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L^p Hardy type inequality in the half-space on the H-type group

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Abstract
In the current work we studied Hardy type and L^p Hardy type inequalities in the half-space on the H-type group, where the Hardy inequality in the upper half-space $\mathbf{R}^n_+$ was proved by Tidblom in (J. Funct. Anal. 221:482-495, 2005).

Keywords: H-type group; Hardy type inequality; Green's function

1 Introduction

In recent years a lot of authors studied the Hardy inequalities (see [1–5]). They are the extensions of the original inequality by Hardy [6]. The Heisenberg group, denoted by $\mathbf{H}_n$, is also very popular in mathematics (see [7–11]). By $\mathbf{H}_{n,+} = \{(z,t) \in \mathbf{H}_n | z \in C^n, t > 0\}$ is denoted the half-space on the Heisenberg group. A Hardy type inequality on $\mathbf{H}_{n,+}$ in [4] is stated as follows. For $u \in C_0^\infty(\mathbf{H}_{n,+})$, we have

$$\int_{\mathbf{H}_{n,+}} |\nabla_{\mathbf{H}_n} u|^2 \,dz\,dt \geq \int_{\mathbf{H}_{n,+}} \frac{|z|^2}{t^2}|u|^2\,dz\,dt + \frac{(Q+2)(Q-2)}{4}\int_{\mathbf{H}_{n,+}} \rho^{-4}|z|^2|u|^2\,dz\,dt,$$

where $\rho = (|z|^4 + t^2)^{\frac{1}{4}}$ and $Q = 2n+2$ is the homogeneous dimension of the Heisenberg group. We know that the H-type group, denoted by $\mathbf{H} = \{(z,t) \in \mathbf{H} | z \in C^n, t \in \mathbf{R}^m\}$, is the nilpotent Lie group introduced by Kaplan (see [12]). We also know that $\mathbf{H}_n$ is a nilpotent Lie group with homogeneous dimension $2n+2$. The homogeneous dimension of $\mathbf{H}$ is $2n+2m$. Kaplan introduced the H-type group as a direct generalization of the Heisenberg group, which motivates us to study the H-type group.

In this paper we prove the Hardy type inequality in the half-space on the H-type group (see Theorem 2.1). The half-space on the H-type group is given by $\mathbf{H}^+ = \{(z,t) \in \mathbf{H} | t_m > 0\}$. For $u \in C_0^\infty(\mathbf{H}^+)$, we have

$$\begin{aligned}\int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} u|^2\,dz\,dt \geq{}& \frac{1}{16}\int_{\mathbf{H}^+} \frac{|z|^2}{t_m^2}|u|^2\,dz\,dt \\ &+ \frac{(Q-2)(Q+2)}{4}\int_{\mathbf{H}^+} d(z,t)^{-4}|z|^2|u|^2\,dz\,dt \\ &- (Q+2)\int_{\mathbf{H}^+} d(z,t)^{-4}\sum_{k=1}^{m-1}\langle U^{(k)}z, U^{(m)}z\rangle \frac{t_k|u|^2}{t_m}\,dz\,dt,\end{aligned}$$

where $d(z,t) = (|z|^4 + 16|t|^2)^{\frac{1}{4}}$ and $Q = 2n + 2m$ is the homogeneous dimension of the H-type group.

In [1], the L^p Hardy inequalities in the upper half-space $\mathbf{R}^n_+$ were studied. So we are also interested in the L^p Hardy type inequalities in the half-space on the H-type group.

In the remainder of this section we give a basic concept of H-type group and a useful theorem.

Let $(z,t),(z',t') \in \mathbf{H}$, $U^{(j)}$ is a $2n \times 2n$ skew-symmetric orthogonal matrix and $U^{(j)}$ satisfy $U^{(i)}U^{(j)} + U^{(j)}U^{(i)} = 0$, $i,j = 1,2,\dots,m$ with $i \neq j$. The group law is given by

$$(z,t)(z',t') = \left(z + z', t + t' + \frac{1}{2}\Im(zz')\right),$$

where $(\Im(zz'))_j = \langle z, U^{(j)}z'\rangle$, $\langle z, U^{(j)}z'\rangle$ is the inner product of z and $U^{(j)}z'$ on $\mathbf{R}^{2n}$.

