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Some inequalities on meromorphic function and its derivative concerning small functions in an angular domain

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Abstract

In this paper, we investigate the value distribution of a meromorphic function and its derivative concerning small functions in an angular domain and obtain some inequalities for a meromorphic function in an angular domain, which improve the previous results.

MSC: 30D30

Keywords: meromorphic function; small function; angular domain

1 Introduction and main results

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}$ $(= \mathbb{C} \cup \{\infty\})$ to denote the extended complex plane, and Ω $(\subset \mathbb{C})$ to denote an angular domain. We will use the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions [1, 2].

The research on the value distribution of q meromorphic function is very active in the field of complex analysis; many mathematicians had done a lot of works in this project by using the Nevanlinna value distribution theory and obtained many famous results, such as the Picard theorem, Julia direction, Borel theorem, Borel direction, Hayman theorem, Yang-Zhang theorem, and so on (see [3–12]).

In 1964, Hayman [1] investigated the value distribution of a meromorphic function concerning its derivative in the complex plane and obtained the following well-known theorem,

Theorem 1.1 (Hayman inequality (see [1])) *Let f be a transcendental meromorphic function on the complex plane. Then, for any positive integer k, we have*

$$T(r,f) < \left(2+\frac{1}{k}\right)N\left(r,\frac{1}{f}\right)+\left(2+\frac{2}{k}\right)N\left(r,\frac{1}{f^{(k)}-1}\right)+S_1(r,f),$$

where $S_1(r,f)$ is the remainder term satisfying

(i) $S_1(r,f)=O(\log r)$ $(r\to\infty)$ *if the order of $f(z)$ is finite*;
(ii) $S_1(r,f)=O(\log(rT(r,f)))$ $(r\to\infty, r\notin E)$ *if the order of $f(z)$ is infinite, where E is a set of finite linear measure.*

In 1990, Yang [13] further investigated the above question and established the well-known Yang Lo inequality, in which the coefficients of the counting functions are more precise than those of the Hayman inequality.

Theorem 1.2 (see [13]) *Let f be a transcendental meromorphic function on the complex plane. Then, for any $\varepsilon > 0$ and positive integer k, we have*

$$T(r,f) < \left(1+\frac{1}{k}\right)N\left(r,\frac{1}{f}\right) + \left(1+\frac{1}{k}\right)N\left(r,\frac{1}{f^{(k)}-1}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + \varepsilon T(r,f) + S_1(r,f),$$

where $S_1(r,f)$ is as in Theorem 1.1.

Remark 1.1 From Theorem 1.1 we know that the characteristic function $T(r,f)$ is under control by only two counting functions, and without the counting function of the derivative function, we cannot obtain the better conclusion than the former one given in Theorem 1.1. Moreover, in contrast to the above two theorems, the coefficients of the counting functions in Theorem 1.1 are larger than those in Theorem 1.2.

Recently, there was also interest in studying the value distribution of a meromorphic function from the whole plane to an angular domain. For example, Zheng studied the uniqueness problem under the condition that five values and four values are shared in some angular domain in $\mathbb{C}$ around 2003 (see [14–16]); in 2012, Long and Wu [17] studied the uniqueness of meromorphic functions of infinite order sharing some values in the Borel direction; in the same year, Zhang *et al.* [18] further investigated the uniqueness of meromorphic functions sharing some values in the Borel direction and improved the results of Long and Wu; in 2013, Zhang [19] also studied the problems of Borel directions of meromorphic functions concerning shared values and obtained that if two meromorphic functions of infinite order share three distinct values, then their Borel directions are same; later, Xu *et al.* [20] and Xu and Yi [21] investigated the exceptional values in its Borel direction concerning multiple values and its derivative, and so on. In the discussion of the above topics, we find that the characteristics of meromorphic functions in the angular domain played an important role (see [13–15, 22, 23]). So, we first introduce the characteristics of meromorphic functions in the angular domain.

Let f be a meromorphic function on the angular domain $\Omega(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leq 2\pi$. Define

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right| + \log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f) = \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+\left|f\left(re^{i\theta}\right)\right| \sin\omega(\theta-\alpha)\,d\theta,$$

$$C_{\alpha,\beta}(r,f) = 2\sum_{1<|b_\mu|<r}\left(\frac{1}{|b_\mu|^\omega} - \frac{|b_\mu|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_\mu - \alpha),$$

$$S_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f) = D_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f),$$

where $\omega = \frac{\pi}{\beta-\alpha}$ and $b_\mu = |b_\mu|e^{i\theta_\mu}$ ($\mu = 1,2,\ldots$) are the poles of f on $\Omega(\alpha,\beta)$ counted according to their multiplicities. $S_{\alpha,\beta}(r,f)$ is called the Nevanlinna angular characteristic,

$C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\Omega(\alpha,\beta)$, and $\overline{C}_{\alpha,\beta}(r,f)$ is the reduced function of $C_{\alpha,\beta}(r,f)$. Similarly, when $a \neq \infty$, we will use the notations $A_{\alpha,\beta}(r,\frac{1}{f-a})$, $B_{\alpha,\beta}(r,\frac{1}{f-a})$, $C_{\alpha,\beta}(r,\frac{1}{f-a})$, $S_{\alpha,\beta}(r,\frac{1}{f-a})$, and so on.

