Kwun et al. *Journal of Inequalities and Applications* (2016) 2016:171
DOI 10.1186/s13660-016-1115-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Relations between generalized von Neumann-Jordan and James constants for quasi-Banach spaces

Young Chel Kwun[1], Qaisar Mehmood[2], Waqas Nazeer[3], Absar Ul Haq[4] and Shin Min Kang[5*]

*Correspondence: smkang@gnu.ac.kr
[5]Department of Mathematics and RINS, Gyeongsang National University, Jinju, 660-701, Korea
Full list of author information is available at the end of the article

Abstract

Let $\mathcal{C}_{NJ}(\mathcal{B})$ and $J(\mathcal{B})$ be the generalized von Neumann-Jordan and James constants of a quasi-Banach space $\mathcal{B}$, respectively. In this paper we shall show the relation between $\mathcal{C}_{NJ}(\mathcal{B})$, $J(\mathcal{B})$, and the modulus of convexity. Also, we show that if $\mathcal{B}$ is not uniform non-square then $J(\mathcal{B}) = \mathcal{C}_{NJ}(\mathcal{B}) = 2$. Moreover, we give an equivalent formula for the generalized von Neumann-Jordan constant.

MSC: Primary 46B20; secondary 46E30

Keywords: the generalized von Neumann-Jordan constant; James constant; uniform non-squareness; quasi-normed spaces

1 Introduction

Among various geometric constants of a Banach space $\mathcal{B}$, the von Neumann-Jordan constant $C_{NJ}(\mathcal{B})$, and the James constant $J(\mathcal{B})$ have been treated most widely. In connection with the famous work [1] (see also [2]) of Jordan and von Neumann concerning inner products, the von Neumann-Jordan constant $C_{NJ}(\mathcal{B})$ for a Banach space $\mathcal{B}$ was introduced by Clarkson [3] as the smallest constant C for which the estimates

$$\frac{1}{C} \le \frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2(\|x_1\|^2 + \|x_2\|^2)} \le C$$

hold for any $x_1, x_2 \in \mathcal{B}$ with $(x_1, x_2) \neq (0, 0)$. Equivalently

$$C_{NJ}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2(\|x_1\|^2 + \|x_2\|^2)} : x_1, x_2 \in \mathcal{B} \text{ with } (x_1, x_2) \neq (0, 0)\right\}.$$

The classical von Neumann-Jordan constant $C_{NJ}(\mathcal{B})$ was investigated in [4–8].

A Banach space $\mathcal{B}$ is said to be uniformly non-square in the sense of James if there exists a positive number $\delta < 2$ such that for any $x_1, x_2 \in S_{\mathcal{B}} = \{x \in \mathcal{B} : \|x\| = 1\}$ we have

$$\min(\|x_1 + x_2\|, \|x_1 - x_2\|) \le \delta.$$

The James constant $J(\mathcal{B})$ of a Banach space $\mathcal{B}$ is defined by

$$J(\mathcal{B}) = \sup\{\min(\|x_1 + x_2\|, \|x_1 - x_2\|) : x_1, x_2 \in S_{\mathcal{B}}\}.$$

It is obvious that $\mathcal{B}$ is uniformly non-square if and only if $J(\mathcal{B}) < 2$.

In [9], the authors introduced the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(\mathcal{B})$, which is defined as

$$C_{NJ}^{p}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + x_2\|^p + \|x_1 - x_2\|^p}{2^{p-1}(\|x_1\|^p + \|x_2\|^p)} : x_1, x_2 \in \mathcal{B} \text{ with } (x_1, x_2) \neq (0,0)\right\},$$

and obtained the relationship between $C_{NJ}^{(p)}(\mathcal{B})$ and $J(\mathcal{B})$. Furthermore, they used the constant $C_{NJ}^{(p)}(\mathcal{B})$ to establish some new equivalent conditions for the uniform non-squareness of a Banach space $\mathcal{B}$. Both the von Neumann-Jordan $\mathcal{C}_{NJ}(\mathcal{B})$ and the James constants $J(\mathcal{B})$ play an important role in the description of various geometric structures. It is therefore worthwhile to clarify the relation between them.

