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Essential norm of generalized weighted composition operators from the Bloch space to the Zygmund space

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Abstract
In this paper, we give some estimates of the essential norm for generalized weighted composition operators from the Bloch space to the Zygmund space. Moreover, we give a new characterization for the boundedness and compactness of the operator.

MSC: 30H30; 47B38

Keywords: Bloch space; Zygmund space; essential norm; generalized weighted composition operator

1 Introduction

Let X and Y be Banach spaces. The essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators K mapping X into Y, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \text{ is compact}\},$$

where $\|\cdot\|_{X\to Y}$ is the operator norm.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. Let φ be a nonconstant analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, and n be a nonnegative integer. The *generalized weighted composition operator*, denoted by $D^n_{\varphi,u}$, is defined on $H(\mathbb{D})$ by

$$\big(D^n_{\varphi,u}f\big)(z) = u(z)f^{(n)}\big(\varphi(z)\big), \quad z \in \mathbb{D}.$$

When $n = 0$, the generalized weighted composition operator $D^n_{\varphi,u}$ is the weighted composition operator, denoted by uC_φ. In particular, when $n = 0$ and $u = 1$, we get the composition operator C_φ. If $n = 1$ and $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$, which was widely studied, for example, in [1–9]. If $u(z) = 1$, then $D^n_{\varphi,u} = C_\varphi D^n$, which was studied, for example, in [1, 5, 10, 11]. For the study of the generalized weighted composition operator on various function spaces see, for example, [12–21]. Recently there has been a huge interest in the study of various related product-type operators containing composition operators; see, *e.g.*, [22–30] and the references therein.

The Bloch space, denoted by $\mathcal{B}$, is defined to be the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \bigl|f(0)\bigr| + \sup_{z\in\mathbb{D}}\bigl(1-|z|^2\bigr)\bigl|f'(z)\bigr| < \infty.$$

$\mathcal{B}$ is a Banach space with the above norm. An $f \in \mathcal{B}$ is said to belong to the little Bloch space $\mathcal{B}_0$ if $\lim_{|z|\to 1}|f'(z)|(1-|z|^2) = 0$. See [31] for more information of Bloch spaces. Composition operators, as well as weighted composition operators mapping into Bloch-type spaces were studied a lot see, for example, [3, 6, 16, 32–45].

The Zygmund space, denoted by $\mathcal{Z}$, is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}} = \bigl|f(0)\bigr| + \bigl|f'(0)\bigr| + \sup_{z\in\mathbb{D}}\bigl(1-|z|^2\bigr)\bigl|f''(z)\bigr| < \infty.$$

It is easy to see that $\mathcal{Z}$ is a Banach space with the above norm $\|\cdot\|_{\mathcal{Z}}$. See [4, 7, 12, 15, 16, 22, 36, 46–50] for some results of the Zygmund space and related operators mapping into the Zygmund space or into some of its generalizations.

In 1995, Madigan and Matheson proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if (see [38])

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)}\bigl|\varphi'(z)\bigr| = 0.$$

In 1999, Montes-Rodrieguez in [40] obtained the exact value for the essential norm of the operator $C_\varphi : \mathcal{B} \to \mathcal{B}$, *i.e.*,

$$\|C_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} = \lim_{s\to 1}\sup_{|\varphi(z)|>s}\frac{(1-|z|^2)|\varphi'(z)|}{(1-|\varphi(z)|^2)}.$$

Tjani in [43] proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{|a|\to 1}\|C_\varphi\sigma_a\|_{\mathcal{B}} = 0$, where $\sigma_a = \frac{a-z}{1-\bar{a}z}$. Wulan *et al.* in [44] showed that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{j\to\infty}\|\varphi^j\|_{\mathcal{B}} = 0$. Ohno *et al.* studied the boundedness and compactness of the operator uC_φ on the Bloch space in [41]. The estimate for the essential norm of the operator uC_φ on the Bloch space was given in [37]. Some new estimates for the essential norm of uC_φ on the Bloch space were given in [33, 39]. In [21], Zhu has obtained some estimates for the essential norm of $D^n_{\varphi,u}$ on the Bloch space when n is a positive integer.

Stević studied the boundedness and compactness of $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ in [16] (see also [50]). In [12], Li and Fu obtained a new characterization for the boundedness, as well as the compactness for $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ by using three families of functions. We combine the results in [12] and [16] as follows.

Theorem A *Let n be a positive integer, $u \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$. Suppose that $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is bounded, then the following statements are equivalent*:

(a) *The operator $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is compact.*

(b)

$$\lim_{|\varphi(w)|\to 1}\bigl\|D^n_{\varphi,u}f_{\varphi(w)}\bigr\|_{\mathcal{Z}} = \lim_{|\varphi(w)|\to 1}\bigl\|D^n_{\varphi,u}g_{\varphi(w)}\bigr\|_{\mathcal{Z}} = \lim_{|\varphi(w)|\to 1}\bigl\|D^n_{\varphi,u}h_{\varphi(w)}\bigr\|_{\mathcal{Z}} = 0,$$

where

$$f_{\varphi(w)}(z) = \frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}, \qquad g_{\varphi(w)}(z) = \frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^2},$$
$$h_{\varphi(w)}(z) = \frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^3}, \quad z \in \mathbb{D}.$$

(c)

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^n} = \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{n+2}}$$
$$= \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1+n}} = 0.$$

Motivated by these observations, the purpose of this paper is to give some estimates of the essential norm for the operator $D^n_{\varphi,u}:\mathcal{B}\to\mathcal{Z}$. Moreover, we give a new characterization for the boundedness, compactness, and essential norm of the operator $D^n_{\varphi,u}:\mathcal{B}\to\mathcal{Z}$.