In 1990, Yang [24] extended Theorem 1.1 to an angular domain and obtained the following result.

Theorem 1.3 (see [24]) *Let f be a transcendental meromorphic function on the complex plane, and $\overline{\Omega}(\alpha,\beta)$ be an angular domain. Then, for any positive integer k, we have*

$$S_{\alpha,\beta}(r,f) \leq \left(2+\frac{1}{k}\right)C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{k}\right)\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-1}\right) + Q_{\alpha,\beta}(r,f),$$

where $Q_{\alpha,\beta}(r,f) = (2+\frac{2}{k})D_{\alpha,\beta}(r,\frac{f^{(k+1)}}{f^{(k)}-1}) + (2+\frac{1}{k})[D_{\alpha,\beta}(r,\frac{f^{(k+1)}}{f^{(k)}}) + D_{\alpha,\beta}(r,\frac{f^{(k)}}{f})] + O(1)$.

In 2010, Yi *et al.* [25] extended Theorem 1.2 to an angular domain and obtained the following result.

Theorem 1.4 (see [25], Theorem 1.7) *Let f be a transcendental meromorphic function on the complex plane, and $\Omega(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}$ be an angular domain with $0 < \beta - \alpha \leq 2\pi$. Then, for any $\varepsilon > 0$ and positive integer k, we have*

$$\begin{aligned} S_{\alpha,\beta}(r,f) &< \left(1+\frac{1}{k}\right)C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \left(1+\frac{1}{k}\right)C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-1}\right) \\ &\quad - C_{\alpha,\beta}\left(r,\frac{1}{f^{(k+1)}}\right) + \varepsilon S_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f). \end{aligned}$$

Throughout, we use $R_{\alpha,\beta}(r,*)$ to denote the quantity satisfying

$$R_{\alpha,\beta}(r,*) = O\bigl(\log\bigl(rT(r,*)\bigr)\bigr), \quad r \notin E,$$

where E is a set of finite linear measure.

Furthermore, when a, b are two finite complex number, $a \neq b$ and $b \neq 0$, and f satisfies

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r,f)}{\log(rT(r,f))} = \infty \quad (r \notin E), \tag{1}$$

we have

$$\delta_{\alpha,\beta}(a,f) + \delta^{k}_{\alpha,\beta}\bigl(b,f^{(k)}\bigr) \leq \frac{k+2}{k+1}.$$

To state our main results, we require the following denotation.

Let $f(z)$ be meromorphic function in an angular domain $\Omega(\alpha,\beta) := \{z : \alpha \leq \arg z \leq \beta\}$ with $\alpha < \beta, \beta - \alpha < 2\pi$. We denote by $\ell(f)$ the set of meromorphic functions φ satisfying $\limsup_{r\to+\infty} \frac{S_{\alpha,\beta}(r,\varphi)}{\log(rT(r,f))} = 0$, that is,

$$\ell(f) := \left\{\varphi : \limsup_{r\to+\infty} \frac{S_{\alpha,\beta}(r,\varphi)}{\log(rT(r,f))} = 0\right\}. \tag{2}$$

In this paper, we further investigate the value distribution of a meromorphic function and its derivative in an angular domain concerning multiple values and small functions and obtain some inequalities of meromorphic functions in an angular domain as follows.

Theorem 1.5 *Let f be a transcendental meromorphic function on the complex plane $\mathbb{C}$, $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$ be an angular domain with $0<\beta-\alpha\leq 2\pi$, and let $\alpha_0(z),\alpha_1(z)\in\ell(f)$ satisfy $\alpha_0(z)\not\equiv 1$, $\alpha_1(z)\not\equiv 0$. Set*

$$\psi(z)=\alpha_0(z)f(z)+\alpha_1(z)f'(z).$$

Then

$$S_{\alpha,\beta}(r,f)<4\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)+3\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\psi-1}\right)+R_{\alpha,\beta}(r,f). \tag{3}$$

By applying Theorem 1.5 we can get the following results.

Theorem 1.6 *Let f be a transcendental meromorphic function on the complex plane, $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$ be an angular domain with $0<\beta-\alpha\leq 2\pi$, and let $\varphi_i(z)\in\ell(f)$, $i=1,2,3$, satisfy $\varphi_1(z)\not\equiv\varphi_2(z)$, $\varphi_1'(z)\not\equiv\varphi_2(z)$, and $\varphi_2'(z)\not\equiv\varphi_3(z)$. Then*

$$S_{\alpha,\beta}(r,f)<4\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)+3\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_2}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_3}\right)+R_{\alpha,\beta}(r,f). \tag{4}$$

Furthermore, if f satisfies (1), *then*

$$4\Theta_{\alpha,\beta}(\varphi_1,f)+3\Theta_{\alpha,\beta}(\varphi_2,f)+\Theta^1_{\alpha,\beta}(\varphi_3,f')<7, \tag{5}$$

where

$$\Theta_{\alpha,\beta}(\varphi,f)=1-\limsup_{r\to+\infty}\frac{\overline{C}_{\alpha,\beta}(r,\frac{1}{f-\varphi})}{S_{\alpha,\beta}(r,f)}\quad\text{and}\quad\Theta^k_{\alpha,\beta}(\varphi,f^{(k)})=1-\limsup_{r\to+\infty}\frac{\overline{C}_{\alpha,\beta}(r,\frac{1}{f^{(k)}-\varphi})}{S_{\alpha,\beta}(r,f)}$$

for $k\in\mathbb{N}_+$ and $\varphi\in\ell(f)$.

