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Inequalities for finite trigonometric sums. An interplay: with some series related to harmonic numbers

Omran Kouba*

*Correspondence:
omran_kouba@hiast.edu.sy
Department of Mathematics, Higher Institute for Applied Sciences and Technology, P.O. Box 31983, Damascus, Syria

Abstract

An interplay between the sum of certain series related to harmonic numbers and certain finite trigonometric sums is investigated. This allows us to express the sum of these series in terms of the considered trigonometric sums, and permits us to find sharp inequalities bounding these trigonometric sums. In particular, this answers positively an open problem of Chen (Excursions in Classical Analysis, 2010).

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1 Introduction

Many identities that evaluate trigonometric sums in closed form can be found in the literature. For example, in a solution to a problem in SIAM Review [2], p.157, Fisher shows that

$$\sum_{k=1}^{p-1}\sec^2\left(\frac{k\pi}{2p}\right)=\frac{2}{3}(p^2-1),$$
$$\sum_{k=1}^{p-1}\sec^4\left(\frac{k\pi}{2p}\right)=\frac{4}{45}(2p^4+5p^2-7).$$

General results giving closed forms for the power sums of secants $\sum_{k=1}^{p-1}\sec^{2n}(\frac{k\pi}{2p})$ and $\sum_{k=1}^{p}\sec^{2n}(\frac{k\pi}{2p+1})$, for many values of the positive integer n, can be found in [3] and [4]. Also, in [5] the author proves that

$$\sum_{k=1}^{p}\sec\left(\frac{2k\pi}{2p+1}\right)=\begin{cases} p & \text{if } p \text{ is even,}\\ -p-1 & \text{if } p \text{ is odd.}\end{cases}$$

However, while there are many cases where closed forms for finite trigonometric sums can be obtained, it seems that there are no such formulas for the sums we are interested in.

In this paper we study the trigonometric sums I_p and J_p defined for positive integers p by the formulas

$$I_p = \sum_{k=1}^{p-1} \frac{1}{\sin(k\pi/p)} = \sum_{k=1}^{p-1} \csc\left(\frac{k\pi}{p}\right), \tag{1.1}$$

$$J_p = \sum_{k=1}^{p-1} k \cot\left(\frac{k\pi}{p}\right), \tag{1.2}$$

with empty sums interpreted as 0.

To the best of the author's knowledge there no closed form for I_p is known, and the same can be said about the sum J_p. Therefore, we will look for asymptotic expansions for these sums and will give some tight inequalities that bound I_p and J_p. This investigation complements the work of Chen in [1], Chapter 7, where it was asked, as an open problem, whether the inequality

$$I_p \le \frac{2p}{\pi}\bigl(\ln p + \gamma - \ln(\pi/2)\bigr)$$

holds true for $p \ge 3$ (here γ is the so-called the Euler-Mascheroni constant). In fact, it will be proved that for every positive integer p and every nonnegative integer n, we have

$$I_p < \frac{2p}{\pi}\bigl(\ln p + \gamma - \ln(\pi/2)\bigr) + \sum_{k=1}^{2n} (-1)^k \frac{(2^{2k}-2)b_{2k}^2}{k \cdot (2k)!} \left(\frac{\pi}{p}\right)^{2k-1}$$

and

$$I_p > \frac{2p}{\pi}\bigl(\ln p + \gamma - \ln(\pi/2)\bigr) + \sum_{k=1}^{2n+1} (-1)^k \frac{(2^{2k}-2)b_{2k}^2}{k \cdot (2k)!} \left(\frac{\pi}{p}\right)^{2k-1},$$

where the b_{2k}'s are Bernoulli numbers (see Theorem 3.4). The corresponding inequalities for J_p are also proved (see Theorem 3.9).

Harmonic numbers play an important role in this investigation. Recall that the nth harmonic number H_n is defined by $H_n = \sum_{k=1}^{n} 1/k$ (with the convention $H_0 = 0$). In this work, a link between our trigonometric sums I_p and J_p and the sum of several series related to harmonic numbers is uncovered. Indeed, the well-known fact that $H_n = \ln n + \gamma + \frac{1}{2n} + \mathcal{O}(\frac{1}{n^2})$ proves the convergence of the numerical series,

$$C_p = \sum_{n=1}^{\infty} \left(H_{pn} - \ln(pn) - \gamma - \frac{1}{2pn}\right),$$

$$D_p = \sum_{n=1}^{\infty} (-1)^{n-1}\bigl(H_{pn} - \ln(pn) - \gamma\bigr),$$

$$E_p = \sum_{n=0}^{\infty} (-1)^n (H_{p(n+1)} - H_{pn}),$$

for every positive integer p.

An interplay between the considered trigonometric sums and the sum of these series will allow us to prove sharp inequalities for I_p and J_p, and to find the expression of the sums C_p, D_p, and E_p in terms of I_p and J_p.

The main tool will be the following formulation ([6], Corollary 8.2) of the Euler-Maclaurin summation formula.