Throughout this paper, we say that $P\lesssim Q$ if there exists a constant C such that $P\le CQ$. The symbol $P\approx Q$ means that $P\lesssim Q\lesssim P$.

2 Essential norm of $D^n_{\varphi,u}:\mathcal{B}\to\mathcal{Z}$

In this section, we give two estimates of the essential norm for the operator $D^n_{\varphi,u}:\mathcal{B}\to\mathcal{Z}$.

Theorem 2.1 *Let n be a positive integer, $u\in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$ such that $D^n_{\varphi,u}:\mathcal{B}\to\mathcal{Z}$ is bounded. Then*

$$\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} \approx \max\{A,B,C\} \approx \max\{E,F,G\},$$

where

$$A := \limsup_{|a|\to 1}\left\|D^n_{\varphi,u}\left(\frac{1-|a|^2}{1-\overline{a}z}\right)\right\|_{\mathcal{Z}}, \qquad B := \limsup_{|a|\to 1}\left\|D^n_{\varphi,u}\left(\frac{(1-|a|^2)^2}{(1-\overline{a}z)^2}\right)\right\|_{\mathcal{Z}},$$
$$C := \limsup_{|a|\to 1}\left\|D^n_{\varphi,u}\left(\frac{(1-|a|^2)^3}{(1-\overline{a}z)^3}\right)\right\|_{\mathcal{Z}}, \qquad F := \limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^n},$$
$$E := \limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{n+1}},$$

and

$$G := \limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{n+2}}.$$

Proof First we prove that $\max\{A,B,C\}\le\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}}$. Let $a\in\mathbb{D}$. Define

$$f_a(z) = \frac{1-|a|^2}{(1-\overline{a}z)}, \qquad g_a(z) = \frac{(1-|a|^2)^2}{(1-\overline{a}z)^2}, \qquad h_a(z) = \frac{(1-|a|^2)^3}{(1-\overline{a}z)^3}, \quad z\in\mathbb{D}.$$

It is easy to check that $f_a, g_a, h_a \in \mathcal{B}_0$ and $\|f_a\|_{\mathcal{B}}\lesssim 1$, $\|g_a\|_{\mathcal{B}}\lesssim 1$, $\|h_a\|_{\mathcal{B}}\lesssim 1$ for all $a\in\mathbb{D}$ and f_a, g_a, h_a converge to 0 weakly in $\mathcal{B}$ as $|a|\to 1$. This follows since a bounded sequence contained in $\mathcal{B}_0$ which converges uniformly to 0 on compact subsets of $\mathbb{D}$ converges weakly to 0 in $\mathcal{B}$ (see [37, 42]). Thus, for any compact operator $K:\mathcal{B}_0\to\mathcal{Z}$, we have

$$\lim_{|a|\to 1}\|Kf_a\|_{\mathcal{Z}}=0,\qquad \lim_{|a|\to 1}\|Kg_a\|_{\mathcal{Z}}=0,\qquad \lim_{|a|\to 1}\|Kh_a\|_{\mathcal{Z}}=0.$$

Hence

$$\begin{aligned}\|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}}&\gtrsim\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-K)f_a\|_{\mathcal{Z}}\\&\geq\limsup_{|a|\to 1}\|D^n_{\varphi,u}f_a\|_{\mathcal{Z}}-\limsup_{|a|\to 1}\|Kf_a\|_{\mathcal{Z}}=A,\\\|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}}&\gtrsim\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-K)g_a\|_{\mathcal{Z}}\\&\geq\limsup_{|a|\to 1}\|D^n_{\varphi,u}g_a\|_{\mathcal{Z}}-\limsup_{|a|\to 1}\|Kg_a\|_{\mathcal{Z}}=B,\end{aligned}$$

and

$$\begin{aligned}\|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}}&\gtrsim\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-K)h_a\|_{\mathcal{Z}}\\&\geq\limsup_{|a|\to 1}\|D^n_{\varphi,u}h_a\|_{\mathcal{Z}}-\limsup_{|a|\to 1}\|Kh_a\|_{\mathcal{Z}}=C.\end{aligned}$$

Therefore, from the definition of the essential norm, we obtain

$$\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}}=\inf_K\|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}}\gtrsim\max\{A,B,C\}.$$

Next, we prove that $\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}}\gtrsim\max\{E,F,G\}$. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\to 1$ as $j\to\infty$. Define

$$\begin{aligned}k_j(z)&=\frac{1-|\varphi(z_j)|^2}{1-\overline{\varphi(z_j)}z}-\frac{2n+5}{(n+1)(n+3)}\frac{(1-|\varphi(z_j)|^2)^2}{(1-\overline{\varphi(z_j)}z)^2}+\frac{2}{(n+1)(n+3)}\frac{(1-|\varphi(z_j)|^2)^3}{(1-\overline{\varphi(z_j)}z)^3},\\l_j(z)&=\frac{1-|\varphi(z_j)|^2}{1-\overline{\varphi(z_j)}z}-\frac{2(n+3)}{2+(n+1)(n+4)}\frac{(1-|\varphi(z_j)|^2)^2}{(1-\overline{\varphi(z_j)}z)^2}+\frac{2}{2+(n+1)(n+4)}\frac{(1-|\varphi(z_j)|^2)^3}{(1-\overline{\varphi(z_j)}z)^3},\end{aligned}$$

and

$$m_j(z)=\frac{1-|\varphi(z_j)|^2}{1-\overline{\varphi(z_j)}z}-\frac{2}{n+1}\frac{(1-|\varphi(z_j)|^2)^2}{(1-\overline{\varphi(z_j)}z)^2}+\frac{2}{(n+1)(n+2)}\frac{(1-|\varphi(z_j)|^2)^3}{(1-\overline{\varphi(z_j)}z)^3}.$$