In this paper, we shall show the relation between the generalized von Neumann-Jordan constant $\mathcal{C}_{NJ}(\mathcal{B})$ and the James constant $J(\mathcal{B})$ and we also show that if $\mathcal{B}$ is not uniform non-square then $J(\mathcal{B}) = \mathcal{C}_{NJ}(\mathcal{B}) = 2$. In the second section, we present basic definitions and define the modulus of convexity of a quasi-Banach space. In the third section, we establish a relationship between the generalized von Neumann-Jordan constant and the modulus of convexity, the James constant and the modulus of convexity, the generalized von Neumann-Jordan constant and the James constant, and we give the equivalent formula of the generalized von Neumann-Jordan constant.

2 Preliminaries

Definition 2.1 [10] A quasi-norm on $\|\cdot\|$ on vector space $\mathcal{B}$ over a field K ($\mathbb{R}$ or $\mathbb{C}$) is a map $\mathcal{B} \longrightarrow [0,\infty)$ with the properties:

- $\|x\| = 0 \Longleftrightarrow x = 0$.
- $\|\alpha x\| = |\alpha|\|x\|$ if $\forall \alpha \in K$, $\forall x \in \mathcal{B}$.
- There is a constant $C \geq 1$ such that if $\forall x_1, x_2 \in \mathcal{B}$ we have

$$\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|).$$

Definition 2.2 The generalized von Neumann-Jordan constant $\mathcal{C}_{NJ}^{(p)}(\mathcal{B})$ for quasi-Banach spaces is defined by

$$\mathcal{C}_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + x_2\|^p + \|x_1 - x_2\|^p}{2^{p-1}C^p(\|x_1\|^p + \|x_2\|^p)} : x_1, x_2 \in \mathcal{B} \text{ with } (x_1, x_2) \neq (0,0)\right\},$$

where $1 \leq p < \infty$.

We will also use the following parametrized formula for the constant $\mathcal{C}_{NJ}^{(p)}(\mathcal{B})$:

$$\mathcal{C}_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{C^p 2^{p-1}(1 + t^p)} : x_1, x_2 \in S_{\mathcal{B}}, 0 \leq t \leq 1\right\},$$

where $S_{\mathcal{B}}$ is the unit sphere. By taking $t = 1$ and $x_1 = x_2$, we obtain the estimate

$$C_{NJ}^{(p)}(\mathcal{B}) \geq \frac{\|2x_1\|^p}{C2^p(1+1)} \geq \frac{2^p}{C2^{p-1}(1+1)} = \frac{1}{C}.$$

Definition 2.3 In a quasi-Banach space $\mathcal{B}$ the James constant is defined as

$$J(\mathcal{B}) = \sup\left\{\frac{1}{C}\min\bigl(\|x_1+x_2\|, \|x_1-x_2\|\bigr) : x_1, x_2 \in S_{\mathcal{B}}\right\}. \tag{2.1}$$

Definition 2.4 A quasi-Banach space $\mathcal{B}$ is said to be uniformly non-square if there exists a positive number $\delta < 2$ such that for any $x_1, x_2 \in S_{\mathcal{B}}$, we have

$$\min\left(\left\|\frac{x_1+x_2}{C}\right\|, \left\|\frac{x_1+x_2}{C}\right\|\right) \leq \delta.$$

Remark 2.5 It is obvious that $\mathcal{B}$ is uniformly non-square if and only if $J(\mathcal{B}) < 2$.