The left invariant vector fields are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{m}\left(\sum_{i=1}^{2n} s_i U^{(k)}_{i,j}\right)\frac{\partial}{\partial t_k}, \quad j = 1,2,\dots,n,$$

$$Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}\sum_{k=1}^{m}\left(\sum_{i=1}^{2n} s_i U^{(k)}_{i,j+n}\right)\frac{\partial}{\partial t_k}, \quad j = 1,2,\dots,n,$$

$$T_k = \frac{\partial}{\partial t_k}, \quad k = 1,2,\dots,m,$$

where $s_i = x_i$ for $i = 1,2,\dots,n$ and $s_i = y_{i-n}$ for $i = n+1, n+2,\dots,2n$. The sub Laplacian $\mathcal{L}$ is defined by

$$\mathcal{L} = -\sum_{j=1}^{n}\left(X_j^2 + Y_j^2\right).$$

We write $\nabla_{\mathbf{H}} = (X_1,\dots,X_n,Y_1,\dots,Y_n)$ and

$$\operatorname{div}_{\mathbf{H}}(f_1,f_2,\dots,f_{2n}) = \sum_{j=1}^{n}(X_jf_j + Y_jf_{j+n}).$$

We define the Kohn Laplacian $\Delta_{\mathbf{H}}$ by

$$\Delta_{\mathbf{H}} = \sum_{j=1}^{n}\left(X_j^2 + Y_j^2\right). \tag{1}$$

On the Heisenberg group, a fundamental solution for the sub Laplacian was studied in [13]. Similarly, we give a fundamental solution for $\Delta_{\mathbf{H}}$ below. For $0 < r < \infty$ and $(z,t) \in \mathbf{H}$, we define $\delta_r(z,t) = (rz, r^2t)$.

Theorem 1.1 *A fundamental solution for* $\Delta_{\mathbf{H}}$ *with source at* 0 *is given by* $c_{n,m}d(z,t)^{-Q+2}$, *where*

$$c_{n,m}^{-1} = 4(n+m+1)(1-n-m)\int_{\mathbf{H}} |z|^2\left(d(z,t)^4 + 1\right)^{\frac{-n-m-3}{2}}\,dz\,dt.$$

For $u(z,t) \in C_0^\infty(\mathbf{H})$, *we have*

$$\langle \Delta_{\mathbf{H}} u(z,t), c_{n,m} d(z,t)^{-Q+2} \rangle_{L^2(\mathbf{H})} = u(0,0).$$

Proof For $\varepsilon > 0$, let $d_\varepsilon(z,t) = (d(z,t)^4 + \varepsilon^4)^{\frac{1}{4}}$, similar to [13], by equation (1) and a direct calculation, we have

$$\Delta_{\mathbf{H}} d_\varepsilon(z,t)^{-Q+2} = \varepsilon^{-Q} \phi\bigl(\delta_{\frac{1}{\varepsilon}}(z,t)\bigr), \tag{2}$$

where

$$\phi(z,t) = 4(n+m+1)(1-n-m)|z|^2 \bigl(d(z,t)^4 + 1\bigr)^{\frac{-n-m-3}{2}}.$$

From this, it follows that, for all $u(z,t) \in C_0^\infty(\mathbf{H})$,

$$\begin{aligned} \langle \Delta_{\mathbf{H}} u(z,t), c_{n,m} d(z,t)^{-Q+2} \rangle_{L^2(\mathbf{H})} &= \lim_{\varepsilon\to 0} \langle \Delta_{\mathbf{H}} u(z,t), c_{n,m} d_\varepsilon(z,t)^{-Q+2} \rangle_{L^2(\mathbf{H})} \\ &= \lim_{\varepsilon\to 0} \langle u(z,t), c_{n,m} \Delta_{\mathbf{H}} d_\varepsilon(z,t)^{-Q+2} \rangle_{L^2(\mathbf{H})} \\ &= u(0,0). \end{aligned}$$ □

For $\varepsilon > 0$, the Green's function on the half-space on the H-type group is given by

$$G(z,t,\varepsilon) = \frac{1}{(|z|^4 + 16\sum_{j=1}^{m-1} t_j^2 + 16(t_m - \varepsilon)^2)^{\frac{Q-2}{4}}} - \frac{1}{(|z|^4 + 16\sum_{j=1}^{m-1} t_j^2 + 16(t_m + \varepsilon)^2)^{\frac{Q-2}{4}}}.$$

2 Result

We give the main results of this paper in this section.

Theorem 2.1 *For* $u \in C_0^\infty(\mathbf{H}^+)$, *we have*

$$\begin{aligned} \int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} u|^2 \,dz\,dt \geq{}& \frac{1}{16} \int_{\mathbf{H}^+} \frac{|z|^2}{t_m^2} |u|^2 \,dz\,dt \\ &+ \frac{(Q-2)(Q+2)}{4} \int_{\mathbf{H}^+} d(z,t)^{-4} |z|^2 |u|^2 \,dz\,dt \\ &- (Q+2) \int_{\mathbf{H}^+} d(z,t)^{-4} \sum_{k=1}^{m-1} \langle U^{(k)} z, U^{(m)} z \rangle \frac{t_k |u|^2}{t_m} \,dz\,dt. \end{aligned}$$

The theorems below show us the L^p Hardy type inequalities in the half-space on the H-type group.