Similarly to the above we see that all k_j, l_j, and m_j belong to $\mathcal{B}_0$ and converge to 0 weakly in $\mathcal{B}$. Moreover,

$$\begin{aligned}&k_j^{(n)}(\varphi(z_j))=0,\qquad k_j^{(n+2)}(\varphi(z_j))=0,\qquad |k_j^{(n+1)}(\varphi(z_j))|=\frac{n!}{n+3}\frac{|\varphi(z_j)|^{n+1}}{(1-|\varphi(z_j)|^2)^{n+1}},\\&l_j^{(n+1)}(\varphi(z_j))=0,\qquad l_j^{(n+2)}(\varphi(z_j))=0,\qquad |l_j^{(n)}(\varphi(z_j))|=\frac{2n!}{2+(n+1)(n+4)}\frac{|\varphi(z_j)|^{n}}{(1-|\varphi(z_j)|^2)^{n}},\\&m_j^{(n)}(\varphi(z_j))=0,\qquad m_j^{(n+1)}(\varphi(z_j))=0,\qquad |m_j^{(n+2)}(\varphi(z_j))|=2n!\frac{|\varphi(z_j)|^{n+2}}{(1-|\varphi(z_j)|^2)^{n+2}}.\end{aligned}$$

Then for any compact operator $K:\mathcal{B}\to\mathcal{Z}$, we obtain

$$\begin{aligned}\|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}} &\gtrsim \limsup_{j\to\infty}\|D^n_{\varphi,u}(k_j)\|_{\mathcal{Z}}-\limsup_{j\to\infty}\|K(k_j)\|_{\mathcal{Z}}\\ &\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)||\varphi(z_j)|^{n+1}}{(1-|\varphi(z_j)|^2)^{n+1}},\\ \|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}} &\gtrsim \limsup_{j\to\infty}\|D^n_{\varphi,u}(l_j)\|_{\mathcal{Z}}-\limsup_{j\to\infty}\|K(l_j)\|_{\mathcal{Z}}\\ &\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)|u''(z_j)||\varphi(z_j)|^{n}}{(1-|\varphi(z_j)|^2)^{n}},\end{aligned}$$

and

$$\begin{aligned}\|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}} &\gtrsim \limsup_{j\to\infty}\|D^n_{\varphi,u}(m_j)\|_{\mathcal{Z}}-\limsup_{j\to\infty}\|K(m_j)\|_{\mathcal{Z}}\\ &\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^{n+2}}{(1-|\varphi(z_j)|^2)^{n+2}}.\end{aligned}$$

From the definition of the essential norm, we obtain

$$\begin{aligned}\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} &= \inf_K\|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}}\\ &\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)||\varphi(z_j)|^{n+1}}{(1-|\varphi(z_j)|^2)^{n+1}}\\ &= \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{n+1}}=E,\\ \|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} &= \inf_K\|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}}\\ &\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)|u''(z_j)||\varphi(z_j)|^{n}}{(1-|\varphi(z_j)|^2)^{n}}\\ &= \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{n}}=F,\end{aligned}$$

and

$$\begin{aligned}\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} &= \inf_K\|D^n_{\varphi,u}-K\|_{\mathcal{B}\to\mathcal{Z}}\\ &\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^{n+2}}{(1-|\varphi(z_j)|^2)^{n+2}}\\ &= \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{n+2}}=G.\end{aligned}$$

Hence

$$\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}}\gtrsim\max\{E,F,G\}.$$

Now, we prove that

$$\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} \lesssim \max\{A,B,C\} \quad \text{and} \quad \|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} \lesssim \max\{E,F,G\}.$$

For $r \in [0,1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $(K_r f)(z) = f_r(z) = f(rz)$, $f \in H(\mathbb{D})$. It is obvious that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, the operator K_r is compact on $\mathcal{B}$ and $\|K_r\|_{\mathcal{B}\to\mathcal{B}} \le 1$ (see [37]). Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for all positive integer j, the operator $D^n_{\varphi,u}K_{r_j} : \mathcal{B} \to \mathcal{Z}$ is compact. By the definition of the essential norm, we get

$$\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} \le \limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{r_j}\|_{\mathcal{B}\to\mathcal{Z}}. \tag{2.1}$$

Therefore, we only need to prove that

$$\limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{r_j}\|_{\mathcal{B}\to\mathcal{Z}} \lesssim \max\{A,B,C\}$$

and

$$\limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{r_j}\|_{\mathcal{B}\to\mathcal{Z}} \lesssim \max\{E,F,G\}.$$

For any $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{B}} \le 1$, we consider

$$\begin{aligned} &\|(D^n_{\varphi,u} - D^n_{\varphi,u}K_{r_j})f\|_{\mathcal{Z}} \\ &\quad = |u(0)f^{(n)}(\varphi(0)) - r_j^n u(0)f^{(n)}(r_j\varphi(0))| \\ &\qquad + |u'(0)(f-f_{r_j})^{(n)}(\varphi(0)) + u(0)(f-f_{r_j})^{(n+1)}(\varphi(0))\varphi'(0)| \\ &\qquad + \|u \cdot (f-f_{r_j})^{(n)} \circ \varphi\|_*, \end{aligned} \tag{2.2}$$

where $\|f\|_* = \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|$.