Remark 1.2 If f satisfies (1) and $\varphi\in\ell(f)$, then we have $\limsup_{r\to+\infty}\frac{S_{\alpha,\beta}(r,\varphi)}{S_{\alpha,\beta}(r,f)}=0$. Thus, we can say that φ is a 'small function' of a meromorphic function f in an angular domain.

Theorem 1.7 *Let f be a transcendental meromorphic function on the complex plane, $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$ be an angular domain with $0<\beta-\alpha\leq 2\pi$, and let $\varphi_i(z)\in\ell(f)$, $i=1,2,3$, satisfy that $\varphi_1'(z)$, $\varphi_2(z)$, $\varphi_3(z)$ are different from one another. Then*

$$S_{\alpha,\beta}(r,f)<\overline{\overline{C}}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_2}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_3}\right)+R_{\alpha,\beta}(r,f), \tag{6}$$

where

$$\overline{\overline{C}}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)=2\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)-\overline{C}^{1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right).$$

Theorem 1.8 *Let f be a transcendental meromorphic function on the complex plane, $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$ be an angular domain with $0<\beta-\alpha\le 2\pi$, and let $\varphi_i(z)\in\ell(f)$, $i=1,2$, satisfy $\varphi_1'(z)\not\equiv\varphi_2(z)$. Then*

$$S_{\alpha,\beta}(r,f)<\overline{\overline{C}}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_2}\right)+\overline{C}_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f). \tag{7}$$

By applying inequalities (4), (6), and (7) we get the following corollaries.

Corollary 1.1 *Under the conditions of Theorem* 1.5, *let $k_j\in\mathbb{N}_+\cup\{\infty\}$, $j=1,2,3$, satisfy*

$$\Delta_1:=\frac{4}{k_1}+\frac{3}{k_2}+\frac{2}{k_3}<1. \tag{8}$$

Then

$$\begin{aligned}(1-\Delta_1)S_{\alpha,\beta}(r,f)<{}&4\left(1-\frac{1}{k_1}\right)\overline{C}^{(k_1-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)+3\left(1-\frac{1}{k_2}\right)\overline{C}^{(k_2-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_2}\right)\\&+\left(1-\frac{1}{k_3}\right)\overline{C}^{(k_3-1)}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_3}\right)+R_{\alpha,\beta}(r,f).\end{aligned} \tag{9}$$

Corollary 1.2 *Under the conditions of Theorem* 1.6, *let $k_j\in\mathbb{N}_+\cup\{\infty\}$, $j=1,2,3$, satisfy*

$$\Delta_2:=2\sum_{j=1}^{3}\frac{1}{k_j}<1. \tag{10}$$

Then

$$\begin{aligned}(1-\Delta_2)S_{\alpha,\beta}(r,f)<{}&\left(1-\frac{2}{k_1}\right)\overline{\overline{C}}^{(k_1-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)+\left(1-\frac{1}{k_2}\right)\overline{C}^{(k_2-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_2}\right)\\&+\left(1-\frac{1}{k_3}\right)\overline{C}^{(k_3-1)}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_3}\right)+R_{\alpha,\beta}(r,f).\end{aligned} \tag{11}$$

Corollary 1.3 *Under the conditions of Theorem* 1.7, *let $k_j\in\mathbb{N}_+\cup\{\infty\}$, $j=1,2,3$, satisfy*

$$\Delta_3:=\frac{2}{k_1}+\frac{2}{k_2}+\left(1+\frac{1}{k_2}\right)\frac{1}{k_3}<1. \tag{12}$$

Then

$$\begin{aligned}(1-\Delta_3)S_{\alpha,\beta}(r,f)<{}&\left(1-\frac{2}{k_1}\right)\overline{C}^{(k_1-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)+\left(1-\frac{2}{k_2}\right)\overline{C}^{(k_2-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_2}\right)\\&+\left(1+\frac{1}{k_2}\right)\left(1-\frac{1}{k_3}\right)\overline{C}^{(k_3-1)}_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f).\end{aligned} \tag{13}$$

2 Some lemmas

To prove our results, we need the following lemmas.

Lemma 2.1 (see [22]) *Let f be a nonconstant meromorphic function on $\Omega(\alpha,\beta)$. Then, for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=S_{\alpha,\beta}(r,f)+\varepsilon(r,a),$$

where $\varepsilon(r,a)=O(1)$ as $r\to+\infty$.