Theorem 2.2 *Let* $u \in C_0^\infty(\mathbf{H}^+)$ *and* $1 < p < \infty$, *then*

$$\begin{aligned} \int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} u|^p \,dz\,dt \geq{}& \left(\frac{p-1}{p}\right)^p \frac{p}{4} \int_{\mathbf{H}^+} \frac{|u|^p |z|^2}{t_m^p} \,dz\,dt \\ &- \left(\frac{p-1}{p}\right)^p \frac{p-1}{2^{\frac{p}{p-1}}} \int_{\mathbf{H}^+} \frac{1}{t_m^p} |z|^{\frac{p}{p-1}} |u|^p \,dz\,dt. \end{aligned} \tag{3}$$

Theorem 2.3 *Let $u \in C_0^\infty(\mathbf{H}^+)$ and $1 < p < \infty$, then*

$$\begin{aligned}\int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} u|^p \, dz\, dt \geq{}& \left(\frac{p-1}{p}\right)^p \frac{p}{4} \int_{\mathbf{H}^+} \frac{|u|^p |z|^2}{t_m^p} \, dz\, dt \\ &- \left(\frac{p-1}{p}\right)^p \frac{p-1}{2^{\frac{p}{p-1}}} \int_{\mathbf{H}^+} \frac{1}{t_m^p} \left|1 - t_m^{p-1}\right|^{\frac{p}{p-1}} |z|^{\frac{p}{p-1}} |u|^p \, dz\, dt. \end{aligned} \tag{4}$$

We also study the L^p Hardy type inequalities in the H-type group.

Theorem 2.4 *Let $u \in C_0^\infty(\mathbf{H})$ and $1 < p < \infty$, then*

$$\begin{aligned}\int_{\mathbf{H}} |\nabla_{\mathbf{H}} u|^p \, dz\, dt \geq{}& \left(\frac{p-1}{p}\right)^p p(Q-2)^2 \int_{\mathbf{H}} \frac{|u|^p |z|^2}{d^{(-Q+2)p+2Q}} \, dz\, dt \\ &- \left(\frac{p-1}{p}\right)^p (p-1)(Q-2)^{\frac{p}{p-1}} \int_{\mathbf{H}} \frac{|u|^p |z|^{\frac{p}{p-1}}}{d^{(-Q+2)p+\frac{p}{p-1}Q}} \, dz\, dt. \end{aligned} \tag{5}$$

Theorem 2.5 *Let $u \in C_0^\infty(\mathbf{H})$ and $1 < p < \infty$. Then*

$$\begin{aligned}&\int_{\mathbf{H}} |\nabla_{\mathbf{H}} u|^p \, dz\, dt \\ &\quad\geq \left(\frac{p-1}{p}\right)^{p-1} c_{n,m}^{-1} \left|u(0)\right|^p + \left(\frac{p-1}{p}\right)^p p(Q-2)^2 \int_{\mathbf{H}} \frac{|u|^p |z|^2}{d^{(-Q+2)p+2Q}} \, dz\, dt \\ &\qquad - \left(\frac{p-1}{p}\right)^p (p-1)(Q-2)^{\frac{p}{p-1}} \int_{\mathbf{H}} \frac{|u|^p |1 - d^{(-Q+2)(p-1)}|^{\frac{p}{p-1}} |z|^{\frac{p}{p-1}}}{d^{(-Q+2)p+\frac{p}{p-1}Q}} \, dz\, dt. \end{aligned} \tag{6}$$

3 Hardy type inequality

This section is to show the Hardy type inequality in $\mathbf{H}^+$.

Proof of Theorem 2.1 Let $v(z,t) = G(z,t,\varepsilon)^{-\frac{1}{2}} u(z,t)$. Write $t^\varepsilon = (0,\ldots,0,\varepsilon)$. We know that $G(0,t^\varepsilon,\varepsilon) = \infty$, so we have $v(0,t^\varepsilon) = 0$ and $u(z,t) = G(z,t,\varepsilon)^{\frac{1}{2}} v(z,t)$. Then we obtain

$$\nabla_{\mathbf{H}} u = \left(\frac{1}{2} \frac{\nabla_{\mathbf{H}} G}{G} + \frac{\nabla_{\mathbf{H}} v}{v}\right) u$$