It is obvious that

$$\lim_{j\to\infty} |u(0)f^{(n)}(\varphi(0)) - r_j^n u(0)f^{(n)}(r_j\varphi(0))| = 0 \tag{2.3}$$

and

$$\lim_{j\to\infty} |u'(0)(f-f_{r_j})^{(n)}(\varphi(0)) + u(0)(f-f_{r_j})^{(n+1)}(\varphi(0))\varphi'(0)| = 0. \tag{2.4}$$

Now, we consider

$$\begin{aligned} &\limsup_{j\to\infty} \|u \cdot (f-f_{r_j})^{(n)} \circ \varphi\|_* \\ &\quad \le \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)|(f-f_{r_j})^{(n+1)}(\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)| \\ &\qquad + \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|(f-f_{r_j})^{(n+1)}(\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)| \end{aligned}$$

$$+\limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n)}\bigl(\varphi(z)\bigr)\bigr|\bigl|u''(z)\bigr|$$
$$+\limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n)}\bigl(\varphi(z)\bigr)\bigr|\bigl|u''(z)\bigr|$$
$$+\limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n+2)}\bigl(\varphi(z)\bigr)\bigr|\bigl|\varphi'(z)\bigr|^2\bigl|u(z)\bigr|$$
$$+\limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n+2)}\bigl(\varphi(z)\bigr)\bigr|\bigl|\varphi'(z)\bigr|^2\bigl|u(z)\bigr|$$
$$=Q_1+Q_2+Q_3+Q_4+Q_5+Q_6, \tag{2.5}$$

where $N\in\mathbb{N}$ is large enough such that $r_j\ge\frac{1}{2}$ for all $j\ge N$,

$$Q_1:=\limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n+1)}\bigl(\varphi(z)\bigr)\bigr|\bigl|2u'(z)\varphi'(z)+u(z)\varphi''(z)\bigr|,$$
$$Q_2:=\limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n+1)}\bigl(\varphi(z)\bigr)\bigr|\bigl|2u'(z)\varphi'(z)+u(z)\varphi''(z)\bigr|,$$
$$Q_3:=\limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n)}\bigl(\varphi(z)\bigr)\bigr|\bigl|u''(z)\bigr|,$$
$$Q_4:=\limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n)}\bigl(\varphi(z)\bigr)\bigr|\bigl|u''(z)\bigr|,$$
$$Q_5:=\limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n+2)}\bigl(\varphi(z)\bigr)\bigr|\bigl|\varphi'(z)\bigr|^2\bigl|u(z)\bigr|,$$

and

$$Q_6:=\limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}\bigl(1-|z|^2\bigr)\bigl|(f-f_{r_j})^{(n+2)}\bigl(\varphi(z)\bigr)\bigr|\bigl|\varphi'(z)\bigr|^2\bigl|u(z)\bigr|.$$

Since $D^n_{\varphi,u}:\mathcal{B}\to\mathcal{Z}$ is bounded, by Theorem 1 of [12], we see that $u\in\mathcal{Z}$,

$$\widetilde{K}_1:=\sup_{z\in\mathbb{D}}\bigl(1-|z|^2\bigr)\bigl|2u'(z)\varphi'(z)+u(z)\varphi''(z)\bigr|<\infty$$

and

$$\widetilde{K}_2:=\sup_{z\in\mathbb{D}}\bigl(1-|z|^2\bigr)\bigl|\varphi'(z)\bigr|^2\bigl|u(z)\bigr|<\infty.$$

Since $r_j^{n+1}f_{r_j}^{(n+1)}\to f^{(n+1)}$, as well as $r_j^{n+2}f_{r_j}^{(n+2)}\to f^{(n+2)}$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, we have

$$Q_1\le\widetilde{K}_1\limsup_{j\to\infty}\sup_{|w|\le r_N}\bigl|f^{(n+1)}(w)-r_j^{n+1}f^{(n+1)}(r_jw)\bigr|=0 \tag{2.6}$$

and

$$Q_5\le\widetilde{K}_2\limsup_{j\to\infty}\sup_{|w|\le r_N}\bigl|f^{(n+2)}(w)-r_j^{n+2}f^{(n+2)}(r_jw)\bigr|=0. \tag{2.7}$$

Similarly, from the fact that $u \in \mathcal{Z}$ we have

$$Q_3 \le \|u\|_{\mathcal{Z}} \limsup_{j\to\infty} \sup_{|w|\le r_N} \left|f^{(n)}(w) - r_j^n f^{(n)}(r_j w)\right| = 0. \tag{2.8}$$

