Bani-Ahmad and Bani-Domi *Journal of Inequalities and Applications* (2016) 2016:175
DOI 10.1186/s13660-016-1108-y

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

New norm equalities and inequalities for operator matrices

Feras Ali Bani-Ahmad[1] and Watheq Bani-Domi[2*]

*Correspondence:
watheq@yu.edu.jo
[2]Department of Mathematics, Yarmouk University, Irbed, Jordan
Full list of author information is available at the end of the article

Abstract

We prove new inequalities for general 2×2 operator matrices. These inequalities, which are based on classical convexity inequalities, generalize earlier inequalities for sums of operators. Some other related results are also presented. Also, we prove a numerical radius equality for a 5×5 tridiagonal operator matrix.

MSC: 47A12; 47A30; 47A63; 47B15; 47B36

Keywords: numerical radius; usual operator norm; operator matrix; tridiagonal; Jensen's inequality; mixed Schwarz inequality

1 Introduction

Let $\mathrm{LB}(H)$ denote the C^*-algebra of all linear bounded operators on a complex separable Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. For $M \in \mathrm{LB}(H)$, let $\omega(M) = \sup\{|\langle Mx,x\rangle| : x \in H \text{ and } \|x\| = 1\}$ and $\|M\| = \sup\{|\langle Mx,y\rangle| : x,y \in H \text{ and } \|x\| = \|y\| = 1\}$ denote the numerical radius and the usual operator norm of M, respectively. It is well known that $\omega(\cdot)$ defines a norm on $\mathrm{LB}(H)$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for every $M \in \mathrm{LB}(H)$,

$$\frac{1}{2}\|M\| \leq \omega(M) \leq \|M\|. \tag{1.1}$$

Here, the first inequality in (1.1) becomes an equality if $M^2 = 0$. Also, the second inequality becomes an equality if M is normal. The property of the numerical radius norm which is important is its weak unitary invariance, that is, for $M \in \mathrm{LB}(H)$,

$$\omega\left(UMU^*\right) = \omega(M) \tag{1.2}$$

for any unitary $U \in \mathrm{LB}(H)$.

Also

$$\omega\left(M^*\right) = \omega(M)$$

for all $M \in \mathrm{LB}(H)$.

For more basic properties of the numerical radius, see [1] and [2]. Kittaneh in [3] and [4] improved the inequalities in (1.1). It has been shown in [3] and [4], respectively, that if

$M \in \mathrm{LB}(H)$, then

$$\omega(M) \le \frac{1}{2}\left\| |M| + |M^*| \right\| \le \frac{1}{2}\left(\|M\| + \left\|M^2\right\|^{\frac{1}{2}}\right), \tag{1.3}$$

where $|M| = (M^*M)^{\frac{1}{2}}$ means the absolute value of M, and

$$\frac{1}{4}\left\|M^*M + MM^*\right\| \le \omega^2(M) \le \frac{1}{2}\left\|M^*M + MM^*\right\|. \tag{1.4}$$

In [5] El-Haddad and Kittaneh generalized some inequalities for powers of the usual operator norm and a related numerical radius for sum of two operators. It has been shown that if $M, N \in \mathrm{LB}(H)$, $0 < \alpha < 1$, and $r \ge 1$, then

$$\|M + N\|^r \le 2^{r-2}\left(\left\| |M|^{2r\alpha} + |N|^{2r\alpha} \right\| + \left\| \left|M^*\right|^{2r(1-\alpha)} + \left|N^*\right|^{2r(1-\alpha)} \right\|\right) \tag{1.5}$$

and

$$\omega^r(M + N) \le 2^{r-2}\left\| |M|^{2r\alpha} + \left|M^*\right|^{2r(1-\alpha)} + |N|^{2r\alpha} + \left|N^*\right|^{2r(1-\alpha)} \right\|. \tag{1.6}$$

In Section 2, we generalize the inequalities (1.5) and (1.6) using some operator inequalities and some classical inequalities for nonnegative real numbers. In Section 3, we establish a numerical radius equality for 5×5 tridiagonal operator matrices.