and

$$\begin{aligned}&\int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} u|^2 \, dz\, dt \\ &\quad= \frac{1}{4} \int_{\mathbf{H}^+} \frac{|\nabla_{\mathbf{H}} G|^2}{G^2} |u|^2 \, dz\, dt + \int_{\mathbf{H}^+} \frac{\langle \nabla_{\mathbf{H}} G, \nabla_{\mathbf{H}} v\rangle}{Gv} |u|^2 \, dz\, dt + \int_{\mathbf{H}^+} \frac{|\nabla_{\mathbf{H}} v|^2}{v^2} |u|^2 \, dz\, dt \\ &\quad= \frac{1}{4} \int_{\mathbf{H}^+} \frac{|\nabla_{\mathbf{H}} G|^2}{G^2} |u|^2 \, dz\, dt + \int_{\mathbf{H}^+} v \langle \nabla_{\mathbf{H}} G, \nabla_{\mathbf{H}} v\rangle \, dz\, dt + \int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} v|^2 G \, dz\, dt \\ &\quad= \frac{1}{4} \int_{\mathbf{H}^+} \frac{|\nabla_{\mathbf{H}} G|^2}{G^2} |u|^2 \, dz\, dt + \frac{1}{2} \int_{\mathbf{H}^+} \langle \nabla_{\mathbf{H}} G, \nabla_{\mathbf{H}} v^2\rangle \, dz\, dt + \int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} v|^2 G \, dz\, dt \\ &\quad= \frac{1}{4} \int_{\mathbf{H}^+} \frac{|\nabla_{\mathbf{H}} G|^2}{G^2} |u|^2 \, dz\, dt + \frac{1}{2} c_{n,m}^{-1} v^2(0,t^\varepsilon) + \int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} v|^2 G \, dz\, dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\int_{\mathbf{H}^+} \frac{|\nabla_{\mathbf{H}} G|^2}{G^2}|u|^2\,dz\,dt + \int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} v|^2 G\,dz\,dt \\
&\geq \frac{1}{4}\int_{\mathbf{H}^+} \frac{|\nabla_{\mathbf{H}} G|^2}{G^2}|u|^2\,dz\,dt.
\end{aligned}
$$

Using L'Hospital's rule, we also have

$$
\lim_{\varepsilon\to 0^+} \frac{G(z,t,\varepsilon)}{\varepsilon} = 16(Q-2)t_m d(z,t)^{-Q-2}
$$

and

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+} \left|\frac{\nabla_{\mathbf{H}} G(z,t,\varepsilon)}{\varepsilon}\right|^2 = {}& \bigl(16(Q-2)\bigr)^2 \bigl(t_m^2 \bigl|\nabla_{\mathbf{H}} d(z,t)^{-Q-2}\bigr|^2 \\
&+ 2d(z,t)^{-Q-2} t_m \bigl\langle \nabla_{\mathbf{H}} d(z,t)^{-Q-2}, \nabla_{\mathbf{H}} t_m\bigr\rangle \\
&+ \bigl(d(z,t)^{-Q-2}\bigr)^2 |\nabla_{\mathbf{H}} t_m|^2\bigr).
\end{aligned}
$$

Because

$$
\nabla_{\mathbf{H}} t_m = \left(\frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,1}^{(m)}, \ldots, \frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,n}^{(m)}, \frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,1+n}^{(m)}, \ldots, \frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,2n}^{(m)}\right),
$$

from this we can see that

$$
\begin{aligned}
|\nabla_{\mathbf{H}} t_m| &= \left(\left(\frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,1}^{(m)}\right)^2 + \cdots + \left(\frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,n}^{(m)}\right)^2 \right. \\
&\quad \left. + \left(\frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,1+n}^{(m)}\right)^2 + \cdots + \left(\frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,2n}^{(m)}\right)^2\right)^{\frac{1}{2}} \\
&= \frac{1}{2}|z|.
\end{aligned}
$$

By a direct calculation, we get

$$
\bigl|\nabla_{\mathbf{H}} d(z,t)\bigr|^2 = \frac{|z|^2}{d(z,t)^2}.
$$

Thus we have

$$
\begin{aligned}
&\bigl|\nabla_{\mathbf{H}} d(z,t)^{-Q-2}\bigr|^2 = (Q+2)^2 d(z,t)^{-2Q-8}|z|^2, \\
&|\nabla_{\mathbf{H}} t_m|^2 = \frac{1}{4}|z|^2,
\end{aligned}
\tag{7}
$$

and

$$
\bigl\langle \nabla_{\mathbf{H}} d(z,t)^{-Q-2}, \nabla_{\mathbf{H}} t_m\bigr\rangle = 2(-Q-2)d(z,t)^{-Q-6}\left(\sum_{k=1}^{m} \bigl\langle U^{(k)}z, U^{(m)}z\bigr\rangle t_k\right).
$$

Consequently, we have

$$\begin{aligned}\lim_{\varepsilon\to0^+}\left|\frac{\nabla_{\mathbf H}G(z,t,\varepsilon)}{\varepsilon}\right|^2 &= \big(16(Q-2)\big)^2\bigg((Q+2)^2 d(z,t)^{-2Q-8}|z|^2t_m^2\\&\quad+4(-Q-2)d(z,t)^{-2Q-8}\big\langle U^{(m)}z,U^{(m)}z\big\rangle t_m^2+\frac14 d(z,t)^{-2Q-4}|z|^2\\&\quad+4(-Q-2)d(z,t)^{-2Q-8}\sum_{k=1}^{m-1}\big\langle U^{(k)}z,U^{(m)}z\big\rangle t_mt_k\bigg).\end{aligned}$$