Next we consider Q_2. We have $Q_2 \le \limsup_{j\to\infty}(S_1 + S_2)$, where

$$S_1 := \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right)\left|f^{(n+1)}\left(\varphi(z)\right)\right|\left|2u'(z)\varphi'(z) + u(z)\varphi''(z)\right|$$

and

$$S_2 := \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right) r_j^{n+1}\left|f^{(n+1)}\left(r_j\varphi(z)\right)\right|\left|2u'(z)\varphi'(z) + u(z)\varphi''(z)\right|.$$

First we estimate S_1. Using the fact that $\|f\|_{\mathcal{B}} \le 1$ and Theorem 5.4 in [31], we have

$$\begin{aligned}
S_1 &= \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right)\left|f^{(n+1)}\left(\varphi(z)\right)\right|\left|2u'(z)\varphi'(z) + u(z)\varphi''(z)\right| \\
&\quad \times \frac{(1-|\varphi(z)|^2)^{n+1}(n+3)}{|\varphi(z)|^{n+1} n!} \frac{|\varphi(z)|^{n+1} n!}{(n+3)(1-|\varphi(z)|^2)^{n+1}} \\
&\lesssim \frac{(n+3)\|f\|_{\mathcal{B}}}{n! r_N^{n+1}} \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right)\left|2u'(z)\varphi'(z) + u(z)\varphi''(z)\right| \\
&\quad \times \frac{n!|\varphi(z)|^{n+1}}{(n+3)(1-|\varphi(z)|^2)^{n+1}} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right)\left|2u'(z)\varphi'(z) + u(z)\varphi''(z)\right| \frac{n!|\varphi(z)|^{n+1}}{(n+3)(1-|\varphi(z)|^2)^{n+1}} \\
&\lesssim \sup_{|a|>r_N} \left\| D^n_{\varphi,u}\left(f_a - \frac{(2n+5)g_a}{(n+1)(n+3)} + \frac{2h_a}{(n+1)(n+3)}\right)\right\|_{\mathcal{Z}} \\
&\lesssim \sup_{|a|>r_N} \left\|D^n_{\varphi,u}(f_a)\right\|_{\mathcal{Z}} + \frac{2n+5}{(n+1)(n+3)} \sup_{|a|>r_N} \left\|D^n_{\varphi,u}(g_a)\right\|_{\mathcal{Z}} \\
&\quad + \frac{2}{(n+1)(n+3)} \sup_{|a|>r_N} \left\|D^n_{\varphi,u}(h_a)\right\|_{\mathcal{Z}}.
\end{aligned} \tag{2.9}$$

Taking the limit as $N \to \infty$ we obtain

$$\begin{aligned}
\limsup_{j\to\infty} S_1 &\lesssim \limsup_{|a|\to 1}\left\|D^n_{\varphi,u}(f_a)\right\|_{\mathcal{Z}} + \limsup_{|a|\to 1}\left\|D^n_{\varphi,u}(g_a)\right\|_{\mathcal{Z}} \\
&\quad + \limsup_{|a|\to 1}\left\|D^n_{\varphi,u}(h_a)\right\|_{\mathcal{Z}} \\
&= A + B + C.
\end{aligned}$$

Similarly, we have $\limsup_{j\to\infty} S_2 \lesssim A + B + C$, *i.e.*, we get

$$Q_2 \lesssim A + B + C \lesssim \max\{A, B, C\}. \tag{2.10}$$

From (2.9), we see that

$$\limsup_{j\to\infty} S_1 \lesssim \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{n+1}} = E.$$

Similarly we have $\limsup_{j\to\infty} S_2 \lesssim E$. Therefore

$$Q_2 \lesssim E. \tag{2.11}$$

Next we consider Q_4. We have $Q_4 \le \limsup_{j\to\infty}(S_3 + S_4)$, where

$$S_3 := \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right)\left|f^{(n)}\left(\varphi(z)\right)\right|\left|u''(z)\right|$$

and

$$S_4 := \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right)r_j^n\left|f^{(n)}\left(r_j\varphi(z)\right)\right|\left|u''(z)\right|.$$

After some calculation, we have

$$\begin{aligned}
S_3 &= \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right)\left|f^{(n)}\left(\varphi(z)\right)\right|\left|u''(z)\right| \\
&\quad \times \frac{(1-|\varphi(z)|^2)^n(2+(n+1)(n+4))}{2n!|\varphi(z)|^n}\frac{2}{2+(n+1)(n+4)}\frac{n!|\varphi(z)|^n}{(1-|\varphi(z)|^2)^n} \\
&\lesssim \frac{2^n(2+(n+1)(n+4))}{2n!}\|f\|_{\mathcal{B}} \sup_{|\varphi(z)|>r_N}\left(1-|z|^2\right)\left|u''(z)\right| \\
&\quad \times \frac{2}{2+(n+1)(n+4)}\frac{n!|\varphi(z)|^n}{(1-|\varphi(z)|^2)^n} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \frac{2n!}{2+(n+1)(n+4)}\frac{(1-|z|^2)|u''(z)||\varphi(z)|^n}{(1-|\varphi(z)|^2)^n} \\
&\lesssim \sup_{|a|>r_N}\left\|D^n_{\varphi,u}(f_a)\right\|_{\mathcal{Z}} + \frac{2(n+3)}{2+(n+1)(n+4)}\sup_{|a|>r_N}\left\|D^n_{\varphi,u}(g_a)\right\|_{\mathcal{Z}} \\
&\quad + \frac{2}{2+(n+1)(n+4)}\sup_{|a|>r_N}\left\|D^n_{\varphi,u}(h_a)\right\|_{\mathcal{Z}} \\
&\lesssim \sup_{|a|>r_N}\left\|D^n_{\varphi,u}(f_a)\right\|_{\mathcal{Z}} + \sup_{|a|>r_N}\left\|D^n_{\varphi,u}(g_a)\right\|_{\mathcal{Z}} + \sup_{|a|>r_N}\left\|D^n_{\varphi,u}(h_a)\right\|_{\mathcal{Z}}.
\end{aligned} \tag{2.12}$$