Lemma 2.2 (see [22], p.138) *Let f be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. Given an angular domain on $\Omega(\alpha,\beta)$, for any $1\le r<R$, we have*

$$A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\le K\left\{\left(\frac{R}{r}\right)^{\omega}\int_1^R\frac{\log^+ T(r,f)}{t^{1+\omega}}\,dt+\log^+\frac{r}{R-r}+\log\frac{R}{r}+1\right\}$$

and

$$B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\le\frac{4\omega}{r^{\omega}}m\left(r,\frac{f'}{f}\right),$$

where $\omega=\frac{\pi}{\beta-\alpha}$, and K is a positive constant not depending on r and R.

Remark 2.1 Nevanlinna conjectured that

$$D_{\alpha,\beta}\left(r,\frac{f'}{f}\right):=A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=o\bigl(S_{\alpha,\beta}(r,f)\bigr)\tag{14}$$

as r tends to $+\infty$ outside an exceptional set of finite linear measure, and he proved that $A_{\alpha,\beta}(r,\frac{f'}{f})+B_{\alpha,\beta}(r,\frac{f'}{f})=O(1)$ when the function f is meromorphic in $\mathbb{C}$ and has finite order. In 1974, Gol'dberg [26] constructed a counterexample to show that (14) is not valid.

Remark 2.2 From [15, 22, 26] we get the following conclusion:

$$D_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=R_{\alpha,\beta}(r,f)=\begin{cases}O(1), & f \text{ is of finite order},\\ O(\log(rT(r,f))),\quad r\notin E, & f \text{ is of infinite order},\end{cases}$$

where $R_{\alpha,\beta}(r,f)$ is as in Theorem 1.4, and E is a set of finite linear measure.

Remark 2.3 From the definition of $A_{\alpha,\beta}(r,f)$, $B_{\alpha,\beta}(r,f)$, $C_{\alpha,\beta}(r,f)$, $S_{\alpha,\beta}(r,f)$ and Lemmas 2.1-2.2 we can see that the properties of $C_{\alpha,\beta}(r,f)$, $(A+B)_{\alpha,\beta}(r,f)$, and $S_{\alpha,\beta}(r,f)$ are the same as for the more familiar quantities $N(r,f)$, $m(r,f)$, and $T(r,f)$, respectively.

Lemma 2.3 (see [27]) *Let $f_1(z), f_2(z)$ be two meromorphic functions in the whole plane $\mathbb{C}$, and $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$ be an angular domain with $0<\beta-\alpha\le 2\pi$. Then*

$$C_{\alpha,\beta}(r,f_1f_2)-C_{\alpha,\beta}\left(r,\frac{1}{f_1f_2}\right)=C_{\alpha,\beta}(r,f_1)+C_{\alpha,\beta}(r,f_2)-C_{\alpha,\beta}\left(r,\frac{1}{f_1}\right)-C_{\alpha,\beta}\left(r,\frac{1}{f_2}\right)+O(1).$$

Lemma 2.4 *Let $f(z)$ be meromorphic function in the whole plane $\mathbb{C}$ satisfying $f(0)\neq 0,1,\infty$, $f'(0)\neq 0$, and $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$ be an angular domain with $0<\beta-\alpha\le$*

2π. *Then*

$$S_{\alpha,\beta}(r,f) \leq \overline{C}_{\alpha,\beta}(r,f) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) + R_{\alpha,\beta}(r,f). \tag{15}$$

Proof Since $\frac{1}{f} = 1 - \frac{f'}{f}\cdot\frac{f-1}{f'}$, we have

$$D_{\alpha,\beta}\left(r,\frac{1}{f}\right) \leq D_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + D_{\alpha,\beta}\left(r,\frac{f-1}{f'}\right) + O(1). \tag{16}$$

From Lemmas 2.1 and 2.3 we have

$$D_{\alpha,\beta}\left(r,\frac{1}{f}\right) = S_{\alpha,\beta}\left(r,\frac{1}{f}\right) - C_{\alpha,\beta}\left(r,\frac{1}{f}\right) - S_{\alpha,\beta}(r,f) - C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + O(1) \tag{17}$$

and

$$\begin{aligned} D_{\alpha,\beta}\left(r,\frac{f-1}{f'}\right) &= D_{\alpha,\beta}\left(r,\frac{f'}{f-1}\right) + C_{\alpha,\beta}\left(r,\frac{f'}{f-1}\right) - C_{\alpha,\beta}\left(r,\frac{f-1}{f'}\right) + O(1) \\ &= D_{\alpha,\beta}\left(r,\frac{f-1}{f'}\right) + C_{\alpha,\beta}(r,f') + C_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) \\ &\quad - C_{\alpha,\beta}(r,f) - C_{\alpha,\beta}\left(r,\frac{1}{f'}\right) + O(1). \end{aligned} \tag{18}$$

Then, from (16)-(18) we have

$$\begin{aligned} S_{\alpha,\beta}(r,f) &\leq C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + C_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) + C_{\alpha,\beta}(r,f') - C_{\alpha,\beta}(r,f) \\ &\quad - C_{\alpha,\beta}\left(r,\frac{1}{f'}\right) + D_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + D_{\alpha,\beta}\left(r,\frac{f'}{f-1}\right) + O(1) \\ &\leq C_{\alpha,\beta}(r,f) + C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + C_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) - C^0_{\alpha,\beta}(r) \\ &\quad + D_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + D_{\alpha,\beta}\left(r,\frac{f'}{f-1}\right) + O(1). \end{aligned} \tag{19}$$

Hence, it follows from (19) and Lemma 2.2 that

$$S_{\alpha,\beta}(r,f) \leq C_{\alpha,\beta}(r,f) + C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + C_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) - C^0_{\alpha,\beta}(r) + R_{\alpha,\beta}(r,f), \tag{20}$$

where $C^0_{\alpha,\beta}(r) = 2C_{\alpha,\beta}(r,f) - C_{\alpha,\beta}(r,f') + C_{\alpha,\beta}(r,\frac{1}{f'})$.