This finishes the proof of the theorem. □

4 L^p Hardy type inequality

In this section, we are going to consider the L^p Hardy type inequalities in $\mathbf{H}^+$ and $\mathbf{H}$, respectively. Let Ω be a domain in $\mathbf{H}$. We write $\varrho(z,t)=\operatorname{dist}((z,t),\partial\Omega)$. Similar to [1], we have the lemma below.

Lemma 4.1 *Let $u\in C_0^\infty(\Omega)$, $l\in\{1,2,3,\dots\}$, $1<p<\infty$, $s\in(-\infty,lp-1)$, $F_j\in C^1(\Omega)$, $j=1,2,\dots,2n$, $F=(F_1,F_2,\dots,F_{2n})$ and $w\in C^1(\Omega)$ be a nonnegative weight function. We write $C(p,l,s)=(\frac{lp-s-1}{p})^p$, then we have*

$$\begin{aligned}\int_\Omega\frac{|\nabla_{\mathbf H}u|^pw}{\varrho^{(l-1)p-s}}\,dz\,dt &\ge C(p,l,s)\int_\Omega\frac{p|u|^p|\nabla_{\mathbf H}\varrho|^2w}{\varrho^{lp-s}}\,dz\,dt\\&\quad-C(p,l,s)\int_\Omega\frac{p|u|^p\Delta_{\mathbf H}\varrho w}{(lp-s-1)\varrho^{lp-s-1}}\,dz\,dt\\&\quad+C(p,l,s)\int_\Omega\frac{p\operatorname{div}_{\mathbf H}F|u|^pw}{lp-s-1}\,dz\,dt\\&\quad-C(p,l,s)\int_\Omega\frac{p-1}{\varrho^{lp-s}}\big|\nabla_{\mathbf H}\varrho-\varrho^{lp-s-1}F\big|^{\frac{p}{p-1}}|u|^pw\,dz\,dt\\&\quad+\left(\frac{lp-s-1}{p}\right)^{p-1}\int_\Omega\nabla_{\mathbf H}w\left(F-\frac{\nabla_{\mathbf H}\varrho}{\varrho^{lp-s-1}}\right)|u|^p\,dz\,dt.\end{aligned}\tag{8}$$

Proof Applying Hölder's inequality, we can deduce that

$$\begin{aligned}&p^p\int_\Omega\frac{|\nabla_{\mathbf H}u|^pw}{\varrho^{(l-1)p-s}}\,dz\,dt\left(\int_\Omega\left|\frac{\nabla_{\mathbf H}\varrho}{\varrho^{l(p-1)+\frac{s}{p}-s}}-\varrho^{l-1-\frac{s}{p}}F\right|^{\frac{p}{p-1}}|u|^pw\,dz\,dt\right)^{p-1}\\&\quad\ge p^p\left|\int_\Omega\left(\frac{\nabla_{\mathbf H}\varrho w}{\varrho^{lp-s-1}}-Fw\right)\big(\operatorname{sign}(u)|u|^{p-1}\big)\nabla_{\mathbf H}u\,dz\,dt\right|^p.\end{aligned}$$

On the other hand, by partial integration we get

$$\begin{aligned}&p^p\left|\int_\Omega\left(\frac{\nabla_{\mathbf H}\varrho w}{\varrho^{lp-s-1}}-Fw\right)\big(\operatorname{sign}(u)|u|^{p-1}\big)\nabla_{\mathbf H}u\,dz\,dt\right|^p\\&\quad=\left|\int_\Omega\left(\left(\frac{(lp-s-1)|\nabla_{\mathbf H}\varrho|^2}{\varrho^{lp-s}}-\frac{\Delta_{\mathbf H}\varrho}{\varrho^{lp-s-1}}+\operatorname{div}_{\mathbf H}F\right)w\right.\right.\\&\qquad\left.\left.+\nabla_{\mathbf H}w\left(F-\frac{\nabla_{\mathbf H}\varrho}{\varrho^{lp-s-1}}\right)\right)|u|^p\,dz\,dt\right|^p.\end{aligned}$$