Taking the limit as $N \to \infty$ we obtain

$$\begin{aligned}
\limsup_{j\to\infty} S_3 &\lesssim \limsup_{|a|\to 1}\left\|D^n_{\varphi,u}(f_a)\right\|_{\mathcal{Z}} + \limsup_{|a|\to 1}\left\|D^n_{\varphi,u}(g_a)\right\|_{\mathcal{Z}} \\
&\quad + \limsup_{|a|\to 1}\left\|D^n_{\varphi,u}(h_a)\right\|_{\mathcal{Z}} \\
&= A + B + C.
\end{aligned}$$

Similarly, we have $\limsup_{j\to\infty} S_4 \lesssim A + B + C$, *i.e.*, we get

$$Q_4 \lesssim A + B + C \lesssim \max\{A, B, C\}. \tag{2.13}$$

From (2.12), we see that

$$\limsup_{j\to\infty} S_3 \lesssim \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |u''(z)|}{(1-|\varphi(z)|^2)^n} = F.$$

Similarly we have $\limsup_{j\to\infty} S_4 \lesssim F$. Therefore

$$Q_4 \lesssim F. \tag{2.14}$$

Finally we consider Q_6. We have $Q_6 \le \limsup_{j\to\infty}(S_5 + S_6)$, where

$$S_5 := \sup_{|\varphi(z)|>r_N} \big(1-|z|^2\big)\big|f^{(n+2)}\big(\varphi(z)\big)\big|\big|\varphi'(z)\big|^2\big|u(z)\big|$$

and

$$S_6 := \sup_{|\varphi(z)|>r_N} \big(1-|z|^2\big)r_j^{n+2}\big|f^{(n+2)}\big(r_j\varphi(z)\big)\big|\big|\varphi'(z)\big|^2\big|u(z)\big|.$$

After some calculation, we have

$$\begin{aligned}
S_5 &\lesssim \frac{2^{n+2}\|f\|_{\mathcal{B}}}{2n!} \sup_{|\varphi(z)|>r_N} \big(1-|z|^2\big)\big|\varphi'(z)\big|^2\big|u(z)\big| \frac{2n!|\varphi(z)|^{n+2}}{(1-|\varphi(z)|^2)^{n+2}} \\
&\lesssim \frac{2^{n+2}}{2n!} \sup_{|\varphi(z)|>r_N} \big(1-|z|^2\big)\big|\varphi'(z)\big|^2\big|u(z)\big| \frac{2n!|\varphi(z)|^{n+2}}{(1-|\varphi(z)|^2)^{n+2}} \\
&\lesssim \sup_{|a|>r_N} \left\| D^n_{\varphi,u}\left(f_a - \frac{2}{n+1}g_a + \frac{2}{(n+1)(n+2)}h_a\right)\right\|_{\mathcal{Z}} \\
&\lesssim \sup_{|a|>r_N} \big\|D^n_{\varphi,u}(f_a)\big\|_{\mathcal{Z}} + \frac{2}{n+1}\sup_{|a|>r_N} \big\|D^n_{\varphi,u}(g_a)\big\|_{\mathcal{Z}} + \frac{2}{(n+1)(n+2)}\sup_{|a|>r_N} \big\|D^n_{\varphi,u}(h_a)\big\|_{\mathcal{Z}} \\
&\le \sup_{|a|>r_N} \big\|D^n_{\varphi,u}(f_a)\big\|_{\mathcal{Z}} + \sup_{|a|>r_N} \big\|D^n_{\varphi,u}(g_a)\big\|_{\mathcal{Z}} + \sup_{|a|>r_N} \big\|D^n_{\varphi,u}(h_a)\big\|_{\mathcal{Z}}.
\end{aligned} \tag{2.15}$$

Taking the limit as $N \to \infty$ we obtain

$$\begin{aligned}
\limsup_{j\to\infty} S_5 &\lesssim \limsup_{|a|\to 1}\big\|D^n_{\varphi,u}(f_a)\big\|_{\mathcal{Z}} + \limsup_{|a|\to 1}\big\|D^n_{\varphi,u}(g_a)\big\|_{\mathcal{Z}} \\
&\quad + \limsup_{|a|\to 1}\big\|D^n_{\varphi,u}(h_a)\big\|_{\mathcal{Z}} \\
&= A + B + C.
\end{aligned}$$

Similarly, we have $\limsup_{j\to\infty} S_6 \lesssim A + B + C$, *i.e.*, we get

$$Q_6 \lesssim A + B + C \lesssim \max\{A, B, C\}. \tag{2.16}$$

From (2.15), we see that

$$\limsup_{j\to\infty} S_5 \lesssim \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|^2|u(z)|}{(1-|\varphi(z)|^2)^{n+2}} = G.$$

