u_i^2}{16}\right) d\mu. \end{aligned} \tag{4.3}$$

From integration by parts, we have

$$\begin{aligned}
&\sum_{\alpha=1}^{n}\int_{\Omega} u_i x_\alpha \frac{\partial u_i}{\partial x_\alpha} e^{-\frac{|x|^2}{4}}\,dv \\
&\quad= \sum_{\alpha=1}^{n}\left\{-\int_{\Omega} u_i^2 e^{-\frac{|x|^2}{4}}\,dv - \int_{\Omega} u_i x_\alpha \frac{\partial u_i}{\partial x_\alpha} e^{-\frac{|x|^2}{4}}\,dv + \frac{1}{2}\int_{\Omega} u_i^2 x_\alpha^2 e^{-\frac{|x|^2}{4}}\,dv\right\} \\
&\quad= -n - \sum_{\alpha=1}^{n} a\int_{\Omega} u_i x_\alpha \frac{\partial u_i}{\partial x_\alpha} e^{-\frac{|x|^2}{4}}\,dv + \frac{1}{2}\int_{\Omega} u_i^2 |x|^2 e^{-\frac{|x|^2}{4}}\,dv,
\end{aligned}$$

which implies that

$$-2\sum_{\alpha=1}^{n}\int_{\Omega} u_i x_\alpha \frac{\partial u_i}{\partial x_\alpha}\,d\mu = n - \frac{1}{2}\int_{\Omega} u_i^2 |x|^2\,d\mu. \tag{4.4}$$

By a similar computation to (3.12), we have

$$\|\nabla u_i\|^2 \le \frac{\lambda_i}{\xi_1}. \tag{4.5}$$

Using (4.4), we have

$$\begin{aligned}
\sum_{\alpha=1}^{n}\int_{\Omega}\left(\langle\nabla x_\alpha, \nabla u_i\rangle - \frac{u_i x_\alpha}{4}\right)^2 d\mu &= \sum_{\alpha=1}^{n}\int_{\Omega}\left(\langle\nabla x_\alpha, \nabla u_i\rangle^2 - \frac{1}{2}\frac{\partial u_i}{\partial x_\alpha} u_i x_\alpha + \frac{u_i^2 x_\alpha^2}{16}\right) d\mu \\
&= \frac{n}{4} + \int_{\Omega}\left(|\nabla u_i|^2 - \frac{1}{16}u_i^2|x|^2\right) d\mu \\
&\le \frac{n}{4} + \frac{\lambda_i}{\xi_1} - \frac{1}{16}\min_{x\in\overline{\Omega}}\{|x|^2\}.
\end{aligned} \tag{4.6}$$

Taking (4.2) and (4.6) into (4.1), we have

$$\begin{aligned}
n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le{}& \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2 n\xi_2 \\
&+ \sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left(\frac{n}{4} + \frac{\lambda_i}{\xi_1} - \frac{1}{16}\min_{x\in\overline{\Omega}}\{|x|^2\}\right).
\end{aligned} \tag{4.7}$$

Taking

$$\delta = \left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\{\frac{n}{4} + \frac{\lambda_i}{\xi_1} - \frac{1}{16}\min_{x\in\overline{\Omega}}\{|x|^2\}\}}{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 n\xi_2}\right\}^{\frac{1}{2}},$$

we obtain (1.7). This finishes the proof of Theorem 1.5. □

Proof of Theorem 1.7 Let $a = 0, b = 1, V \equiv 0, \rho \equiv 1$ in (2.1), Set $\overline{x} = (x,t) \in \Omega$, where $x \in \Sigma, t \in \mathbb{R}$. By a direct computation, we know that $|\nabla t| = 1, \Delta_f t = -\kappa t$, then taking $h = t$, we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\int_\Omega tu_ip_i\,d\mu + \sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\delta}\int_\Omega\left(\langle\nabla t,\nabla u_i\rangle-\frac{\kappa tu_i}{2}\right)^2 d\mu, \tag{4.8}$$

where $p_i = -2\langle\nabla t, A\nabla u_i\rangle + u_i\mathfrak{L}_f t$.

In the following, let us estimate the right side of (4.8), first of all, by a similar computation to (3.11), we infer from $A \le \xi_2 I$ that

$$\int_\Omega tu_ip_i\,d\mu = \int_\Omega tu_i\big(-2\langle\nabla t, A\nabla u_i\rangle + u_i\mathfrak{L}_f t\big)\,d\mu = \int_\Omega u_i^2\langle\nabla t, A\nabla t\rangle\,d\mu \le \xi_2. \tag{4.9}$$

By a direct computation,

$$\int_\Omega tu_i\langle\nabla t,\nabla u_i\rangle\,d\mu = -\int_\Omega u_i^2|\nabla t|^2\,d\mu - \int_\Omega tu_i\langle\nabla t,\nabla u_i\rangle\,d\mu - \int_\Omega tu_i^2\Delta_f t\,d\mu,$$

which implies that

$$2\int_\Omega tu_i\langle\nabla t,\nabla u_i\rangle\,d\mu = -1 + \kappa\int_\Omega t^2u_i^2\,d\mu. \tag{4.10}$$

From the above equality, we have

$$\begin{aligned}
2\int_\Omega\left(\langle\nabla t,\nabla u_i\rangle-\frac{\kappa tu_i}{2}\right)^2 d\mu &= \int_\Omega\left(\langle\nabla t,\nabla u_i\rangle^2-\kappa tu_i\langle\nabla t,\nabla u_i\rangle+\frac{\kappa^2}{4}t^2u_i^2\right)d\mu\\
&= \int_\Omega\left(\langle\nabla t,\nabla u_i\rangle^2+1-\frac{\kappa^2}{4}t^2u_i^2\right)d\mu\\
&\le 1+\|\nabla u_i\|^2-\frac{\kappa^2}{4}\min_{(x,t)\in\overline{\Omega}}\big\{|t|^2\big\}\\
&\le 1+\frac{\lambda_i}{\xi_1}-\frac{\kappa^2}{4}\min_{(x,t)\in\overline{\Omega}}\big\{|t|^2\big\}.
\end{aligned} \tag{4.11}$$

Taking (4.9) and (4.11) into (4.8), we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\xi_2 + \sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\delta}\left(1+\frac{\lambda_i}{\xi_1}-\frac{\kappa^2}{4}\min_{(x,t)\in\overline{\Omega}}\big\{|t|^2\big\}\right). \tag{4.12}$$

Taking

$$\delta = \left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\{1+\frac{\lambda_i}{\xi_1}-\frac{\kappa^2}{4}\min_{(x,t)\in\overline{\Omega}}\{|t|^2\}\}}{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\xi_2}\right\}^{\frac{1}{2}},$$

we obtain (1.9). This finishes the proof of Theorem 1.7. □

Competing interests
All authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Acknowledgements
Y Zhu and G Liu were partially supported by NSF of Hubei Provincial Department of Education (Grant No. Q20154301). F Du was partially supported by the NSF of China (Grant No. 11401131), and NSF of Hubei Provincial Department of Education (Grant No. Q20154301).

Received: 9 March 2016 Accepted: 20 July 2016 Published online: 02 August 2016

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