Thus we obtain

$$\begin{aligned} &p^p \int_\Omega \frac{|\nabla_{\mathbf{H}} u|^p w}{\varrho^{(l-1)p-s}}\,dz\,dt \\ &\geq \left| \int_\Omega \left(\left(\frac{(lp-s-1)|\nabla_{\mathbf{H}}\varrho|^2}{\varrho^{lp-s}} - \frac{\Delta_{\mathbf{H}}\varrho}{\varrho^{lp-s-1}} + \operatorname{div}_{\mathbf{H}} F \right) w \right.\right. \\ &\quad \left.\left. + \nabla_{\mathbf{H}} w \left(F - \frac{\nabla_{\mathbf{H}}\varrho}{\varrho^{lp-s-1}} \right) \right) |u|^p\,dz\,dt \right|^p \\ &\quad \times \left(\int_\Omega \left| \frac{\nabla_{\mathbf{H}}\varrho}{\varrho^{l(p-1)+\frac{s}{p}-s}} - \varrho^{l-1-\frac{s}{p}} F \right|^{\frac{p}{p-1}} |u|^p w\,dz\,dt \right)^{-p+1}. \end{aligned}$$

It is clear that $\frac{|a|^p}{b^{p-1}} \geq pa - (p-1)b$ for $b > 0$. Then we have equation (8). □

For $F = 0$, we have

$$\begin{aligned} \int_\Omega \frac{|\nabla_{\mathbf{H}} u|^p w}{\varrho^{(l-1)p-s}}\,dz\,dt &\geq C(p,l,s) \int_\Omega \frac{p|u|^p |\nabla_{\mathbf{H}}\varrho|^2 w}{\varrho^{lp-s}}\,dz\,dt \\ &\quad - C(p,l,s) \int_\Omega \frac{p|u|^p \Delta_{\mathbf{H}}\varrho w}{(lp-s-1)\varrho^{lp-s-1}}\,dz\,dt \\ &\quad - C(p,l,s) \int_\Omega \frac{p-1}{\varrho^{lp-s}} |\nabla_{\mathbf{H}}\varrho|^{\frac{p}{p-1}} |u|^p w\,dz\,dt \\ &\quad - \left(\frac{lp-s-1}{p} \right)^{p-1} \int_\Omega \frac{\nabla_{\mathbf{H}} w \nabla_{\mathbf{H}}\varrho}{\varrho^{lp-s-1}} |u|^p\,dz\,dt. \end{aligned}$$

Now, let us discuss the L^p Hardy type inequalities in $\mathbf{H}^+$. Let $l = 1$, $s = 0$, and $w = 1$, we have by equation (8)

$$\begin{aligned} \int_\Omega |\nabla_{\mathbf{H}} u|^p\,dz\,dt &\geq \left(\frac{p-1}{p} \right)^p \int_\Omega \frac{p|u|^p |\nabla_{\mathbf{H}}\varrho|^2}{\varrho^p}\,dz\,dt \\ &\quad - \left(\frac{p-1}{p} \right)^p \int_\Omega \frac{p|u|^p \Delta_{\mathbf{H}}\varrho}{(p-1)\varrho^{p-1}}\,dz\,dt \\ &\quad + \left(\frac{p-1}{p} \right)^p \int_\Omega \frac{p \operatorname{div}_{\mathbf{H}} F |u|^p}{p-1}\,dz\,dt \\ &\quad - \left(\frac{p-1}{p} \right)^p \int_\Omega \frac{p-1}{\varrho^p} \left| \nabla_{\mathbf{H}}\varrho - \varrho^{p-1} F \right|^{\frac{p}{p-1}} |u|^p\,dz\,dt. \end{aligned} \tag{9}$$

For $\Omega = \mathbf{H}^+$, we have $\varrho = t_m$. So we get

$$\begin{aligned} \int_{\mathbf{H}^+} |\nabla_{\mathbf{H}} u|^p\,dz\,dt &\geq \left(\frac{p-1}{p} \right)^p \int_{\mathbf{H}^+} \frac{p|u|^p |\nabla_{\mathbf{H}} t_m|^2}{t_m^p}\,dz\,dt \\ &\quad - \left(\frac{p-1}{p} \right)^p \int_{\mathbf{H}^+} \frac{p|u|^p \Delta_{\mathbf{H}} t_m}{(p-1)t_m^{p-1}}\,dz\,dt \\ &\quad + \left(\frac{p-1}{p} \right)^p \int_{\mathbf{H}^+} \frac{p \operatorname{div}_{\mathbf{H}} F |u|^p}{p-1}\,dz\,dt \\ &\quad - \left(\frac{p-1}{p} \right)^p \int_{\mathbf{H}^+} \frac{p-1}{t_m^p} \left| \nabla_{\mathbf{H}} t_m - t_m^{p-1} F \right|^{\frac{p}{p-1}} |u|^p\,dz\,dt. \end{aligned} \tag{10}$$