Theorem 1.1 *Consider a positive integer m, and a function f that has a continuous $(2m-1)$st derivative on $[0,1]$. If $f^{(2m-1)}$ is* decreasing, *then*

$$\int_0^1 f(t)\,dt = \frac{f(1)+f(0)}{2} - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!}\delta f^{(2k-1)} + (-1)^{m+1} R_m$$

with

$$R_m = \int_0^{1/2} \frac{|B_{2m-1}(t)|}{(2m-1)!} \bigl(f^{(2m-1)}(t) - f^{(2m-1)}(1-t)\bigr)\,dt$$

and

$$0 \le R_m \le \frac{6}{(2\pi)^{2m}} \bigl(f^{(2m-1)}(0) - f^{(2m-1)}(1)\bigr),$$

where the b_{2k}'s are Bernoulli numbers, B_{2m-1} is the Bernoulli polynomial of degree $2m-1$, and the notation δg for a function $g : [0,1] \to \mathbb{C}$ means $g(1) - g(0)$.

For more information on the Bernoulli polynomials, Bernoulli numbers, and the Euler-Maclaurin formula, the reader may refer to [6–10], and the references therein. This paper is organized as follows. In Section 2 we find the asymptotic expansions of C_p and D_p for large p. In Section 3, the inequalities the trigonometric sums I_p and J_p are proved.

2 Asymptotic expansions for the sum of certain series related to harmonic numbers

In the next lemma, the asymptotic expansion of $(H_n)_{n\in\mathbb{N}}$ is presented. It can be found implicitly in Chapter 9 of [11]; we present a proof for convenience of the reader.

Lemma 2.1 *For every positive integer n and nonnegative integer m, we have*

$$H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{m-1} \frac{b_{2k}}{2k} \cdot \frac{1}{n^{2k}} + (-1)^m R_{n,m},$$

with

$$R_{n,m} = \int_0^{1/2} \bigl|B_{2m-1}(t)\bigr| \sum_{j=n}^{\infty} \left(\frac{1}{(j+t)^{2m}} - \frac{1}{(j+1-t)^{2m}} \right) dt.$$

Moreover, $0 < R_{n,m} < \frac{|b_{2m}|}{2m\cdot n^{2m}}$.

Proof Note that for $j \geq 1$ we have

$$\frac{1}{j} - \ln\left(1 + \frac{1}{j}\right) = \int_0^1 \left(\frac{1}{j} - \frac{1}{j+t}\right) dt = \int_0^1 \frac{t}{j(j+t)}\, dt.$$

Adding these equalities as j varies from 1 to $n-1$ we conclude that

$$H_n - \ln n - \frac{1}{n} = \int_0^1 \left(\sum_{j=1}^{n-1} \frac{t}{j(j+t)}\right) dt.$$

Thus, letting n tend to ∞, and using the monotone convergence theorem, we conclude

$$\gamma = \int_0^1 \left(\sum_{j=1}^{\infty} \frac{t}{j(j+t)}\right) dt.$$

It follows that

$$\gamma + \ln n - H_n + \frac{1}{n} = \int_0^1 \left(\sum_{j=n}^{\infty} \frac{t}{j(j+t)}\right) dt.$$

So, let us consider the function $f_n : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f_n(t) = \sum_{j=n}^{\infty} \frac{t}{j(j+t)}.$$

Note that $f_n(0) = 0$, $f_n(1) = 1/n$, and that f_n is infinitely continuously derivable with

$$\frac{f_n^{(k)}(t)}{k!} = (-1)^{k+1} \sum_{j=n}^{\infty} \frac{1}{(j+t)^{k+1}}, \quad \text{for } k \geq 1.$$

In particular,

$$\frac{f_n^{(2k-1)}(t)}{(2k-1)!} = \sum_{j=n}^{\infty} \frac{1}{(j+t)^{2k}}, \quad \text{for } k \geq 1.$$

So, $f_n^{(2m-1)}$ is decreasing on the interval $[0,1]$, and

$$\frac{\delta f_n^{(2k-1)}}{(2k-1)!} = \sum_{j=n}^{\infty} \frac{1}{(j+1)^{2k}} - \sum_{j=n}^{\infty} \frac{1}{j^{2k}} = -\frac{1}{n^{2k}}.$$

Applying Theorem 1.1 to f_n, and using the above data, we get

$$\gamma + \ln n - H_n + \frac{1}{2n} = \sum_{k=1}^{m-1} \frac{b_{2k}}{2kn^{2k}} + (-1)^{m+1} R_{n,m}$$

with

$$R_{n,m} = \int_0^{1/2} \left|B_{2m-1}(t)\right| \sum_{j=n}^{\infty} \left(\frac{1}{(j+t)^{2m}} - \frac{1}{(j+1-t)^{2m}}\right) dt$$

and

$$0 < R_{n,m} < \frac{6 \cdot (2m-1)!}{(2\pi)^{2m} n^{2m}}.$$

The important estimate is the lower bound, *i.e.* $R_{n,m} > 0$. In fact, considering separately the cases m odd and m even, we obtain, for every nonnegative integer m':

$$H_n < \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{2m'} \frac{b_{2k}}{2k} \cdot \frac{1}{n^{2k}}$$

and

$$H_n > \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{2m'+1} \frac{b_{2k}}{2k} \cdot \frac{1}{n^{2k}}.$$