2 Generalization of inequalities (1.5) and (1.6) to general 2×2 operator matrices

In this section we generalize the inequalities (1.5) and (1.6). To prove our generalized theorem, we need several well-known lemmas. The first lemma is important and it has been proved by Kittaneh [6].

Lemma 2.1 *Let $M \in \mathrm{LB}(H)$ and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(s)g(s) = s$ for all $s \in [0, \infty)$. Then*

$$\left|\langle Mx, y\rangle\right| \le \left\|f\left(|M|\right)x\right\| \left\|g\left(\left|M^*\right|\right)y\right\| \quad \text{for all } x, y \in H.$$

The second lemma is a consequence of Jensen's inequality, concerning the convexity or the concavity of certain power functions. It is a special case of Schlömilch's inequality for weighted means of nonnegative real numbers (see, *e.g.*, [7], p.26).

Lemma 2.2 *For $a, b \ge 0$, $0 < \alpha < 1$, and $r \ne 0$, let $K_r(a, b, \alpha) = (\alpha a^r + (1 - \alpha)b^r)^{\frac{1}{r}}$ and let $K_0(a, b, \alpha) = a^\alpha b^{1-\alpha}$. Then*

$$K_r(a, b, \alpha) \le K_s(a, b, \alpha) \quad \text{for } 0 \le r \le s.$$

Now, from the spectral theorem for positive operators and Jensen's inequality, we give the third lemma [6].

Lemma 2.3 *Let $M \in \mathrm{LB}(H)$ be positive, and let $x \in H$ be any unit vector. Then*

(a) $\langle Mx, x\rangle^s \leq \langle M^s x, x\rangle$ *for* $s \geq 1$,
(b) $\langle M^s x, x\rangle \leq \langle Mx, x\rangle^s$ *for* $0 < s \leq 1$.

The fourth lemma is a simple consequence of Jensen's inequality, concerning the concavity of the function $h(x) = x^s$, $0 \leq s \leq 1$ on $x \in [0, \infty)$.

Lemma 2.4 *If* $x_1, x_2, \ldots, x_n$ *are nonnegative real numbers, then*

$$\left(x_1^s + x_2^s + \cdots + x_n^s\right) \leq n^{1-s}(x_1 + x_2 + \cdots + x_n)^s \quad \textit{for } 0 \leq s \leq 1.$$

The fifth and last lemma contains three parts. Part (i) was proved in [4], while (ii) and (iii) were proved in [8].

Lemma 2.5 *Let* $M, N \in \mathrm{LB}(H)$. *Then*

$$\text{(i)} \quad \omega\left(\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}\right) = \max\{\omega(M), \omega(N)\},$$

$$\text{(ii)} \quad \omega\left(\begin{bmatrix} M & N \\ N & M \end{bmatrix}\right) = \max\{\omega(M+N), \omega(M-N)\},$$

$$\text{(iii)} \quad \omega\left(\begin{bmatrix} M & N \\ -N & M \end{bmatrix}\right) = \max\{\omega(M+iN), \omega(M-iN)\}.$$

3 Results and discussion

3.1 Result for 2 × 2 operator matrices

In the following theorem we prove a generalization of the inequalities (1.5) and (1.6).

Theorem 3.1 *Let* $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{LB}(H \oplus H)$, *and let* f *and* g *be nonnegative functions on* $[0, \infty)$, *which are continuous and that satisfy the relation* $f(s)g(s) = s$ *for all* $s \in [0, \infty)$, *and* $r \geq 1$. *Then*
(a) $\|T\|^r \leq 2^{r-2}(\max\{\|\rho\|, \|\beta\|\} + \max\{\|\gamma\|, \|\delta\|\})$,
(b) $\omega^r(T) \leq 2^{r-2}\max\{\omega(\rho+\gamma), \omega(\beta+\delta)\}$,
where

$$\rho = f^{2r}(|A|) + f^{2r}(|C|), \qquad \beta = f^{2r}(|D|) + f^{2r}(|B|),$$
$$\gamma = g^{2r}(|A^*|) + g^{2r}(|B^*|), \qquad \delta = g^{2r}(|D^*|) + g^{2r}(|C^*|).$$