Definition 2.6 The modulus of uniform smoothness of the quasi-Banach space $\mathcal{B}$ is defined as

$$\rho_{\mathcal{B}}(t) = \sup\left\{\frac{\|x_1+tx_2\| + \|x_1-tx_2\|}{2C} - \frac{1}{C} : x_1, x_2 \in S_{\mathcal{B}}, t \geq 0\right\}.$$

It is clear that $\rho_{\mathcal{B}}(t)$ is a convex function on the interval $[0,\infty)$ satisfying $\rho_{\mathcal{B}}(0) = 0$, whence it follows that $\rho_{\mathcal{B}}(t)$ is nondecreasing on $[0,\infty)$.

Definition 2.7 A quasi-Banach space $\mathcal{B}$ is said to be uniformly smooth if $(\rho_{\mathcal{B}})'_+(0) = \lim_{t\to 0^+}\frac{\rho_{\mathcal{B}}(t)}{t} = 0$.

Definition 2.8 Given any quasi-Banach space $\mathcal{B}$ and a number $p \in [0,\infty)$, another function $J_{\mathcal{B},p}(t)$ is defined by

$$J_{\mathcal{B},p}(t) = \sup\left\{\left(\frac{\|x_1+tx_2\|^p + \|x_1-tx_2\|^p}{2C^p}\right)^{\frac{1}{p}} : x_1, x_2 \in S_{\mathcal{B}}\right\}.$$

Definition 2.9 The modulus of convexity of a quasi-Banach space $\mathcal{B}$ is defined as

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x_1+x_2}{2C}\right\| : \left\|\frac{x_1-x_2}{C}\right\| \geq \epsilon; \forall x_1, x_2 \in S_{\mathcal{B}}\right\}, \quad 0 \leq \epsilon \leq 2.$$

3 Main results

Lemma 3.1 *For any number* $0 \leq a \leq 2$, $0 \leq b \leq 2$ *we have*

$$\left(\frac{a+b}{2C} - \frac{1}{C}\right)^2 + \frac{1}{C^2} \geq \frac{a^2+b^2}{4C^2}. \tag{3.1}$$

Proof

$$\begin{aligned}\left(\frac{a+b}{2C} - \frac{1}{C}\right)^2 + \frac{1}{C^2} &= \frac{a^2+b^2+2ab}{4C^2} + \frac{1}{C^2} - \frac{(a+b)}{C^2} + \frac{1}{C^2} \\ &= \frac{a^2+b^2}{4C^2} + \frac{ab}{2C^2} + \frac{1}{C^2} - \frac{(a+b)}{C^2} + \frac{1}{C^2}\end{aligned}$$

$$\begin{aligned} &= \frac{a^2+b^2}{4C^2} + \frac{1}{C^2}\left[\frac{ab}{2} + 2 - (a+b)\right] \\ &\geq \frac{a^2+b^2}{4C^2}. \end{aligned}$$

□

Lemma 3.2 *Let a be a real number and let* $b > 0$, *then*

$$\frac{at^2+bt}{1+t^2} \leq \frac{a+\sqrt{a^2+b^2}}{2}, \quad \forall t \geq 0.$$

The first theorem is a relation between $\mathcal{C}_{NJ}(\mathcal{B})$ and the modulus of uniform smoothness.

Theorem 3.3 *Let* $\mathcal{B}$ *be a quasi-Banach space, then*

$$\mathcal{C}_{NJ}(\mathcal{B}) \leq 1 + C\rho_{\mathcal{B}}(1)\left[\sqrt{\left\{1 - C\rho_{\mathcal{B}}(1)\right\}^2 + 1} - \left\{1 - C\rho_{\mathcal{B}}(1)\right\}\right]. \tag{3.2}$$

Proof We know that

$$\mathcal{C}_{NJ}(\mathcal{B}) = \sup\left\{\frac{\|x_1+tx_2\|^2 + \|x_1-tx_2\|^2}{2C^2(1+t^2)} : \forall x_1, x_2 \in S_{\mathcal{B}} \text{ with } (x_1,x_2) \neq (0,0), 0 \leq t \leq 1\right\}.$$