This yields the following more precise estimate for the error term:

$$0 < (-1)^m \left(H_n - \ln n - \gamma - \frac{1}{2n} + \sum_{k=1}^{m-1} \frac{b_{2k}}{2k \cdot n^{2k}} \right) < \frac{|b_{2m}|}{2m \cdot n^{2m}},$$

which is valid for every positive integer m. □

Now, consider the two sequences $(c_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ defined by

$$c_n = H_n - \ln n - \gamma - \frac{1}{2n} \quad \text{and} \quad d_n = H_n - \ln n - \gamma.$$

For a positive integer p, we know according to Lemma 2.1 that $c_{pn} = \mathcal{O}(\frac{1}{n^2})$, it follows that the series $\sum_{n=1}^{\infty} c_{pn}$ is convergent. Similarly, since $d_{pn} = c_{pn} + \frac{1}{2pn}$ and the series $\sum_{n=1}^{\infty} (-1)^{n-1}/n$ is convergent, we conclude that $\sum_{n=1}^{\infty} (-1)^{n-1} d_{pn}$ is also convergent. In the following we aim to find asymptotic expansions (for large p) of the following sums:

$$C_p = \sum_{n=1}^{\infty} c_{pn} = \sum_{n=1}^{\infty} \left(H_{pn} - \ln(pn) - \gamma - \frac{1}{2pn} \right), \tag{2.1}$$

$$D_p = \sum_{n=1}^{\infty} (-1)^{n-1} d_{pn} = \sum_{n=1}^{\infty} (-1)^{n-1} \big(H_{pn} - \ln(pn) - \gamma \big), \tag{2.2}$$

$$E_p = \sum_{n=0}^{\infty} (-1)^n (H_{p(n+1)} - H_{pn}). \tag{2.3}$$

Theorem 2.2 *If p and m are positive integers and C_p is defined by* (2.1), *then*

$$C_p = -\sum_{k=1}^{m-1} \frac{b_{2k} \zeta(2k)}{2k \cdot p^{2k}} + (-1)^m \frac{\zeta(2m)}{2m \cdot p^{2m}} \varepsilon_{p,m}, \quad \textit{with } 0 < \varepsilon_{p,m} < |b_{2m}|,$$

where ζ is the well-known Riemann zeta function.

Proof Indeed, we conclude from Lemma 2.1 that

$$H_{pn} - \ln(pn) - \gamma - \frac{1}{2pn} = -\sum_{k=1}^{m-1} \frac{b_{2k}}{2k \cdot p^{2k}} \cdot \frac{1}{n^{2k}} + \frac{(-1)^m}{2m \cdot p^{2m}} \cdot \frac{r_{pn,m}}{n^{2m}},$$

with $0 < r_{pn,m} \le |b_{2m}|$. It follows that

$$C_p = -\sum_{k=1}^{m-1} \frac{b_{2k}}{2kp^{2k}} \cdot \left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}}\right) + \frac{(-1)^m}{2m \cdot p^{2m}} \cdot \tilde{r}_{p,m},$$

where $\tilde{r}_{p,m} = \sum_{n=1}^{\infty} \frac{r_{pn,m}}{n^{2m}}$.

Hence

$$0 < \tilde{r}_{p,m} = \sum_{n=1}^{\infty} \frac{r_{pn,m}}{n^{2m}} < |b_{2m}| \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = |b_{2m}|\zeta(2m)$$

and the desired conclusion follows with $\varepsilon_{p,m} = \tilde{r}_{p,m}/\zeta(2m)$. □

For example, taking $m = 3$, we obtain

$$\sum_{n=1}^{\infty}\left(H_{pn} - \ln(pn) - \gamma - \frac{1}{2pn}\right) = -\frac{\pi^2}{72p^2} + \frac{\pi^4}{10{,}800p^4} + \mathcal{O}\left(\frac{1}{p^6}\right).$$

In the next proposition we have the analogous result corresponding to D_p.

Theorem 2.3 *If p and m are positive integers and D_p is defined by* (2.2), *then*

$$D_p = \frac{\ln 2}{2p} - \sum_{k=1}^{m-1} \frac{b_{2k}\eta(2k)}{2k \cdot p^{2k}} + (-1)^m \frac{\eta(2m)}{2m \cdot p^{2m}} \varepsilon'_{p,m}, \quad \text{with } 0 < \varepsilon'_{p,m} < |b_{2m}|,$$

where η is the Dirichlet eta function [12].