Proof (a) Let $X = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$ be any two unit vectors in $(H \oplus H)$. Then using the triangle inequality, Lemma 2.1, Lemma 2.2, Lemma 2.3(a), and Lemma 2.4, we have

$$\begin{aligned}
&\left|\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} X, Y \right\rangle\right| \\
&\quad = \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} X, Y \right\rangle + \left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} X, Y \right\rangle\right| \\
&\quad \leq \left|\left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} X, Y \right\rangle\right| + \left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} X, Y \right\rangle\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| f\left(\left|\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right|\right) X \right\| \left\| g\left(\left|\begin{bmatrix} A^* & 0 \\ 0 & D^* \end{bmatrix}\right|\right) Y \right\| \\
&\quad + \left\| f\left(\left|\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right|\right) X \right\| \left\| g\left(\left|\begin{bmatrix} 0 & C^* \\ B^* & 0 \end{bmatrix}\right|\right) Y \right\| \\
&= \left\langle f^2\left(\begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix}\right) X, X \right\rangle^{\frac{1}{2}} \left\langle g^2\left(\begin{bmatrix} |A^*| & 0 \\ 0 & |D^*| \end{bmatrix}\right) Y, Y \right\rangle^{\frac{1}{2}} \\
&\quad + \left\langle f^2\left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}\right) X, X \right\rangle^{\frac{1}{2}} \left\langle g^2\left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix}\right) Y, Y \right\rangle^{\frac{1}{2}} \\
&= \left\langle \begin{bmatrix} f^2(|A|) & 0 \\ 0 & f^2(|D|) \end{bmatrix} X, X \right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(|A^*|) & 0 \\ 0 & g^2(|D^*|) \end{bmatrix} Y, Y \right\rangle^{\frac{1}{2}} \\
&\quad + \left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} X, X \right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} Y, Y \right\rangle^{\frac{1}{2}} \\
&\leq 2^{\frac{-1}{r}} \left(\left\langle \begin{bmatrix} f^2(|A|) & 0 \\ 0 & f^2(|D|) \end{bmatrix} X, X \right\rangle^{r} + \left\langle \begin{bmatrix} g^2(|A^*|) & 0 \\ 0 & g^2(|D^*|) \end{bmatrix} Y, Y \right\rangle^{r} \right)^{\frac{1}{r}} \\
&\quad + 2^{\frac{-1}{r}} \left(\left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} X, X \right\rangle^{r} + \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} Y, Y \right\rangle^{r} \right)^{\frac{1}{r}} \\
&\leq 2^{\frac{-1}{r}} \left(\left\langle \begin{bmatrix} f^{2r}(|A|) & 0 \\ 0 & f^{2r}(|D|) \end{bmatrix} X, X \right\rangle + \left\langle \begin{bmatrix} g^{2r}(|A^*|) & 0 \\ 0 & g^{2r}(|D^*|) \end{bmatrix} Y, Y \right\rangle \right)^{\frac{1}{r}} \\
&\quad + 2^{\frac{-1}{r}} \left(\left\langle \begin{bmatrix} f^{2r}(|C|) & 0 \\ 0 & f^{2r}(|B|) \end{bmatrix} X, X \right\rangle + \left\langle \begin{bmatrix} g^{2r}(|B^*|) & 0 \\ 0 & g^{2r}(|C^*|) \end{bmatrix} Y, Y \right\rangle \right)^{\frac{1}{r}} \\
&\leq 2^{\frac{-1}{r}} \left(2^{1-\frac{1}{r}}\right) \left(\langle \Phi X, X \rangle + \langle \Psi Y, Y \rangle + \langle \Omega X, X \rangle + \langle \Theta Y, Y \rangle \right)^{\frac{1}{r}} \\
&= 2^{(1-\frac{2}{r})} \left(\left\langle (\Phi + \Omega) X, X \right\rangle + \left\langle (\Psi + \Theta) Y, Y \right\rangle \right)^{\frac{1}{r}},
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi &= \begin{bmatrix} f^{2r}(|A|) & 0 \\ 0 & f^{2r}(|D|) \end{bmatrix}, \\
\Psi &= \begin{bmatrix} g^{2r}(|A^*|) & 0 \\ 0 & g^{2r}(|D^*|) \end{bmatrix}, \\
\Omega &= \begin{bmatrix} f^{2r}(|C|) & 0 \\ 0 & f^{2r}(|B|) \end{bmatrix}, \\
\Theta &= \begin{bmatrix} g^{2r}(|B^*|) & 0 \\ 0 & g^{2r}(|C^*|) \end{bmatrix}.
\end{aligned}
$$