By using Lemma 3.1

$$\begin{aligned} &\frac{\|x_1+x_2\|^2 + \|x_1-x_2\|^2}{4C^2} \leq \left\{\left(\frac{\|x_1+x_2\| + \|x_1-x_2\|}{2C} - \frac{1}{C}\right)^2 + \frac{1}{C^2}\right\}, \\ &\|x_1+x_2\|^2 + \|x_1-x_2\|^2 \leq 4C^2\left\{\rho_{\mathcal{B}}^2(1) + \frac{1}{C^2}\right\}. \end{aligned} \tag{3.3}$$

Also we have

$$\|x_1+x_2\| + \|x_1-x_2\| \leq 2C\left\{\rho_{\mathcal{B}}(1) + \frac{1}{C}\right\}. \tag{3.4}$$

Since

$$\begin{aligned} \|x_1+tx_2\| &= \left\|t(x_1+x_2) + (1-t)x_1\right\| \leq C\bigl(t\|x_1+x_2\| + (1-t)\bigr), \\ \|x_1-tx_2\| &= \left\|t(x_1-x_2) + (1-t)x_1\right\| \leq C\bigl(t\|x_1-x_2\| + (1-t)\bigr), \end{aligned}$$

we have

$$\begin{aligned} \|x_1+tx_2\|^2 + \|x_1-tx_2\|^2 &\leq C^2\left[\bigl(t\|x_1+x_2\| + (1-t)\bigr)^2 + \bigl(t\|x_1-x_2\| + (1-t)\bigr)^2\right] \\ &= C^2\left[\bigl(\|x_1+x_2\|^2 + \|x_1-x_2\|^2\bigr)t^2\right. \\ &\quad \left. + \bigl(\|x_1+x_2\| + \|x_1-x_2\|\bigr)2t(1-t) + 2(1-t)^2\right] \\ &\leq C^2\left[4C^2\left\{\rho_{\mathcal{B}}^2(1) + \frac{1}{C^2}\right\}t^2\right. \\ &\quad \left. + 2\left(2C\left(\frac{1}{C} + \rho_{\mathcal{B}}(1)\right)t(1-t) + 2(1-t)^2\right)\right] \\ &= C^2\left[4Ct^2\rho_{\mathcal{B}}(1)\bigl(C\rho_{\mathcal{B}}(1) - 1\bigr) + 4Ct\rho_{\mathcal{B}}(1) + \bigl(1+t^2\bigr)\right], \end{aligned}$$

$$\begin{aligned}\frac{\|x_1+tx_2\|^2+\|x_1-tx_2\|^2}{2C^2(1+t^2)} &\leq \frac{2Ct^2\rho_{\mathcal{B}}(1)(C\rho_{\mathcal{B}}(1)-1)+2Ct\rho_{\mathcal{B}}(1)+2(1+t^2)}{1+t^2}\\ &\leq C\rho_{\mathcal{B}}(1)\bigl\{C\rho_{\mathcal{B}}(1)-1\bigr\}\\ &\quad+\sqrt{C^2\rho_{\mathcal{B}}^2(1)\bigl\{C\rho_{\mathcal{B}}(1)-1\bigr\}^2+C^2\rho_{\mathcal{B}}^2(1)}+1\\ &\quad\text{(by using Lemma 3.2)}\\ &=1+C\rho_{\mathcal{B}}(1)\sqrt{\bigl\{C\rho_{\mathcal{B}}(1)-1\bigr\}^2+1}-\bigl\{1-C\rho_{\mathcal{B}}(1)\bigr\}. \qquad \square\end{aligned}$$

The next result is the relation between the James constant and the modulus of convexity.