Now, we will estimate $C^0_{\alpha,\beta}(r)$. If z_0 is a pole of f of order k in Ω, then z_0 is a pole of f of order $k+1$ in Ω; if z_0 is a zero of f of order k, then z_0 is a zero of f' of order $k-1$ in Ω. Thus, we have

$$\begin{aligned} &C_{\alpha,\beta}(r,f) + C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + C_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) - C^0_{\alpha,\beta}(r) \\ &\quad \leq \overline{C}_{\alpha,\beta}(r,f) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right). \end{aligned}$$

From this inequality and (20) we easily get (15). □

Lemma 2.5 *For all $a, b \in \mathbb{C}$, we have*

$$\left|a^2 - ab\right| \geq \frac{1}{2}\left(|a|^2 - |b|^2\right).$$

Proof If $|a| \leq |b|$, then the inequality is obvious.

If $|a| > |b|$, then we have

$$\left|a^2 - ab\right| \geq |a|^2 - |ab| \geq |a|^2 - \frac{1}{2}\left(|a|^2 + |b|^2\right) = \frac{1}{2}\left(|a|^2 - |b|^2\right).$$

This completes the proof of Lemma 2.5. □

3 Proof of Theorem 1.5

Set

$$F(z) = \frac{1-\psi}{f^2(1-\alpha_0)} = \frac{1-\alpha_0 f - \alpha_1 f'}{f^2(1-\alpha_0)} = \frac{1}{1-\alpha_0}\left[\frac{1}{f^2} - \frac{1}{f}\left(\alpha_0 + \alpha_1\frac{f'}{f}\right)\right].$$

Then we have

$$S_{\alpha,\beta}(r,F) \leq 5S_{\alpha,\beta}(r,f) + 2S_{\alpha,\beta}(r,\alpha_0) + S_{\alpha,\beta}(r,\alpha_1) + D_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + O(1),$$

and from Lemma 2.2 and $\alpha_0, \alpha_1 \in \ell(f)$ it follows that

$$R_{\alpha,\beta}(r,F) = R_{\alpha,\beta}(r,f). \tag{21}$$

By Lemma 2.5 we have

$$\begin{aligned} 2D_{\alpha,\beta}\left(r,\frac{1}{f}\right) &\leq 3\left[D_{\alpha,\beta}(r,\alpha_0) + D_{\alpha,\beta}(r,\alpha_1)\right] + 2D_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + D_{\alpha,\beta}(r,F) + O(1) \\ &= D_{\alpha,\beta}(r,F) + R_{\alpha,\beta}(r,f). \end{aligned} \tag{22}$$

Since $\frac{1}{F} = 1 - \frac{F-1}{F'} \cdot \frac{F'}{F}$, it follows by Lemma 2.4 that

$$S_{\alpha,\beta}(r,F) \leq \overline{C}_{\alpha,\beta}(r,F) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{F}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{F-1}\right) + R_{\alpha,\beta}(r,F),$$

that is,

$$D_{\alpha,\beta}(r,F) \leq \overline{C}_{\alpha,\beta}\left(r,\frac{1}{F}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{F-1}\right) - C^1_{\alpha,\beta}(r,F) + R_{\alpha,\beta}(r,F), \tag{23}$$

where $C^1_{\alpha,\beta}(r,F) = C_{\alpha,\beta}(r,F) - \overline{C}_{\alpha,\beta}(r,F)$. Thus, it follows from (21)-(23) that

$$\begin{aligned} S_{\alpha,\beta}(r,f) \leq{}& C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \frac{1}{2}\left[\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{F-1}\right) - C^1_{\alpha,\beta}(r,F)\right] \\ &+ R_{\alpha,\beta}(r,f). \end{aligned} \tag{24}$$

We consider three cases to estimate $\overline{C}_{\alpha,\beta}(r,\frac{1}{F})$, $\overline{C}_{\alpha,\beta}(r,\frac{1}{F-1})$, $C^{1}_{\alpha,\beta}(r,F)$ in (24), respectively.

Case 1. $\overline{C}_{\alpha,\beta}(r,\frac{1}{F})$.