Thus,

$$
\left| \left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} X, Y \right\rangle \right|^{r} \leq 2^{r-2} \left(\left\langle (\Phi + \Omega) X, X \right\rangle + \left\langle (\Psi + \Theta) Y, Y \right\rangle \right)
$$

and so

$$\begin{aligned}
&\sup\bigl\{\bigl|\langle TX,Y\rangle\bigr|^r : X,Y\in(H\oplus H), \|X\|=\|Y\|=1\bigr\}\\
&\quad\le 2^{r-2}\sup\bigl(\bigl\langle(\Phi+\Omega)X,X\bigr\rangle+\bigl\langle(\Psi+\Theta)Y,Y\bigr\rangle\bigr)\\
&\quad\le 2^{r-2}\bigl(\sup\bigl(\bigl\langle(\Phi+\Omega)X,X\bigr\rangle\bigr)+\bigl(\sup\bigl\langle(\Psi+\Theta)Y,Y\bigr\rangle\bigr)\bigr).
\end{aligned}$$

Hence,

$$\left\|\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right\|^r \le 2^{r-2}\bigl(\max\bigl\{\|\rho\|,\|\beta\|\bigr\}+\max\bigl\{\|\gamma\|,\|\delta\|\bigr\}\bigr).$$

(b) The result follows from the proof of part (a) by letting $X=Y$. □

The above theorem includes several norm inequalities of numerical radius and the usual operator norm for operator matrices. Samples of inequalities are demonstrated in the following remarks.

Remark 3.2 Let $A=B=D$, $C=-A$, $f(t)=t^\alpha$, and $g(t)=t^{1-\alpha}$ with $\alpha\in[0,1]$ in part (b) of Theorem 3.1. Then by using Lemma 2.5(iii) we get the following:

$$\begin{aligned}
\omega^r\left(\begin{bmatrix} A & A\\ -A & A\end{bmatrix}\right) &= \bigl(\sqrt{2}\omega(A)\bigr)^r = 2^{\frac{r}{2}}\omega^r(A)\\
&\le 2^{r-1}\bigl\||A|^{2\alpha r}+\bigl|A^*\bigr|^{2(1-\alpha)r}\bigr\|,
\end{aligned}$$

and so

$$\omega^r(A)\le 2^{\frac{r}{2}-1}\bigl\||A|^{2\alpha r}+\bigl|A^*\bigr|^{2(1-\alpha)r}\bigr\|.$$

Remark 3.3 Let $f(t)=t^\alpha$ and $g(t)=t^{1-\alpha}$, $\alpha\in[0,1]$, in part (a) of Theorem 3.1. Then we get the following inequality:

$$\left\|\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right\|^r \le 2^{r-2}\bigl(\max\bigl\{\|\rho\|,\|\beta\|\bigr\}+\max\bigl\{\|\gamma\|,\|\delta\|\bigr\}\bigr),$$

where

$$\begin{aligned}
&\rho=|A|^{2\alpha r}+|C|^{2\alpha r}, \qquad \beta=|D|^{2\alpha r}+|B|^{2\alpha r},\\
&\gamma=\bigl|A^*\bigr|^{2(1-\alpha)r}+\bigl|B^*\bigr|^{2(1-\alpha)r}, \qquad \delta=\bigl|D^*\bigr|^{2(1-\alpha)r}+\bigl|C^*\bigr|^{2(1-\alpha)r}.
\end{aligned}$$