Proof of Theorem 2.2 We know that

$$|\nabla_{\mathbf{H}} t_m| = \frac{1}{2}|z|$$

and

$$\Delta_{\mathbf{H}} t_m = 0.$$

Set $F = 0$, using equation (10), then we obtain equation (3). □

Proof of Theorem 2.3 Set $F = \nabla_{\mathbf{H}} t_m$. Since $U^{(m)}$ is a $2n \times 2n$ skew-symmetric orthogonal matrix, we have

$$\begin{aligned}\operatorname{div}_{\mathbf{H}} F &= \sum_{j=1}^{n} X_j \frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,j}^{(m)} + \sum_{j=1}^{n} Y_j \frac{1}{2}\sum_{i=1}^{2n} s_i U_{i,j+n}^{(m)} \\ &= \sum_{j=1}^{n} \frac{1}{2} U_{j,j}^{(m)} + \sum_{j=1}^{n} \frac{1}{2} U_{j+n,j+n}^{(m)} \\ &= 0.\end{aligned}$$

Using equation (10), we have equation (4). □

Now we are going to deal with the L^p Hardy type inequalities in $\mathbf{H}$.

Lemma 4.2 *Let* $u \in C_0^\infty(\mathbf{H})$, $l \in \{1,2,3,\ldots\}$, $1 < p < \infty$, $s \in (-\infty, lp-1)$, $F_j \in C^1(\mathbf{H})$, $j = 1,2,\ldots,2n$, $F = (F_1, F_2, \ldots, F_{2n})$ *and* $w \in C^1(\mathbf{H})$ *be a nonnegative weight function. Then we have*

$$\begin{aligned}&\int_{\mathbf{H}} \frac{|\nabla_{\mathbf{H}} u|^p w}{(d^{-Q+2})^{(l-1)p-s}}\,dz\,dt \\ &\quad\geq C(p,l,s)\int_{\mathbf{H}} \frac{p|u|^p|\nabla_{\mathbf{H}} d^{-Q+2}|^2 w}{(d^{-Q+2})^{lp-s}}\,dz\,dt + C(p,l,s)\int_{\mathbf{H}} \frac{p \operatorname{div}_{\mathbf{H}} F|u|^p w}{lp-s-1}\,dz\,dt \\ &\qquad - C(p,l,s)\int_{\mathbf{H}} \frac{p-1}{(d^{-Q+2})^{lp-s}}\big|\nabla_{\mathbf{H}} d^{-Q+2} - \big(d^{-Q+2}\big)^{lp-s-1}F\big|^{\frac{p}{p-1}}|u|^p w\,dz\,dt \\ &\qquad + \left(\frac{lp-s-1}{p}\right)^{p-1}\int_{\mathbf{H}} \nabla_{\mathbf{H}} w\left(F - \frac{\nabla_{\mathbf{H}} d^{-Q+2}}{(d^{-Q+2})^{lp-s-1}}\right)|u|^p\,dz\,dt,\end{aligned} \tag{11}$$

where $C(p,l,s) = (\frac{lp-s-1}{p})^p$.

Proof Similar to Lemma 4.1, we have

$$\begin{aligned}&\int_{\mathbf{H}} \frac{|\nabla_{\mathbf{H}} u|^p w}{(d^{-Q+2})^{(l-1)p-s}}\,dz\,dt \\ &\quad\geq C(p,l,s)\int_{\mathbf{H}} \frac{p|u|^p|\nabla_{\mathbf{H}} d^{-Q+2}|^2 w}{(d^{-Q+2})^{lp-s}}\,dz\,dt \\ &\qquad - C(p,l,s)\int_{\mathbf{H}} \frac{p|u|^p \Delta_{\mathbf{H}} d^{-Q+2} w}{(lp-s-1)(d^{-Q+2})^{lp-s-1}}\,dz\,dt\end{aligned}$$

$$
\begin{aligned}
&+ C(p,l,s)\int_{\mathbf{H}} \frac{p \operatorname{div}_{\mathbf{H}} F|u|^p w}{lp-s-1}\,dz\,dt \\
&- C(p,l,s)\int_{\mathbf{H}} \frac{p-1}{(d^{-Q+2})^{lp-s}}\left|\nabla_{\mathbf{H}} d^{-Q+2} - \left(d^{-Q+2}\right)^{lp-s-1}F\right|^{\frac{p}{p-1}}|u|^p w\,dz\,dt \\
&+ \left(\frac{lp-s-1}{p}\right)^{p-1}\int_{\mathbf{H}} \nabla_{\mathbf{H}} w\left(F - \frac{\nabla_{\mathbf{H}} d^{-Q+2}}{(d^{-Q+2})^{lp-s-1}}\right)|u|^p\,dz\,dt.
\end{aligned} \tag{12}
$$