Proof Indeed, let us define $a_{n,m}$ by the formula

$$a_{n,m} = H_n - \ln n - \gamma - \frac{1}{2n} + \sum_{k=1}^{m-1} \frac{b_{2k}}{2k \cdot n^{2k}}$$

with empty sum equal to 0. We have shown in the proof of Lemma 2.1 that

$$(-1)^m a_{n,m} = \int_0^{1/2} \left|B_{2m-1}(t)\right| g_{n,m}(t)\,dt,$$

where $g_{n,m}$ is the positive decreasing function on $[0, 1/2]$ defined by

$$g_{n,m}(t) = \sum_{j=n}^{\infty}\left(\frac{1}{(j+t)^{2m}} - \frac{1}{(j+1-t)^{2m}}\right).$$

Now, for every $t \in [0,1/2]$ the sequence $(g_{np,m}(t))_{n\geq 1}$ is positive and decreasing to 0. So, using the alternating series criterion [13], Theorem 7.8 and Corollary 7.9, we see that, for every $N \geq 1$ and $t \in [0,1/2]$,

$$\left|\sum_{n=N}^{\infty}(-1)^{n-1}g_{np,m}(t)\right| \leq g_{Np,m}(t) \leq g_{Np,m}(0) = \frac{1}{(Np)^{2m}}.$$

This proves the uniform convergence on $[0,1/2]$ of the series

$$G_{p,m}(t) = \sum_{n=1}^{\infty}(-1)^{n-1}g_{np,m}(t).$$

Consequently

$$(-1)^m\sum_{n=1}^{\infty}(-1)^{n-1}a_{pn,m} = \int_0^{1/2}\left|B_{2m-1}(t)\right|G_{p,m}(t)\,dt.$$

Now using the properties of an alternating series, we see that for $t \in (0,1/2)$ we have

$$0 < G_{p,m}(t) < g_{p,m}(t) < g_{p,m}(0) = \sum_{j=p}^{\infty}\left(\frac{1}{j^{2m}} - \frac{1}{(j+1)^{2m}}\right) = \frac{1}{p^{2m}}.$$

Thus,

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_{pn,m} = \frac{(-1)^m}{p^{2m}}\rho_{p,m}$$

with $0 < \rho_{p,m} < \int_0^{1/2}|B_{2m-1}(t)|\,dt$.

On the other hand we have

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}a_{pn,m} &= D_p - \frac{1}{2p}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} + \sum_{k=1}^{m-1}\frac{b_{2k}}{2kp^{2k}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{2k}} \\ &= D_p - \frac{\ln 2}{2p} + \sum_{k=1}^{m-1}\frac{b_{2k}\eta(2k)}{2k\cdot p^{2k}}.\end{aligned}$$

Thus

$$D_p = \frac{\ln 2}{2p} - \sum_{k=1}^{m-1}\frac{b_{2k}\eta(2k)}{2k\cdot p^{2k}} + \frac{(-1)^m}{p^{2m}}\rho_{p,m}$$

Now, the important estimate for $\rho_{p,m}$ is the lower bound, *i.e.* $\rho_{p,m} > 0$. In fact, considering separately the cases m odd and m even, we obtain, for every nonnegative integer m':

$$D_p < \frac{\ln 2}{2p} - \sum_{k=1}^{2m'}\frac{b_{2k}\eta(2k)}{2k\cdot p^{2k}}$$

and

$$D_p > \frac{\ln 2}{2p} - \sum_{k=1}^{2m'+1} \frac{b_{2k}\eta(2k)}{2k \cdot p^{2k}}.$$

This yields the following more precise estimate for the error term:

$$0 < (-1)^m \left(D_p - \frac{\ln 2}{2p} + \sum_{k=1}^{m-1} \frac{b_{2k}\eta(2k)}{2kp^{2k}} \right) < \frac{|b_{2m}|\eta(2m)}{2m \cdot p^{2m}},$$

and the desired conclusion follows. □

The case of E_p, which is the sum of another alternating series (2.3), is discussed in the next lemma where it is shown that E_p can easily be expressed in terms of D_p.

Lemma 2.4 *For a positive integer p, we have*

$$E_p = \ln p + \gamma - \ln\left(\frac{\pi}{2}\right) + 2D_p,$$

where D_p is the sum defined by (2.2).

Proof Indeed

$$\begin{aligned}
2D_p &= d_p + \sum_{n=2}^{\infty}(-1)^{n-1} d_{pn} + \sum_{n=1}^{\infty}(-1)^{n-1} d_{pn} \\
&= d_p + \sum_{n=1}^{\infty}(-1)^{n} d_{p(n+1)} + \sum_{n=1}^{\infty}(-1)^{n-1} d_{pn} \\
&= d_p + \sum_{n=1}^{\infty}(-1)^{n-1} (d_{pn} - d_{p(n+1)}) \\
&= d_p + \sum_{n=1}^{\infty}(-1)^{n} (H_{p(n+1)} - H_{pn}) + \sum_{n=1}^{\infty}(-1)^{n-1} \ln\left(\frac{n+1}{n}\right) \\
&= -\ln p - \gamma + \sum_{n=0}^{\infty}(-1)^{n} (H_{p(n+1)} - H_{pn}) + \sum_{n=1}^{\infty}(-1)^{n-1} \ln\left(\frac{n+1}{n}\right).
\end{aligned}$$

Using the Wallis formula for π [8], Formula 0.262, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^{n-1} \ln\left(\frac{n+1}{n}\right) &= \sum_{n=1}^{\infty} \ln\left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right) \\
&= -\ln \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) = \ln\left(\frac{\pi}{2}\right)
\end{aligned}$$

and the desired formula follows. □

3 Inequalities for trigonometric sums

As we mentioned in the introduction, we are interested in the sum of cosecants I_p defined by (1.1) and the sum of cotangents J_p defined by (1.2). Many other trigonometric sums can be expressed in terms of I_p and J_p. The next lemma lists some of these identities.