Remark 3.4 Let $A=D$, $B=C$, $f(t)=t^\alpha$, and $g(t)=t^{1-\alpha}$ with $\alpha\in[0,1]$ in Theorem 3.1. Then by using Lemma 2.5(ii) we get the inequalities (1.6) and (1.5)

$$\begin{aligned}
\omega^r(A+B)\le\omega^r\left(\begin{bmatrix} A & B\\ B & A\end{bmatrix}\right) &= \max\bigl\{\omega^r(A+B),\omega^r(A-B)\bigr\}\\
&\le 2^{r-2}\max\bigl(\bigl\||A|^{2\alpha r}+\bigl|A^*\bigr|^{2(1-\alpha)r}+|B|^{2\alpha r}+\bigl|B^*\bigr|^{2(1-\alpha)r}\bigr\|,
\end{aligned}$$

$$\begin{aligned}&\big\| |A|^{2\alpha r} + \big|A^*\big|^{2(1-\alpha)r} + |B|^{2\alpha r} + \big|B^*\big|^{2(1-\alpha)r} \big\|\big) \\ &= 2^{r-2} \big\| |A|^{2\alpha r} + \big|A^*\big|^{2(1-\alpha)r} + |B|^{2\alpha r} + \big|B^*\big|^{2(1-\alpha)r} \big\|.\end{aligned}$$

Now, in the last inequality letting $A = B$ we get the following inequality:

$$\omega^r(A) \le \frac{1}{2} \big\| |A|^{2\alpha r} + \big|A^*\big|^{2(1-\alpha)r} \big\|.$$

Also,

$$\begin{aligned}\|A+B\|^r &\le \left\| \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right\|^r = \max\big\{ \|A+B\|^r, \|A-B\|^r \big\} \\ &\le 2^{r-2}\big(\max\big(\big\| |A|^{2\alpha r} + |B|^{2\alpha r} \big\|, \big\| |A|^{2\alpha r} + |B|^{2\alpha r} \big\|\big)\big) \\ &\quad + 2^{r-2}\big(\max\big(\big\| \big|A^*\big|^{2(1-\alpha)r} + \big|B^*\big|^{2(1-\alpha)r} \big\|, \big\| \big|A^*\big|^{2(1-\alpha)r} + \big|B^*\big|^{2(1-\alpha)r} \big\|\big)\big) \\ &= 2^{r-2}\big(\big\| |A|^{2\alpha r} + |B|^{2\alpha r} \big\| + \big\| \big|A^*\big|^{2(1-\alpha)r} + \big|B^*\big|^{2(1-\alpha)r} \big\|\big).\end{aligned}$$

Remark 3.5 Let $r = 1$ in Theorem 3.1. Then we get

$$\omega(T) \le \frac{1}{2} \max\big\{ \|a\|, \|b\| \big\},$$

where

$$a = f^2\big(|A|\big) + f^2\big(|C|\big) + g^2\big(\big|A^*\big|\big) + g^2\big(\big|B^*\big|\big)$$

and

$$b = f^2\big(|D|\big) + f^2\big(|B|\big) + g^2\big(\big|D^*\big|\big) + g^2\big(\big|C^*\big|\big),$$

and this result is proved in Theorem 4 in [9].