Similarly we have $\limsup_{j\to\infty} S_6 \lesssim G$. Therefore

$$Q_6 \lesssim G. \tag{2.17}$$

Hence, by (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), (2.10), (2.13), and (2.16) we get

$$\begin{aligned}
&\limsup_{j\to\infty}\|D^n_{\varphi,u}-D^n_{\varphi,u}K_{r_j}\|_{\mathcal{B}\to\mathcal{Z}}\\
&\quad=\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{B}}\le 1}\|(D^n_{\varphi,u}-D^n_{\varphi,u}K_{r_j})f\|_{\mathcal{Z}}\\
&\quad=\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{B}}\le 1}\|u\cdot(f-f_{r_j})^{(n)}\circ\varphi\|_{*}\lesssim\max\{A,B,C\}.
\end{aligned}\tag{2.18}$$

Similarly, by (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), (2.11), (2.14), and (2.17) we get

$$\limsup_{j\to\infty}\|D^n_{\varphi,u}-D^n_{\varphi,u}K_{r_j}\|_{\mathcal{B}\to\mathcal{Z}}\lesssim\max\{E,F,G\}.\tag{2.19}$$

Therefore, by (2.1), (2.18), and (2.19), we obtain

$$\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}}\lesssim\max\{A,B,C\}\quad\text{and}\quad\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}}\lesssim\max\{E,F,G\}.$$

This completes the proof of Theorem 2.1. □

3 New characterization of $D^n_{\varphi,u}:\mathcal{B}\to\mathcal{Z}$

In this section, we give a new characterization for the boundedness, compactness, and essential norm of the operator $D^n_{\varphi,u}:\mathcal{B}\to\mathcal{Z}$. For this purpose, we present some definitions and some lemmas which will be used later.

The weighted space, denoted by H^∞_ν, consists of all $f\in H(\mathbb{D})$ such that

$$\|f\|_\nu=\sup_{z\in\mathbb{D}}\nu(z)|f(z)|<\infty,$$

where $\nu:\mathbb{D}\to R_+$ is a continuous, strictly positive, and bounded function. H^∞_ν is a Banach space under the norm $\|\cdot\|_\nu$. The weighted ν is called radial if $\nu(z)=\nu(|z|)$ for all $z\in\mathbb{D}$. The associated weight $\tilde{\nu}$ of ν is as follows:

$$\tilde{\nu}=\left(\sup\left\{|f(z)|:f\in H^\infty_\nu,\|f\|_\nu\le 1\right\}\right)^{-1},\quad z\in\mathbb{D}.$$

When $\nu=\nu_\alpha(z)=(1-|z|^2)^\alpha$ $(0<\alpha<\infty)$, it is well known that $\tilde{\nu}_\alpha(z)=\nu_\alpha(z)$. In this case, we denote H^∞_ν by $H^\infty_{\nu_\alpha}$.

Lemma 3.1 [33] *For* $\alpha>0$, *we have* $\lim_{k\to\infty}k^\alpha\|z^{k-1}\|_{\nu_\alpha}=(\frac{2\alpha}{e})^\alpha$.

Lemma 3.2 [51] *Let* v *and* w *be radial, non-increasing weights tending to zero at the boundary of* $\mathbb{D}$. *Then the following statements hold*.

(a) *The weighted composition operator* $uC_\varphi:H^\infty_v\to H^\infty_w$ *is bounded if and only if* $\sup_{z\in\mathbb{D}}\frac{w(z)}{\tilde{v}(\varphi(z))}|u(z)|<\infty$. *Moreover, the following holds*:

$$\|uC_\varphi\|_{H^\infty_v\to H^\infty_w}=\sup_{z\in\mathbb{D}}\frac{w(z)}{\tilde{v}(\varphi(z))}|u(z)|.$$

(b) *Suppose* $uC_\varphi : H_v^\infty \to H_w^\infty$ *is bounded. Then*

$$\|uC_\varphi\|_{e,H_v^\infty\to H_w^\infty} = \lim_{s\to 1^-} \sup_{|\varphi(z)|>s} \frac{w(z)}{\tilde{v}(\varphi(z))}\big|u(z)\big|.$$

Lemma 3.3 [52] *Let v and w be radial, non-increasing weights tending to zero at the boundary of* $\mathbb{D}$. *Then the following statements hold.*

(a) $uC_\varphi : H_v^\infty \to H_w^\infty$ *is bounded if and only if* $\sup_{k\ge 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty$, *with the norm comparable to the above supremum.*

(b) *Suppose* $uC_\varphi : H_v^\infty \to H_w^\infty$ *is bounded. Then*

$$\|uC_\varphi\|_{e,H_v^\infty\to H_w^\infty} = \limsup_{k\to\infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}.$$

Theorem 3.1 *Let n be a positive integer,* $u \in H(\mathbb{D})$, *and* φ *be an analytic self-map of* $\mathbb{D}$. *Then the operator* $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ *is bounded if and only if*

$$\begin{cases} \sup_{j\ge 1} j^{n+1}\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_1} < \infty, \\ \sup_{j\ge 1} j^n\|u''\varphi^{j-1}\|_{v_1} < \infty, \\ \sup_{j\ge 1} j^{n+2}\|u\varphi'^2\varphi^{j-1}\|_{v_1} < \infty. \end{cases} \tag{3.1}$$

Proof By [16], $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is bounded if and only if

$$\begin{cases} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{n+1}} < \infty, \\ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^n} < \infty, \\ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{n+2}} < \infty. \end{cases} \tag{3.2}$$