Lemma 3.1 *For a positive integer p let*

$$K_p = \sum_{k=1}^{p-1} \tan\left(\frac{k\pi}{2p}\right), \qquad \widetilde{K}_p = \sum_{k=1}^{p-1} \cot\left(\frac{k\pi}{2p}\right),$$
$$L_p = \sum_{k=1}^{p-1} \frac{k}{\sin(k\pi/p)}, \qquad M_p = \sum_{k=1}^{p} (2k-1)\cot\left(\frac{(2k-1)\pi}{2p}\right).$$

Then:

(i) $K_p = \widetilde{K}_p = I_p$.
(ii) $L_p = (p/2)I_p$.
(iii) $M_p = (p/2)J_{2p} - 2J_p = -pI_p$.

Proof First, note that the change of summation variable $k \leftarrow p-k$ proves that $K_p = \widetilde{K}_p$. So, using the trigonometric identity $\tan\theta + \cot\theta = 2\csc(2\theta)$ we obtain (i) as follows:

$$2K_p = K_p + \widetilde{K}_p = \sum_{k=1}^{p-1}\left(\tan\left(\frac{k\pi}{2p}\right) + \cot\left(\frac{k\pi}{2p}\right)\right) = 2\sum_{k=1}^{p-1}\csc\left(\frac{k\pi}{p}\right) = 2I_p.$$

Similarly, (ii) follows from the change of summation variable $k \leftarrow p-k$ in L_p:

$$L_p = \sum_{k=1}^{p-1} \frac{p-k}{\sin(k\pi/p)} = pI_p - L_p.$$

Also,

$$\begin{aligned} M_p &= \sum_{\substack{1\le k<2p\\ k \text{ odd}}} k\cot\left(\frac{k\pi}{2p}\right) = \sum_{k=1}^{2p-1} k\cot\left(\frac{k\pi}{2p}\right) - \sum_{\substack{1\le k<2p\\ k \text{ even}}} k\cot\left(\frac{k\pi}{2p}\right) \\ &= \sum_{k=1}^{2p-1} k\cot\left(\frac{k\pi}{2p}\right) - 2\sum_{k=1}^{p-1} k\cot\left(\frac{k\pi}{p}\right) = J_{2p} - 2J_p. \end{aligned}$$

But

$$\begin{aligned} J_{2p} &= \sum_{k=1}^{p-1} k\cot\left(\frac{k\pi}{2p}\right) + \sum_{k=p+1}^{2p-1} k\cot\left(\frac{k\pi}{2p}\right) \\ &= \sum_{k=1}^{p-1} k\cot\left(\frac{k\pi}{2p}\right) - \sum_{k=1}^{p-1} (2p-k)\cot\left(\frac{k\pi}{2p}\right) \\ &= 2\sum_{k=1}^{p-1} k\cot\left(\frac{k\pi}{2p}\right) - 2p\widetilde{K}_p. \end{aligned}$$

Thus, using (i) and the trigonometric identity $\cot(\theta/2) - \cot\theta = \csc\theta$ we obtain

$$\begin{aligned} M_p = J_{2p} - 2J_p &= 2\sum_{k=1}^{p-1} k\left(\cot\left(\frac{k\pi}{2p}\right) - \cot\left(\frac{k\pi}{p}\right)\right) - 2pI_p \\ &= 2\sum_{k=1}^{p-1} k\csc\left(\frac{k\pi}{p}\right) - 2pI_p = 2L_p - 2pI_p = -pI_p. \end{aligned}$$

This concludes the proof of (iii). □

Proposition 3.2 *For $p \geq 2$, let I_p be the sum of cosecants defined by the* (1.1). *Then*

$$\begin{aligned} I_p &= -\frac{2\ln 2}{\pi} + \frac{2p}{\pi}E_p \\ &= -\frac{2\ln 2}{\pi} + \frac{2p}{\pi}\big(\ln p + \gamma - \ln(\pi/2)\big) + \frac{4p}{\pi}D_p, \end{aligned}$$

where D_p and E_p are defined by Eqs. (2.2) *and* (2.3), *respectively.*

Proof Indeed, our starting point will be the 'simple fraction' expansion ([14], Chapter 5, Section 2) of the cosecant function:

$$\frac{\pi}{\sin(\pi\alpha)} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\alpha - n} = \frac{1}{\alpha} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\alpha-n} + \frac{1}{\alpha+n}\right),$$

which is valid for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$. Using this formula with $\alpha = k/p$ for $k = 1, 2, \ldots, p-1$ and adding, we conclude that