From the definition of $F(z)$ we find that the zero of $F(z)$ comes from the zero of $\psi-1$, or the pole of f, or the pole of α_0. Then

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F}\right)\le\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\psi-1}\right)+\overline{C}_{\alpha,\beta}(r,f)+\overline{C}_{\alpha,\beta}(r,\alpha_0).\tag{25}$$

Since $\overline{C}_{\alpha,\beta}(r,f)=\overline{C}^{(2}_{\alpha,\beta}(r,f)+\overline{C}^{1)}_{\alpha,\beta}(r,f)$, by Lemma 2.4 we have

$$\overline{C}^{(2}_{\alpha,\beta}(r,f)\le C^{1}_{\alpha,\beta}(r,f)\le\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+R_{\alpha,\beta}(r,f).\tag{26}$$

If z_0 is a pole of f of order 1 in Ω, and not the zero of α_0, α_0-1, the pole of α_0, and also not the zero and pole of α_1, then we have

$$\begin{aligned}
&\alpha_0(z)=a_0+b_0(z-z_0)^{p_0}+\cdots\quad(a_0\ne0,1,b_0\ne0,p_0\ge1),\\
&\alpha_1(z)=a_1+b_1(z-z_0)^{p_1}+\cdots\quad(a_1\ne0,b_1\ne0,p_1\ge1),\\
&f(z)=\frac{a}{z-z_0}+b(z-z_0)^{p}+\cdots\quad(a\ne0,b\ne0,p\ge0).
\end{aligned}$$

Substituting all this into $F(z)$, we get

$$F(z)=\frac{a_1}{a(1-a_0)}+\cdots,$$

which shows that z_0 is not a zero and a pole of $F(z)$ in Ω.

If z_0 is a pole of f of order 1 in Ω and a zero of anyone of α_0, α_0-1, and $\frac{1}{\alpha_0}$, or z_0 is a zero of anyone of α_1 and $\frac{1}{\alpha_1}$, then z_0 maybe one zero of $F(z)$. Thus, we have

$$\begin{aligned}
\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F}\right)&\le\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\psi-1}\right)+\overline{C}^{(2}_{\alpha,\beta}(r,f)+\overline{C}_{\alpha,\beta}(r,\alpha_0)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\alpha_0-1}\right)\\
&\quad+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\alpha_0}\right)+\overline{C}_{\alpha,\beta}(r,\alpha_1)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\alpha_1}\right)\\
&\le\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\psi-1}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+R_{\alpha,\beta}(r,f).
\end{aligned}\tag{27}$$

Case 2. $\overline{C}_{\alpha,\beta}(r,\frac{1}{F-1})$. Set

$$F_1=(F-1)\cdot\frac{f}{f-1}=\frac{(1-f^2)-\alpha_0f(1-f)-\alpha_1f'}{f(f-1)(1-\alpha_0)}.$$

Then $F-1=F_1\cdot\frac{f-1}{f}$. It follows that the zero of $F-1$ comes from the zero of F_1 or the zero of $f-1$, that is,

$$\begin{aligned}
\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F-1}\right)&\le\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F_1}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)\\
&\le C_{\alpha,\beta}\left(r,\frac{1}{F_1}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right).
\end{aligned}\tag{28}$$

Since

$$F_1 = \frac{1}{1-\alpha_0}\left[-1 - \frac{1}{f} + \alpha_0 + \alpha_1\left(\frac{f'}{f} - \frac{f'}{f-1}\right)\right] := \frac{1}{1-\alpha_0}F_2,$$

it follows that

$$D_{\alpha,\beta}(r,F_1) \le D_{\alpha,\beta}\left(r,\frac{1}{f}\right) + R_{\alpha,\beta}(r,f), \tag{29}$$

$$C_{\alpha,\beta}(r,F_1) \le C_{\alpha,\beta}(r,F_2) + C_{\alpha,\beta}\left(r,\frac{1}{\alpha_0 - 1}\right) = C_{\alpha,\beta}(r,F_2) + R_{\alpha,\beta}(r,f). \tag{30}$$

From the definitions of F_1, F_2 by Lemma 2.1 we have

$$\begin{aligned} C_{\alpha,\beta}(r,F_2) &\le C_{\alpha,\beta}(r,\alpha_0) + C_{\alpha,\beta}(r,\alpha_1) + C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) \\ &\le C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) + R_{\alpha,\beta}(r,f) \end{aligned} \tag{31}$$

and

$$C_{\alpha,\beta}\left(r,\frac{1}{F_1}\right) \le S_{\alpha,\beta}\left(r,\frac{1}{F_1}\right) = D_{\alpha,\beta}(r,F_1) + C_{\alpha,\beta}(r,F_1) + O(1). \tag{32}$$

Thus, it follows from (28)-(32) that

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F-1}\right) \le S_{\alpha,\beta}(r,f) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + 2\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) + R_{\alpha,\beta}(r,f). \tag{33}$$

Case 3. $-C^1_{\alpha,\beta}(r,F)$. If z_0 is a pole of f of order $k \ge 2$, then from the definition of $F(z)$ we see that the pole of $F(z)$ only occurs at the pole of α_1.