Theorem 3.4 *Let $\mathcal{B}$ be a quasi-Banach space then*

$$J(\mathcal{B})=\sup\left\{\varepsilon:\delta(\varepsilon)\leq 1-\frac{\varepsilon}{2}\right\}.$$

Proof Let

$$\alpha=\sup\left\{\varepsilon:\delta(\varepsilon)\leq 1-\frac{\varepsilon}{2}\right\}.$$

We shall show that $J(\mathcal{B})\leq\alpha$. For this purpose, if $\alpha=2$, then there is nothing to prove. So, we may assume that $\alpha<2$. For any $\beta>\alpha$, we have for any $x_1,x_2\in S_{\mathcal{B}}$ and $\frac{\|x_1-x_2\|}{C}\geq\beta$

$$\delta(\beta)>1-\frac{\beta}{2}.$$

From the definition of δ, we have

$$1-\frac{\|x_1+x_2\|}{2C}>1-\frac{\beta}{2},$$

which implies that

$$\frac{\|x_1+x_2\|}{C}<\beta;$$

therefore

$$\min\left(\frac{\|x_1+x_2\|}{C},\frac{\|x_1-x_2\|}{C}\right)\leq\beta.$$

As $J(\mathcal{B})\leq\beta$ and since β was arbitrary we have $J(\mathcal{B})\leq\alpha$.

For the reverse we use the definition of δ, so $\forall\gamma>0$ there exist $x_1,x_2\in S_{\mathcal{B}}$ such that

$$\frac{\|x_1-x_2\|}{C}\geq\varepsilon$$

and

$$1-\frac{\|x_1+x_2\|}{2C}\leq\delta(\varepsilon)+\gamma,$$

where we have $\varepsilon = \alpha - \gamma$, so

$$\frac{\|x_1 + x_2\|}{C} > 2 - 2\delta(\varepsilon) - 2\gamma;$$

therefore

$$\begin{aligned} J(\mathcal{B}) &\geq \min\left(\frac{\|x_1 + x_2\|}{C}, \frac{\|x_1 - x_2\|}{C}\right) \\ &\geq \min\bigl(2\bigl(1 - \delta(\varepsilon) - \gamma\bigr), \varepsilon\bigr) \\ &\geq \min(\varepsilon - 2\gamma, \varepsilon) \\ &= \varepsilon - 2\gamma \\ &= \alpha - 3\gamma, \end{aligned}$$

where γ was arbitrary, so we have $J(\mathcal{B}) \geq \alpha$. □

Corollary 3.5 *For any quasi-Banach space $\mathcal{B}$, we have*

$$J(\mathcal{B}) \geq \sqrt{2}.$$

Corollary 3.6 *Let $\mathcal{B}$ be any quasi-Banach space and $J(\mathcal{B}) \leq 2$, then*

$$\delta\bigl(J(\mathcal{B})\bigr) = 1 - \frac{J(\mathcal{B})}{2}.$$

Now, we are going to give an equivalent formula of the generalized von Neumann-Jordan constant.

Theorem 3.7 *We have*

$$\mathcal{C}_{NJ}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + x_2\|^p + \|x_1 - x_2\|^p}{2^{p-1}C^p(\|x_1\|^p + \|x_2\|^p)} : x_1, x_2 \in \mathcal{B} \textit{ with } \|x_1\| = 1, \|x_2\| \leq 1\right\}, \tag{3.5}$$

where $1 \leq p < \infty$.

Proof If $0 \neq \|x_1\| \geq \|x_2\|$

$$\begin{aligned} &\|x_1 + x_2\|^p = \|x_1\|^p \left\| \frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_1\|} \right\|^p, \\ &\|x_1 - x_2\|^p = \|x_1\|^p \left\| \frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_1\|} \right\|^p, \\ &\|x_1 + x_2\|^p + \|x_1 - x_2\|^p = \|x_1\|^p \left[\left\| \frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_1\|} \right\|^p + \left\| \frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_1\|} \right\|^p \right], \\ &\frac{\|x_1 + x_2\|^p + \|x_1 - x_2\|^p}{2^{p-1}C^p(\|x_1\|^p + \|x_2\|^p)} = \frac{\| \frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_1\|} \|^p + \| \frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_1\|} \|^p}{2^{p-1}C^p(1 + (\frac{x_2}{\|x_1\|})^p)}. \end{aligned}$$