$$\begin{aligned} \frac{\pi}{p}I_p &= \sum_{k=1}^{p-1}\frac{1}{k} + \sum_{n=1}^{\infty}(-1)^n\sum_{k=1}^{p-1}\left(\frac{1}{k-np} + \frac{1}{k+np}\right) \\ &= \sum_{k=1}^{p-1}\frac{1}{k} + \sum_{n=1}^{\infty}(-1)^n\left(-\sum_{j=p(n-1)+1}^{pn-1}\frac{1}{j} + \sum_{j=pn+1}^{p(n+1)-1}\frac{1}{j}\right), \end{aligned}$$

and this result can be expressed in terms of the harmonic numbers as follows:

$$\begin{aligned} \frac{\pi}{p}I_p &= H_{p-1} + \sum_{n=1}^{\infty}(-1)^n(-H_{pn-1} + H_{p(n-1)} + H_{p(n+1)-1} - H_{pn}) \\ &= H_{p-1} + \sum_{n=1}^{\infty}(-1)^n(H_{p(n+1)} - 2H_{pn} + H_{p(n-1)}) + \frac{1}{p}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= H_{p-1} + \sum_{n=1}^{\infty}(-1)^n(H_{p(n+1)} - 2H_{pn} + H_{p(n-1)}) + \frac{1}{p}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\right) \\ &= H_p + \sum_{n=1}^{\infty}(-1)^n(H_{p(n+1)} - 2H_{pn} + H_{p(n-1)}) - \frac{2}{p}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} \\ &= H_p - \frac{2\ln 2}{p} + \sum_{n=1}^{\infty}(-1)^n(H_{p(n+1)} - 2H_{pn} + H_{p(n-1)}). \end{aligned}$$

Thus

$$\begin{aligned}\frac{\pi}{p}I_p+\frac{2\ln 2}{p}&=H_p+\sum_{n=1}^{\infty}(-1)^n(H_{p(n+1)}-H_{pn})+\sum_{n=1}^{\infty}(-1)^n(H_{p(n-1)}-H_{pn})\\&=\sum_{n=0}^{\infty}(-1)^n(H_{p(n+1)}-H_{pn})+\sum_{n=1}^{\infty}(-1)^n(H_{p(n-1)}-H_{pn})\\&=E_p+E_p=2E_p,\end{aligned}$$

and the desired formula follows according to Lemma 2.4. □

Combining Proposition 3.2 and Theorem 2.3, we obtain the following.

Proposition 3.3 *For $p\geq 2$ and $m\geq 1$, we have*

$$\pi I_p=2p\ln p+2\bigl(\gamma-\ln(\pi/2)\bigr)p-\sum_{k=1}^{m-1}\frac{2b_{2k}\eta(2k)}{k\cdot p^{2k-1}}+(-1)^m\frac{2\eta(2m)}{m\cdot p^{2m-1}}\varepsilon'_{p,m}$$

with $0<\varepsilon'_{p,m}<|b_{2m}|$.

Using the well-known result ([8], [12], Formula 9.542):

$$\eta(2k)=\bigl(1-2^{1-2k}\bigr)\zeta(2k)=\frac{(2^{2k-1}-1)\pi^{2k}(-1)^{k-1}b_{2k}}{(2k)!},$$

and considering separately the cases m even and m odd we obtain the following result.

Theorem 3.4 *For every positive integer p and every nonnegative integer n, the sum of cosecants I_p defined by* (1.1) *satisfies the following inequalities*:

$$\begin{aligned}I_p&<\frac{2p}{\pi}\bigl(\ln p+\gamma-\ln(\pi/2)\bigr)+\sum_{k=1}^{2n}(-1)^k\frac{(2^{2k}-2)b_{2k}^2}{k\cdot(2k)!}\left(\frac{\pi}{p}\right)^{2k-1},\\I_p&>\frac{2p}{\pi}\bigl(\ln p+\gamma-\ln(\pi/2)\bigr)+\sum_{k=1}^{2n+1}(-1)^k\frac{(2^{2k}-2)b_{2k}^2}{k\cdot(2k)!}\left(\frac{\pi}{p}\right)^{2k-1}.\end{aligned}$$

As an example, for $n=0$ we obtain the following inequality, valid for every $p\geq 1$:

$$\frac{2p}{\pi}\bigl(\ln p+\gamma-\ln(\pi/2)\bigr)-\frac{\pi}{36p}<I_p<\frac{2p}{\pi}\bigl(\ln p+\gamma-\ln(\pi/2)\bigr).$$

This answers positively the open problem proposed in Section 7.4 of [1].

Remark 3.5 The asymptotic expansion of I_p was proposed as an exercise in [9], Exercise 13, p.460, and it was attributed to Waldvogel, but the result there is less precise than Theorem 3.4 because here we have inequalities valid in the whole range of p.

Now we turn our attention to the other trigonometric sum, J_p. The first step is to find an analogous result to Proposition 3.2 for the trigonometric sum J_p, is the next lemma, where an asymptotic expansion for J_p is proved but it has a harmonic number as an undesired term; later it will be removed.