If z_1 is a zero of f of order $k \ge 2$ and not the pole of α_0, α_1, then we get that z_1 is a pole of $F(z)$ of order $\ge 2k$; if z_1 is a pole of α_0 of order $s_0 \ge 0$ or a pole of α_1 of order $s_1 \ge 0$, then we have

$$\max\{2k - s_0, k, k + 1 - s_0 + s_1\} \ge 2k - s_0 - s_1.$$

Let $C^{1'}_{\alpha,\beta}(r,F)$ be a part of $C_{\alpha,\beta}(r,F)$ corresponding to the zero and pole of f of order $k \ge 2$, Then we have

$$\begin{aligned} C^1_{\alpha,\beta}(r,F) &> C^{1'}_{\alpha,\beta}(r,F) \\ &\ge 2C^{(2}_{\alpha,\beta}\left(r,\frac{1}{f}\right) - \overline{C}^{(2}_{\alpha,\beta}\left(r,\frac{1}{f}\right) - C_{\alpha,\beta}(r,\alpha_0) - C_{\alpha,\beta}(r,\alpha_1) + C_{\alpha,\beta}(r,\alpha_1) \\ &\ge 2C_{\alpha,\beta}\left(r,\frac{1}{f}\right) - \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) - \overline{C}^{1)}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + R_{\alpha,\beta}(r,f), \end{aligned}$$

that is,

$$-C^1_{\alpha,\beta}(r,F) < -2C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + C^{1)}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + R_{\alpha,\beta}(r,f). \tag{34}$$

Then, substituting (27), (33), and (34) into (24), we have

$$\begin{aligned} S_{\alpha,\beta}(r,f) &< C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \frac{1}{2}\left[\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\psi-1}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)\right. \\ &\quad + S_{\alpha,\beta}(r,f) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + 2\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) - 2C_{\alpha,\beta}\left(r,\frac{1}{f}\right) \\ &\quad \left. + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + C^{1)}_{\alpha,\beta}\left(r,\frac{1}{f}\right)\right] + R_{\alpha,\beta}(r,f), \end{aligned}$$

that is,

$$S_{\alpha,\beta}(r,f) < 3\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + 3\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{\psi-1}\right) + C^{1)}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + R_{\alpha,\beta}(r,f),$$

and since $C^{1)}_{\alpha,\beta}(r,\frac{1}{f}) \le \overline{C}_{\alpha,\beta}(r,\frac{1}{f})$, thus it follows that

$$S_{\alpha,\beta}(r,f) < 4\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + 3\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{\psi-1}\right) + R_{\alpha,\beta}(r,f).$$

This completes the proof of Theorem 1.5.

4 Proofs of Theorem 1.6 and Corollary 1.1

4.1 The proof of Theorem 1.6

Let

$$\alpha_0 = \frac{\varphi_2' - \varphi_1'}{\varphi_3 - \varphi_1'}, \qquad \alpha_1 = \frac{\varphi_2 - \varphi_1}{\varphi_3 - \varphi_1'}, \qquad F(z) = \frac{f - \varphi_1}{\varphi_2 - \varphi_1}.$$

Then

$$\psi(z) = \alpha_0 F + \alpha_1 F' = \frac{f' - \varphi_1'}{\varphi_3 - \varphi_1'}.$$

Since $\varphi_i \in \ell(f)$, $i = 1, 2, 3$, it follows that $\alpha_0, \alpha_1 \in \ell(f)$ and

$$\begin{aligned} &S_{\alpha,\beta}(r,f) \le S_{\alpha,\beta}(r,F) + R_{\alpha,\beta}(r,f), \\ &\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F}\right) \le \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right) + R_{\alpha,\beta}(r,f), \\ &\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F-1}\right) \le \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_2}\right) + R_{\alpha,\beta}(r,f), \\ &\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\psi-1}\right) \le \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_3}\right) + R_{\alpha,\beta}(r,f). \end{aligned}$$

Then, from this and from Theorem 1.5 we easily get (4). Furthermore, if f satisfies (1), then $\limsup_{r\to+\infty} \frac{R_{\alpha,\beta}(r,f)}{S_{\alpha,\beta}(r,f)} = 0$. Thus, we can easily get (5) from (4).

This completes the proof of Theorem 1.6.

4.2 The proof of Corollary 1.1

For $\varphi \in \ell(f)$ and any positive integer $k \geq 2$, we have

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi}\right) \leq \left(1-\frac{1}{k}\right)\overline{C}^{k-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi}\right) + \frac{1}{k}S_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f); \tag{35}$$

$$\begin{aligned}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right) &= \left(1-\frac{1}{k}\right)\overline{C}^{k-1)}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right) \\ &\quad + \frac{1}{k}\left(\overline{C}^{k-1)}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right) + k\overline{C}^{(k}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right)\right) \\ &\leq \left(1-\frac{1}{k}\right)\overline{C}^{k-1)}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right) + \frac{1}{k}S_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right) \\ &\leq \left(1-\frac{1}{k}\right)\overline{C}^{k-1)}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right) + \frac{2}{k}S_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f).\end{aligned} \tag{36}$$

Hence, it follows from (4), (35), and (36) that

$$\begin{aligned}(1-\Delta_1)S_{\alpha,\beta}(r,f) &< 4\left(1-\frac{1}{k_1}\right)\overline{C}^{k_1-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right) + 3\left(1-\frac{1}{k_2}\right)\overline{C}^{k_2-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_2}\right) \\ &\quad + \left(1-\frac{1}{k_3}\right)\overline{C}^{k_3-1)}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_3}\right) + R_{\alpha,\beta}(r,f).\end{aligned}$$

This completes the proof of Corollary 1.1.