This shows that

$$\mathcal{C}_{NJ}(\mathcal{B}) = \sup\left\{ \frac{\|x_1 + x_2\|^p + \|x_1 - x_2\|^p}{2^{p-1}C^p(\|x_1\|^p + \|x_2\|^p)} : x_1, x_2 \in \mathcal{B} \text{ with } \|x_1\| = 1, \|x_2\| \le 1 \right\}. \qquad \square$$

The next theorems show the relation between the generalized von Neumann-Jordan and James constants.

Theorem 3.8 *For any quasi-Banach space $\mathcal{B}$, we have*

$$\frac{1}{2}J(\mathcal{B})^2 \le \mathcal{C}_{NJ}(\mathcal{B}) \le \frac{J^2(\mathcal{B})}{(J(\mathcal{B}) - 1)^2 + 1}. \tag{3.6}$$

Proof For any $x_1, x_2 \in S_{\mathcal{B}}$, we have

$$\begin{aligned}
2\bigl(\min\bigl\{\|x_1 + x_2\|, \|x_1 - x_2\|\bigr\}\bigr)^2 &\le 2\left(\frac{\|x_1 + x_2\| + \|x_1 - x_2\|}{2}\right)^2 \\
&\le 2\left(\frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2}\right) \\
&= \frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2C^2(\|x_1\|^2 + \|x_2\|^2)} \cdot 2C^2\bigl(\|x_1\|^2 + \|x_2\|^2\bigr) \\
&\le 2C^2\mathcal{C}_{NJ}(\mathcal{B})\bigl(\|x_1\|^2 + \|x_2\|^2\bigr) \\
&= 4C^2\mathcal{C}_{NJ}(\mathcal{B}),
\end{aligned}$$

$$\bigl(\min\bigl\{\|x_1 + x_2\|, \|x_1 - x_2\|\bigr\}\bigr)^2 \le 2C^2\mathcal{C}_{NJ}(\mathcal{B}),$$

$$\frac{1}{C}\min\bigl\{\|x_1 + x_2\|, \|x_1 - x_2\|)\bigr\} \le 2\sqrt{\mathcal{C}_{NJ}(\mathcal{B})}.$$

Hence

$$\sup\left(\frac{1}{C}\min\bigl\{\|x_1 + x_2\|, \|x_1 - x_2\|\bigr\}\right) \le 2\sqrt{\mathcal{C}_{NJ}(\mathcal{B})}.$$

Therefore

$$\frac{1}{2}J^2(\mathcal{B}) \le \mathcal{C}_{NJ}(\mathcal{B}).$$

To prove the right hand side we use Theorem 3.7, so we only take $\|x_1\| = 1$ and $\|x_2\| \le 1$.

Case 1: If $\|x_2\| = t \ge J(\mathcal{B}) - 1$, then

$$\begin{aligned}
\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 &\le \bigl[C\bigl(\|x_1\| + \|x_2\|\bigr)\bigr]^2 + \left(\frac{C}{C}\min\bigl\{\|x_1 + x_2\|, \|x_1 - x_2\|\bigr\}\right)^2 \\
&= C^2\left[\bigl(\|x_1\| + \|x_2\|\bigr)^2 + \left(\frac{1}{C}\min\bigl\{\|x_1 + x_2\|, \|x_1 - x_2\|\bigr\}\right)^2\right]
\end{aligned}$$

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 \le C^2\bigl[\bigl(\|x_1\| + \|x_2\|\bigr)^2 + J^2(\mathcal{B})\bigr],$$

$$\begin{aligned}
\frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2C^2(\|x_1\|^2 + \|x_2\|^2)} &\le \frac{(\|x_1\| + \|x_2\|)^2 + J^2(\mathcal{B})}{2(\|x_1\|^2 + \|x_2\|^2)} \\
&= \frac{(1 + t)^2 + J^2(\mathcal{B})}{2(1 + t^2)}.
\end{aligned}$$