Lemma 3.6 *For every positive integers p, there is a real number $\theta_p \in (0,1)$ such that*

$$\pi J_p = -p^2 H_p + \ln(2\pi)p^2 - \frac{p}{2} - \theta_p.$$

Proof Indeed, let φ be the function defined by

$$\varphi(x) = \pi x \cot(\pi x) + \frac{1}{1-x}.$$

According to the partial fraction expansion formula for the cotangent function ([14], Chapter 5, Section 2) we know that

$$\varphi(x) = 2 + \frac{x}{x+1} + \sum_{n=2}^{\infty}\left(\frac{x}{x-n} + \frac{x}{x+n}\right).$$

Thus, φ is defined and analytic on the interval $(-1,2)$. Let us show that φ is concave on this interval. Indeed, it is straightforward to check that, for $-1 < x < 2$, we have

$$\varphi''(x) = -\frac{2}{(1+x)^3} - 2\sum_{n=2}^{\infty}\left(\frac{n}{(n-x)^3} + \frac{n}{(n+x)^3}\right) < 0.$$

So, we can use Theorem 1.1 with $m = 1$ applied to the function $x \mapsto \varphi(\frac{x+k}{p})$ for $1 \le k < p$ to get

$$0 < p\int_{k/p}^{(k+1)/p} \varphi(x)\,dx - \frac{1}{2}\left(\varphi\left(\frac{k+1}{p}\right) + \varphi\left(\frac{k}{p}\right)\right) \le \frac{3}{2p\pi^2}\left(\varphi'\left(\frac{k}{p}\right) - \varphi'\left(\frac{k+1}{p}\right)\right).$$

Adding these inequalities and noting that $\varphi(0) = 2$, $\varphi'(0) = 1$, $\varphi(1) = 1$, and $\varphi'(1) = -\pi^2/3$, we get

$$0 < p\int_0^1 \varphi(x)\,dx - \frac{\pi}{p}J_p - pH_p - \frac{1}{2} \le \frac{3+\pi^2}{2\pi^2 p} < \frac{1}{p}.$$

Also, for $x \in [0,1)$, we have

$$\int_0^x \varphi(t)\,dt = -\ln(1-x) + x\ln\sin(\pi x) - \int_0^x \ln\sin(\pi t)\,dt,$$

and, letting x tend to 1 we obtain

$$\int_0^1 \varphi(t)\,dt = \ln(\pi) - \int_0^1 \ln\sin(\pi t)\,dt = \ln(2\pi),$$

where we used the fact $\int_0^1 \ln\sin(\pi t)\,dt = -\ln 2$ (see [8], 4.224 Formula 3). So, we have proved that

$$0 < p\ln(2\pi) - \frac{\pi}{p}J_p - pH_p - \frac{1}{2} < \frac{1}{p},$$

which is equivalent to the desired conclusion. □

The next proposition gives an analogous result to Proposition 3.2 for the trigonometric sum J_p.

Proposition 3.7 *For a positive integer p, let J_p be the sum of cotangents defined by* (1.2). *Then*

$$\pi J_p = -p^2 \ln p + \bigl(\ln(2\pi) - \gamma\bigr)p^2 - p + 2p^2 C_p,$$

where C_p is given by (2.1).

Proof Recall that $c_n = H_n - \ln n - \gamma - \frac{1}{2n}$ satisfies $c_n = \mathcal{O}(1/n^2)$. Thus, the two series

$$C_p = \sum_{n=1}^{\infty} c_{pn} \quad \text{and} \quad \widetilde{C}_p = \sum_{n=1}^{\infty} (-1)^{n-1} c_{pn}$$

are convergent. Further, we note that $\widetilde{C}_p = D_p - \frac{\ln 2}{2p}$ where D_p is defined by (2.2).

According to Proposition 3.2 we have

$$\widetilde{C}_p = \frac{\ln(\pi/2) - \gamma - \ln p}{2} + \frac{\pi}{4p} I_p. \tag{3.1}$$

Now, noting that

$$\begin{aligned} C_p &= \sum_{\substack{n\geq 1 \\ n \text{ odd}}} c_{pn} + \sum_{\substack{n\geq 1 \\ n \text{ even}}} c_{pn} = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} c_{pn} + \sum_{n=1}^{\infty} c_{2pn}, \\ \widetilde{C}_p &= \sum_{\substack{n\geq 1 \\ n \text{ odd}}} c_{pn} - \sum_{\substack{n\geq 1 \\ n \text{ even}}} c_{pn} = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} c_{pn} - \sum_{n=1}^{\infty} c_{2pn} \end{aligned}$$

we conclude that $C_p - \widetilde{C}_p = 2C_{2p}$, or equivalently

$$C_p - 2C_{2p} = \widetilde{C}_p. \tag{3.2}$$

On the other hand, for a positive integer p let us define F_p by

$$F_p = \frac{\ln p + \gamma - \ln(2\pi)}{2} + \frac{1}{2p} + \frac{\pi}{2p^2} J_p. \tag{3.3}$$