5 Proofs of Theorems 1.7, 1.8 and Corollaries 1.2, 1.3

5.1 The proof of Theorem 1.7

Since $f-\varphi_1 = (f'-\varphi_1')\frac{f-\varphi_1}{f'-\varphi_1'}$ and $\varphi_i \in \ell(f)$, by Lemmas 2.1-2.3 we have

$$\begin{aligned}D_{\alpha,\beta}(r,f-\varphi_1) &\leq D_{\alpha,\beta}\left(r,f'-\varphi_1'\right) + D_{\alpha,\beta}\left(r,\frac{f-\varphi_1}{f'-\varphi_1'}\right) + O(1) \\ &\leq D_{\alpha,\beta}\left(r,f'-\varphi_1'\right) + C_{\alpha,\beta}\left(r,\frac{f'-\varphi_1'}{f-\varphi_1}\right) - C_{\alpha,\beta}\left(r,\frac{f-\varphi_1}{f'-\varphi_1'}\right) \\ &\quad + D_{\alpha,\beta}\left(r,\frac{f'-\varphi_1'}{f-\varphi_1}\right) + O(1) \\ &\leq S_{\alpha,\beta}\left(r,f'-\varphi_1'\right) + C_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right) - C_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_1'}\right) \\ &\quad - C_{\alpha,\beta}(r,f-\varphi_1) + R_{\alpha,\beta}(r,f),\end{aligned}$$

that is,

$$S_{\alpha,\beta}(r,f) \leq S_{\alpha,\beta}\left(r,f'\right) + C_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right) - C_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_1'}\right) + R_{\alpha,\beta}(r,f). \tag{37}$$

On the other hand, letting

$$F_3 = \frac{f'-\varphi_1'}{f'-\varphi_2}\cdot\frac{\varphi_3-\varphi_2}{\varphi_3-\varphi_1'},$$

it follows that $R_{\alpha,\beta}(r,F_3)=R_{\alpha,\beta}(r,f)$ and

$$\begin{aligned}
&S_{\alpha,\beta}\big(r,f'\big)\le S_{\alpha,\beta}(r,F_3)+R_{\alpha,\beta}(r,f),\\
&\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F_3}\right)\le \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_1'}\right)+R(_{\alpha,\beta}r,f),\\
&\overline{C}(r,F_3)\le \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_2}\right)+R_{\alpha,\beta}(r,f),\\
&\overline{C}_{\alpha,\beta}\left(r,\frac{1}{F_3-1}\right)\le \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_3}\right)+R_{\alpha,\beta}(r,f).
\end{aligned}$$

Then, from these inequalities and from (37) by applying Lemma 2.4 for F_3 we have

$$\begin{aligned}
S_{\alpha,\beta}(r,f)&<C_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)-C_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_1'}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_1'}\right)\\
&\quad+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_2}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_3}\right)+R_{\alpha,\beta}(r,f)\\
&<\overline{\overline{C}}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi_1}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_2}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi_3}\right)+R_{\alpha,\beta}(r,f).
\end{aligned}$$

This completes the proof of Theorem 1.7.

5.2 The proof of Theorem 1.8

Using the same argument as in Theorem 1.7 and letting $F_3=\frac{f'-\varphi_1'}{f'-\varphi_2}$, we easily get the conclusions of Theorem 1.8.

5.3 Proofs of Corollaries 1.2 and 1.3

For $\varphi\in\ell(f)$ and any positive integer $k\ge 2$, we have

$$\begin{aligned}
&\overline{C}_{\alpha,\beta}(r,f)\left(1-\frac{1}{k}\right)\overline{C}^{k-1)}_{\alpha,\beta}(r,f)+\frac{1}{k}S_{\alpha,\beta}(r,f),\\
&\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi}\right)\le\left(1-\frac{1}{k}\right)\overline{C}^{k-1)}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi}\right)+\frac{1}{k}S_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f),\\
&\overline{\overline{C}}_{\alpha,\beta}\left(r,\frac{1}{f-\varphi}\right)\le\left(1-\frac{2}{k}\right)\overline{C}^{k-1)}\left(r,\frac{1}{f-\varphi}\right)+\frac{2}{k}S(r,f)+R(r,f),\\
&\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right)\le\left(1-\frac{1}{k}\right)\overline{C}^{k-1)}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right)+\frac{1}{k}S_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right)+R_{\alpha,\beta}(r,f)\\
&\qquad\le\left(1-\frac{1}{k}\right)\overline{C}^{k-1)}_{\alpha,\beta}\left(r,\frac{1}{f'-\varphi}\right)+\frac{2}{k}S_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f).
\end{aligned}$$

Applying these inequalities and Theorems 1.7 and 1.8, we easily prove Corollaries 1.2 and 1.3.

Competing interests
The authors declare that none of the authors have any competing interests in the manuscript.

Authors' contributions
HYX, XMZ, and LPZ completed the main part of this article. XMZ, HYX corrected the main theorems. All authors read and approved the final manuscript.

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