Remark 3.6 Let $B = C = 0$, $f(t) = t^\alpha$, and $g(t) = t^{1-\alpha}$ with $\alpha \in [0,1]$ in part (b) of Theorem 3.1. Then by using Lemma 2.5(i) we get the following:

$$\begin{aligned}\omega^r(A) &\le \omega^r\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) = \max\big\{ \omega^r(A), \omega^r(D) \big\} \\ &\le 2^{r-2} \max\big(\omega\big(|A|^{2\alpha r} + \big|A^*\big|^{2(1-\alpha)r}\big), \omega\big(|D|^{2\alpha r} + \big|D^*\big|^{2(1-\alpha)r}\big)\big).\end{aligned}$$

Also, if $D = 0$, then

$$\begin{aligned}\omega^r(A) &\le 2^{r-2}\big(\omega\big(|A|^{2\alpha r} + \big|A^*\big|^{2(1-\alpha)r}\big)\big) \\ &= 2^{r-2} \big\| |A|^{2\alpha r} + \big|A^*\big|^{2(1-\alpha)r} \big\|.\end{aligned}$$

3.2 Numerical radius equality for 5 × 5 tridiagonal operator matrix

Here, we prove a numerical radius equality for a special 5×5 tridiagonal operator matrix and then we prove a more general numerical radius inequality for the general 5×5 tridiagonal operator matrix.

Theorem 3.7 *Let* $A, B \in \mathrm{LB}(H)$ *and*

$$T = \begin{bmatrix} A & B & 0 & 0 & 0 \\ B & A & B & 0 & 0 \\ 0 & B & A & B & 0 \\ 0 & 0 & B & A & B \\ 0 & 0 & 0 & B & A \end{bmatrix}$$

be a tridiagonal operator matrix in $\mathrm{LB}(H^5)$. *Then*

$$\omega(T) = \max\bigl\{\omega(A + \sqrt{3}B), \omega(A + B), \omega(A), \omega(A - B), \omega(A - \sqrt{3}B)\bigr\}.$$

Proof Let

$$U = \begin{bmatrix} \frac{1}{2}I & \frac{\sqrt{3}}{2}I & I & \frac{\sqrt{3}}{2}I & \frac{1}{2}I \\ \frac{\sqrt{3}}{2}I & \frac{\sqrt{3}}{2}I & 0 & -\frac{\sqrt{3}}{2}I & -\frac{\sqrt{3}}{2}I \\ I & 0 & -I & 0 & I \\ \frac{\sqrt{3}}{2}I & -\frac{\sqrt{3}}{2}I & 0 & \frac{\sqrt{3}}{2}I & -\frac{\sqrt{3}}{2}I \\ \frac{1}{2}I & -\frac{\sqrt{3}}{2}I & I & -\frac{\sqrt{3}}{2}I & \frac{1}{2}I \end{bmatrix}$$

be a partitioned operator matrix in $\mathrm{LB}(H^5)$, where I is the identity operator in $\mathrm{LB}(H)$. Then it is easy to show that U is a unitary operator in $\mathrm{LB}(H^5)$ and

$$UTU^* = \begin{bmatrix} A + \sqrt{3}B & 0 & 0 & 0 & 0 \\ 0 & A + B & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & A - B & 0 \\ 0 & 0 & 0 & 0 & A - \sqrt{3}B \end{bmatrix}.$$

Hence, from the invariance property of weakly unitarily invariant norms and Lemma 2.5(i), we have the desired result. □

Here, we give some special cases in the following remark.

Remark 3.8

(1) If $B = 0$, then

$$\omega\left(\begin{bmatrix} A & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & A & 0 \\ 0 & 0 & 0 & 0 & A \end{bmatrix}\right) = \omega(A).$$

(2) If $A = 0$, then

$$\omega\left(\begin{bmatrix} 0 & B & 0 & 0 & 0 \\ B & 0 & B & 0 & 0 \\ 0 & B & 0 & B & 0 \\ 0 & 0 & B & 0 & B \\ 0 & 0 & 0 & B & 0 \end{bmatrix}\right) = \sqrt{3}\omega(B).$$

(3) If $A = B$, then

$$\omega\left(\begin{bmatrix} A & A & 0 & 0 & 0 \\ A & A & A & 0 & 0 \\ 0 & A & A & A & 0 \\ 0 & 0 & A & A & A \\ 0 & 0 & 0 & A & A \end{bmatrix}\right) = (1 + \sqrt{3})\omega(A).$$

(4) If $B = iA$, then

$$\omega\left(\begin{bmatrix} A & iA & 0 & 0 & 0 \\ iA & A & iA & 0 & 0 \\ 0 & iA & A & iA & 0 \\ 0 & 0 & iA & A & iA \\ 0 & 0 & 0 & iA & A \end{bmatrix}\right) = 2\omega(A).$$

Now, the second result in this section is an inequality for a more general tridiagonal operator matrix than the previous one.