We know that $c_{n,m}d(z,t)^{-Q+2}$ is a fundamental solution for $\Delta_{\mathbf{H}}$. So we have

$$
\int_{\mathbf{H}} \frac{|u|^p \Delta_{\mathbf{H}} d^{-Q+2} w}{(d^{-Q+2})^{lp-s-1}}\,dz\,dt = c_{n,m}^{-1}\left|u(0)\right|^p w(0) d(0)^{(Q-2)(lp-s-1)} = 0. \qquad \square
$$

For $l = 1$, $s = 0$, and $w = 1$, we have

$$
\begin{aligned}
&\int_{\mathbf{H}} |\nabla_{\mathbf{H}} u|^p\,dz\,dt \\
&\quad \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbf{H}} \frac{p|u|^p|\nabla_{\mathbf{H}} d^{-Q+2}|^2}{(d^{-Q+2})^p}\,dz\,dt \\
&\qquad + \left(\frac{p-1}{p}\right)^p \int_{\mathbf{H}} \frac{p \operatorname{div}_{\mathbf{H}} F|u|^p}{p-1}\,dz\,dt \\
&\qquad - \left(\frac{p-1}{p}\right)^p \int_{\mathbf{H}} \frac{p-1}{(d^{-Q+2})^p}\left|\nabla_{\mathbf{H}} d^{-Q+2} - \left(d^{-Q+2}\right)^{p-1}F\right|^{\frac{p}{p-1}}|u|^p\,dz\,dt.
\end{aligned} \tag{13}
$$

Set $F = 0$, then we get

$$
\begin{aligned}
\int_{\mathbf{H}} |\nabla_{\mathbf{H}} u|^p\,dz\,dt \geq{}& \left(\frac{p-1}{p}\right)^p \int_{\mathbf{H}} \frac{p|u|^p|\nabla_{\mathbf{H}} d^{-Q+2}|^2}{(d^{-Q+2})^p}\,dz\,dt \\
&- \left(\frac{p-1}{p}\right)^p \int_{\mathbf{H}} \frac{p-1}{(d^{-Q+2})^p}\left|\nabla_{\mathbf{H}} d^{-Q+2}\right|^{\frac{p}{p-1}}|u|^p\,dz\,dt.
\end{aligned} \tag{14}
$$

Proof of Theorem 2.4 It is obvious that

$$
|\nabla_{\mathbf{H}} d|^2 = \frac{|z|^2}{d^2}.
$$

So we have

$$
\left|\nabla_{\mathbf{H}} d^{-Q+2}\right|^2 = (Q-2)^2 d^{2(-Q+1)}\frac{|z|^2}{d^2}. \tag{15}
$$

From this together with (14), we get equation (5). $\square$

Proof of Theorem 2.5 Let $F = \nabla_{\mathbf{H}} d^{-Q+2}$. Then we have $\operatorname{div}_{\mathbf{H}} F = \operatorname{div}_{\mathbf{H}} \nabla_{\mathbf{H}} d^{-Q+2} = \Delta_{\mathbf{H}} d^{-Q+2}$. From equations (13) and (15), it follows that

$$
\begin{aligned}
&\int_{\mathbf{H}} |\nabla_{\mathbf{H}} u|^p\,dz\,dt \\
&\quad \geq \left(\frac{p-1}{p}\right)^p p(Q-2)^2 \int_{\mathbf{H}} \frac{|u|^p|z|^2}{d^{(-Q+2)p+2Q}}\,dz\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\frac{p-1}{p}\right)^p \int_{\mathbf{H}} \frac{p\Delta_{\mathbf{H}} d^{-Q+2}|u|^p}{p-1}\,dz\,dt \\
&- \left(\frac{p-1}{p}\right)^p (p-1)(Q-2)^{\frac{p}{p-1}} \int_{\mathbf{H}} \frac{|u|^p|1-d^{(-Q+2)(p-1)}|^{\frac{p}{p-1}}|z|^{\frac{p}{p-1}}}{d^{(-Q+2)p+\frac{p}{p-1}Q}}\,dz\,dt,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\int_{\mathbf{H}} |\nabla_{\mathbf{H}} u|^p\,dz\,dt \\
&\quad\geq \left(\frac{p-1}{p}\right)^{p-1} c_{n,m}^{-1}\bigl|u(0)\bigr|^p + \left(\frac{p-1}{p}\right)^p p(Q-2)^2 \int_{\mathbf{H}} \frac{|u|^p|z|^2}{d^{(-Q+2)p+2Q}}\,dz\,dt \\
&\quad\quad - \left(\frac{p-1}{p}\right)^p (p-1)(Q-2)^{\frac{p}{p-1}} \int_{\mathbf{H}} \frac{|u|^p|1-d^{(-Q+2)(p-1)}|^{\frac{p}{p-1}}|z|^{\frac{p}{p-1}}}{d^{(-Q+2)p+\frac{p}{p-1}Q}}\,dz\,dt.
\end{aligned}
$$

So we have equation (6). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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