The function

$$f(t) = \frac{(1+t)^2 + J^2(\mathcal{B})}{2(1+t^2)}$$

is increasing on $(0,\mu)$ and decreasing on $(\mu,1)$ where

$$\mu = \frac{-J^2(\mathcal{B}) + \sqrt{J^4(\mathcal{B})+4}}{2}.$$

Since $J(\mathcal{B})-1 \geq \mu$ and $J(\mathcal{B})-1 \leq t$, we have $f(t) \leq f(J(\mathcal{B})-1)$

$$\frac{\|x_1+x_2\|^2 + \|x_1-x_2\|^2}{2C^2(\|x_1\|^2+\|x_2\|^2)} \leq \frac{J^2(\mathcal{B})}{1+(J(\mathcal{B})-1)^2},$$

taking the supremum, we get

$$\mathcal{C}_{NJ}(\mathcal{B}) \leq \frac{J^2(\mathcal{B})}{1+(J(\mathcal{B})-1)^2}. \tag{3.7}$$

Case 2: If $\|x_2\| = t \leq J(\mathcal{B})-1$, then

$$\begin{aligned} \|x_1+x_2\|^2 + \|x_1-x_2\|^2 &\leq 2C^2\big(\|x_1\|+\|x_2\|\big)^2 \\ &= 2C^2(1+t)^2, \end{aligned}$$

$$\frac{\|x_1+x_2\|^2 + \|x_1-x_2\|^2}{2C^2(\|x_1\|^2+\|x_2\|^2)} \leq \frac{(1+t)^2}{(1+t^2)}.$$

Let

$$g(t) = \frac{(1+t)^2}{(1+t^2)},$$

since g is increasing on $(0,1]$

$$g(t) \leq g\big(J(\mathcal{B})-1\big),$$

hence

$$\mathcal{C}_{NJ}(\mathcal{B}) \leq \frac{J^2(\mathcal{B})}{1+(J(\mathcal{B})-1)^2}. \tag{3.8}$$

□

Corollary 3.9 *If $\mathcal{B}$ is not uniformly non-square then*

$$J(\mathcal{B}) = \mathcal{C}_{NJ}(\mathcal{B}) = 2.$$

Theorem 3.10 *For any quasi-Banach space $\mathcal{B}$, we have*

$$\mathcal{C}_{NJ}(\mathcal{B}) \leq 1 + \left(\frac{CJ(\mathcal{B})}{2}\right)^2. \tag{3.9}$$

Proof Since $\mathcal{C}_{NJ}(\mathcal{B}) = \sup\{\mathcal{C}_{NJ}(t,\mathcal{B}) : t \in [0,1]\}$ where

$$\mathcal{C}_{NJ}(t,\mathcal{B}) = \sup\left\{\frac{\|x_1 + tx_2\|^2 + \|x_1 - tx_2\|^2}{2C^2(1+t^2)} : \forall x_1, x_2 \in S_{\mathcal{B}}\right\}.$$

First we prove that

$$\mathcal{C}_{NJ}(t,\mathcal{B}) \leq 1 + \frac{C^2t^2J^2(\mathcal{B}) + 2Ct(1-t)J(\mathcal{B})}{2(1+t^2)}. \tag{3.10}$$

For this purpose

$$\begin{aligned}\|x_1 + tx_2\|^2 + \|x_1 - tx_2\|^2 &\leq \bigl[C(1+t)\bigr]^2 + \bigl(\min\bigl\{\|x_1 + tx_2\|, \|x_1 - tx_2\|\bigr\}\bigr)^2 \\ &\leq C^2\bigl[(1+t)^2 + J^2(t,\mathcal{B})\bigr], \\ \frac{\|x_1 + tx_2\|^2 + \|x_1 - tx_2\|^2}{2C^2(1+t^2)} &\leq \frac{(1+t)^2 + J^2(t,\mathcal{B})}{2(1+t^2)}.\end{aligned}$$