Theorem 3.9 *Let*

$$S = \begin{bmatrix} A & B & 0 & 0 & 0 \\ C & A & B & 0 & 0 \\ 0 & C & A & B & 0 \\ 0 & 0 & C & A & B \\ 0 & 0 & 0 & C & A \end{bmatrix}$$

be a partitioned operator in $\mathrm{LB}(H^5)$. *Then*

$$\omega(S) \geq \frac{1}{2}\max\bigl\{\omega\bigl(2A + \sqrt{3}(B + C)\bigr), \omega(2A + B + C), \omega(2A), \omega(2A - B - C), \omega\bigl(A - \sqrt{3}(B + C)\bigr)\bigr\}.$$

Proof It is easy to show that

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a unitary operator in $\mathrm{LB}(H^5)$ and

$$S + USU^* = \begin{bmatrix} 2A & B+C & 0 & 0 & 0 \\ B+C & 2A & B+C & 0 & 0 \\ 0 & B+C & 2A & B+C & 0 \\ 0 & 0 & B+C & 2A & B+C \\ 0 & 0 & 0 & B+C & 2A \end{bmatrix}.$$

Hence, from the invariance property of weakly unitarily invariant norms, we have

$$\omega\big(S + USU^*\big) = \max\big\{\omega\big(2A + \sqrt{3}(B+C)\big), \omega(2A + B + C), \omega(2A), \\ \omega(2A - B - C), \omega\big(A - \sqrt{3}(B+C)\big)\big\}.$$

Thus,

$$\omega(S) \geq \frac{1}{2}\max\big\{\omega\big(2A + \sqrt{3}(B+C)\big), \omega(2A + B + C), \omega(2A), \\ \omega(2A - B - C), \omega\big(A - \sqrt{3}(B+C)\big)\big\},$$

as required. □

4 Conclusion

New inequalities for general 2×2 operator matrices were derived. These inequalities, which are based on some classical convexity inequalities for the nonnegative real numbers, generalize earlier inequalities for sums of operators. Some other related results were also presented. Also, a numerical radius equality for a 5×5 tridiagonal operator matrix was given.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
FAB-A gave and proved some results in the paper. WB-D gave and proved other results and comments on the first author's results. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Hashemite University, Zarqa, Jordan. [2]Department of Mathematics, Yarmouk University, Irbed, Jordan.

Acknowledgements
The authors would like to thank the anonymous reviewers for their helpful and constructive comments. The author Watheq Bani-Domi would like to thank Yarmouk University for partial support.

Received: 28 December 2015 Accepted: 6 June 2016 Published online: 07 July 2016

References

1. Gustafson, KE, Rao, DKM: Numerical Range. Springer, New York (1997)
2. Halmos, PR: A Hilbert Space Problem Book, 2nd edn. Springer, New York (1982)
3. Kittaneh, F: A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. Stud. Math. **158**, 11-17 (2003)
4. Kittaneh, F: Numerical radius inequalities for Hilbert space operators. Stud. Math. **168**, 73-80 (2005)
5. El-Haddad, M, Kittaneh, F: Numerical radius inequalities for Hilbert space operators. II. Stud. Math. **182**, 133-140 (2007)
6. Kittaneh, F: Notes on some inequalities for Hilbert space operators. Publ. Res. Inst. Math. Sci. **24**, 283-293 (1988)
7. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1988)
8. Bani-Domi, W, Kittaneh, F: Norm equalities and inequalities for operator matrices. Linear Algebra Appl. **429**, 57-67 (2008)
9. Bani-Domi, W: Generalized numerical radius inequalities for 2 × 2 operator matrices. Ital. J. Pure Appl. Math. (to appear)