It is easy to check, using Lemma 3.1(iii), that

$$\begin{aligned} F_p - 2F_{2p} &= \frac{\ln(\pi/2) - \ln p - \gamma}{2} - \frac{\pi}{4p^2}(J_{2p} - 2J_p) \\ &= \frac{\ln(\pi/2) - \ln p - \gamma}{2} + \frac{\pi}{4p} I_p. \end{aligned} \tag{3.4}$$

We conclude from (3.2) and (3.4) that $C_p - 2C_{2p} = F_p - 2F_{2p}$, or equivalently

$$C_p - F_p = 2(C_{2p} - F_{2p}).$$

Hence,

$$\forall m \geq 1, \quad C_p - F_p = 2^m (C_{2^m p} - F_{2^m p}). \tag{3.5}$$

Now, using Lemma 2.1 to replace H_p in Lemma 3.6, we obtain

$$\begin{aligned}\frac{\pi}{p^2} J_p &= \ln(2\pi) - H_p - \frac{1}{2p} + \mathcal{O}\left(\frac{1}{p^2}\right) \\ &= \ln(2\pi) - \ln p - \gamma - \frac{1}{p} + \mathcal{O}\left(\frac{1}{p^2}\right).\end{aligned}$$

Thus $F_p = \mathcal{O}(\frac{1}{p^2})$. Similarly, from the fact that $c_n = \mathcal{O}(\frac{1}{n^2})$ we conclude also that $C_p = \mathcal{O}(\frac{1}{p^2})$. Consequently, there exists a constant κ such that, for large values of p, we have $|C_p - F_p| \leq \kappa/p^2$. So, from (3.5), we see that for large values of m we have

$$|C_p - F_p| \leq \frac{\kappa}{2^m p^2}$$

and letting m tend to $+\infty$ we obtain $C_p = F_p$, which is equivalent to the announced result. □

Combining Proposition 3.7 and Theorem 2.2, we obtain the following.

Proposition 3.8 *For $p \geq 2$ and $m \geq 1$, we have*

$$\pi J_p = -p^2 \ln p + \bigl(\ln(2\pi) - \gamma\bigr)p^2 - p - \sum_{k=1}^{m-1} \frac{b_{2k}\zeta(2k)}{k \cdot p^{2k-2}} + (-1)^m \frac{\zeta(2m)}{m \cdot p^{2m-2}} \varepsilon_{p,m},$$

with $0 < \varepsilon_{p,m} < |b_{2m}|$, where ζ is the well-known Riemann zeta function.

Using the values of the $\zeta(2k)$'s [8], Formula 9.542, and considering separately the cases m even and m odd, we obtain the next result.

Theorem 3.9 *For every positive integer p and every nonnegative integer n, the sum of cotangents J_p defined by* (1.2) *satisfies the following inequalities*:

$$\begin{aligned}J_p &< \frac{1}{\pi}\bigl(-p^2 \ln p + \bigl(\ln(2\pi) - \gamma\bigr)p^2 - p\bigr) + 2\pi \sum_{k=1}^{2n} (-1)^k \frac{b_{2k}^2}{k \cdot (2k)!} \left(\frac{2\pi}{p}\right)^{2k-2}, \\ J_p &> \frac{1}{\pi}\bigl(-p^2 \ln p + \bigl(\ln(2\pi) - \gamma\bigr)p^2 - p\bigr) + 2\pi \sum_{k=1}^{2n+1} (-1)^k \frac{b_{2k}^2}{k \cdot (2k)!} \left(\frac{2\pi}{p}\right)^{2k-2}.\end{aligned}$$

As an example, for $n = 0$ we obtain the following double inequality, which is valid for $p \geq 1$:

$$0 < \frac{1}{\pi}\bigl(-p^2 \ln p + \bigl(\ln(2\pi) - \gamma\bigr)p^2 - p\bigr) - J_p < \frac{\pi}{36}.$$

Remark 3.10 Note that we have proved the following results. For a positive integer p:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\bigl(H_{pn}-\ln(pn)-\gamma\bigr)=\frac{\ln(\pi/2)-\gamma-\ln p}{2}+\frac{\ln 2}{2p}+\frac{\pi}{4p}\sum_{k=1}^{p-1}\csc\left(\frac{k\pi}{p}\right),$$

$$\sum_{n=0}^{\infty}(-1)^{n}(H_{p(n+1)}-H_{pn})=\frac{\ln 2}{p}+\frac{\pi}{2p}\sum_{k=1}^{p-1}\csc\left(\frac{k\pi}{p}\right),$$

$$\sum_{n=1}^{\infty}\left(H_{pn}-\ln(pn)-\gamma-\frac{1}{2pn}\right)=\frac{\ln p+\gamma-\ln(2\pi)}{2}+\frac{1}{2p}+\frac{\pi}{2p^2}\sum_{k=1}^{p-1}k\cot\left(\frac{k\pi}{p}\right).$$

These results are to be compared with those in [15]; see also [16].

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