Taking the supremum we get

$$\mathcal{C}_{NJ}(t,\mathcal{B}) \leq \frac{(1+t)^2 + J^2(t,\mathcal{B})}{2(1+t^2)}. \tag{3.11}$$

Also note that

$$\begin{aligned}&\min\bigl\{\|x_1 + tx_2\| + \|x_1 - tx_2\|\bigr\} \leq C\min\bigl\{t\|x_1 + x_2\| + (1-t), t\|x_1 - x_2\| + (1-t)\bigr\}, \\ &J(t,\mathcal{B}) \leq tCJ(\mathcal{B}) + (1-t).\end{aligned} \tag{3.12}$$

Using (3.11) and (3.12), we get

$$\mathcal{C}_{NJ}(t,\mathcal{B}) \leq \frac{[CtJ(\mathcal{B}) + (1-t)]^2 + (1+t)^2}{2(1+t^2)} \tag{3.13}$$

$$= 1 + \frac{C^2t^2J^2(\mathcal{B}) + 2t(1-t)J(\mathcal{B})}{2(1+t^2)}. \tag{3.14}$$

Taking the supremum over t, we get

$$\mathcal{C}_{NJ}(\mathcal{B}) \leq 1 + \left(\frac{CJ(\mathcal{B})}{2}\right)^2. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Dong-A University, Busan, 604-714, Korea. [2]Department of Mathematics, Science College, Wahdat Road, Lahore, Pakistan. [3]Division of Science and Technology, University of Education, Lahore, Pakistan. [4]Abdus Salam School of Mathmatical Sciences, GC University, Lahore, 54000, Pakistan. [5]Department of Mathematics and RINS, Gyeongsang National University, Jinju, 660-701, Korea.

Acknowledgements
This study was supported by research funds from Dong-A University.

Received: 17 February 2016 Accepted: 21 June 2016 Published online: 05 July 2016

References

1. Jordan, P, von Neumann, J: On inner products in linear metric spaces. Ann. Math. **36**, 719-723 (1935)
2. Zuo, Z, Cui, Y: Some sufficient conditions for fixed points of multivalued nonexpansive mappings. Fixed Point Theory Appl. **2009**, Article ID 319804 (2009). doi:10.1155/2009/319804
3. Clarkson, JA: The von Neumann-Jordan constant for the Lebesgue space. Ann. Math. **38**, 114-115 (1937)
4. Kato, M, Maligranda, L, Takahashi, Y: On James and Jordan-von Neumann constants and the normal structure coefficient of Banach spaces. Stud. Math. **144**(3), 275-295 (2001)
5. Kato, M, Maligranda, L: On James and Jordan-von Neumann constants of Lorentz sequence spaces. J. Math. Anal. Appl. **258**(2), 457-465 (2001)
6. Takahashi, Y, Kato, M: Von Neumann-Jordan constant and uniformly non-square Banach spaces. Nihonkai Math. J. **9**, 155-169 (1998)
7. Yang, C: Jordan-von Neumann constant for Banas Fraczek space. Banach J. Math. Anal. **8**(2), 185-192 (2014)
8. He, C, Cui, Y: Some properties and concerning Milman's moduli. J. Math. Anal. Appl. **329**(2), 1260-1272 (2007)
9. Cui, Y, Huang, W, Hudzik, H, Kaczmarek, R: Generalized von Neumann-Jordan constant and it relationship to the fixed point property. Fixed Point Theory Appl. **2015**, 40 (2015). doi:10.1186/s13663-015-0288-3
10. Bennett, C, Sharpley, R: Interpolation of Operators. Academic Press, New York (1988)