By Lemma 3.2, the first inequality in (3.2) is equivalent to the weighted composition operator $(2u'\varphi' + u\varphi'')C_\varphi : H^\infty_{v_{n+1}} \to H^\infty_{v_1}$ is bounded. By Lemma 3.3, this is equivalent to

$$\sup_{j\ge 1} \frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{1+n}}} < \infty.$$

The second inequality in (3.2) is equivalent to the operator $u''C_\varphi : H^\infty_{v_n} \to H^\infty_{v_1}$ is bounded. By Lemma 3.3, this is equivalent to

$$\sup_{j\ge 1} \frac{\|u''\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_n}} < \infty.$$

The third inequality in (3.2) is equivalent to the operator $u\varphi'^2C_\varphi : H^\infty_{v_{n+2}} \to H^\infty_{v_1}$ is bounded. By Lemma 3.3, this is equivalent to

$$\sup_{j\ge 1} \frac{\|u\varphi'^2\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{n+2}}} < \infty.$$

By Lemma 3.1, we see that $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is bounded if and only if

$$\sup_{j\ge 1} j^{n+1}\big\|\big(2u'\varphi' + u\varphi''\big)\varphi^{j-1}\big\|_{v_1} \approx \sup_{j\ge 1} \frac{j^{n+1}\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_1}}{j^{n+1}\|z^{j-1}\|_{v_{1+n}}} < \infty,$$

$$\sup_{j\geq 1} j^n \left\| u''\varphi^{j-1}\right\|_{v_1} \approx \sup_{j\geq 1} \frac{j^n \|u''\varphi^{j-1}\|_{v_1}}{j^n \|z^{j-1}\|_{v_n}} < \infty,$$

and

$$\sup_{j\geq 1} j^{n+2} \left\| u\varphi'^2\varphi^{j-1}\right\|_{v_1} \approx \sup_{j\geq 1} \frac{j^{n+2} \|u\varphi'^2\varphi^{j-1}\|_{v_1}}{j^{n+2} \|z^{j-1}\|_{v_{n+2}}} < \infty.$$

The proof is completed. □

Theorem 3.2 *Let n be a positive integer, $u \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$ such that $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is bounded. Then*

$$\left\| D^n_{\varphi,u}\right\|_{e,\mathcal{B}\to\mathcal{Z}} \approx \max\{M_1, M_2, M_3\},$$

where

$$M_1 := \limsup_{j\to\infty} j^{1+n} \left\| \left(2u'\varphi' + u\varphi''\right)\varphi^{j-1}\right\|_{v_1},$$

$$M_2 := \limsup_{j\to\infty} j^n \left\| u''\varphi^{j-1}\right\|_{v_1}, \qquad M_3 := \limsup_{j\to\infty} j^{n+2} \left\| u\left(\varphi'\right)^2\varphi^{j-1}\right\|_{v_1}.$$

Proof From the proof of Theorem 3.1 we know that the boundedness of $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is equivalent to the boundedness of the operators $(2u'\varphi' + u\varphi'')C_\varphi : H^\infty_{v_{1+n}} \to H^\infty_{v_1}$, $u''C_\varphi : H^\infty_{v_n} \to H^\infty_{v_1}$, and $u\varphi'^2 C_\varphi : H^\infty_{v_{n+2}} \to H^\infty_{v_1}$.

The upper estimate. By Lemmas 3.1 and 3.3, we get

$$\begin{aligned}\left\| \left(2u'\varphi' + u\varphi''\right)C_\varphi\right\|_{e,H^\infty_{v_{1+n}}\to H^\infty_{v_1}} &= \limsup_{j\to\infty} \frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{1+n}}} \\ &= \limsup_{j\to\infty} \frac{j^{1+n}\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_1}}{j^{1+n}\|z^{j-1}\|_{v_{1+n}}} \\ &\approx \limsup_{j\to\infty} j^{1+n} \left\| \left(2u'\varphi' + u\varphi''\right)\varphi^{j-1}\right\|_{v_1},\end{aligned}$$

$$\begin{aligned}\left\| u''C_\varphi\right\|_{e,H^\infty_{v_n}\to H^\infty_{v_1}} &= \limsup_{j\to\infty} \frac{\|u''\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_n}} = \limsup_{j\to\infty} \frac{j^{\alpha+n-1}\|u''\varphi^{j-1}\|_{v_1}}{j^{\alpha+n-1}\|z^{j-1}\|_{v_n}} \\ &\approx \limsup_{j\to\infty} j^n \left\| u''\varphi^{j-1}\right\|_{v_1},\end{aligned}$$

and

$$\begin{aligned}\left\| u\varphi'^2 C_\varphi\right\|_{e,H^\infty_{v_{n+2}}\to H^\infty_{v_1}} &= \limsup_{j\to\infty} \frac{\|u\varphi'^2\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{n+2}}} = \limsup_{j\to\infty} \frac{j^{n+2}\|u\varphi'^2\varphi^{j-1}\|_{v_1}}{j^{n+2}\|z^{j-1}\|_{v_{n+2}}} \\ &\approx \limsup_{j\to\infty} j^{n+2} \left\| u\varphi'^2\varphi^{j-1}\right\|_{v_1}.\end{aligned}$$

It follows that

$$\begin{aligned}\left\| D^n_{\varphi,u}\right\|_{e,\mathcal{B}\to\mathcal{Z}} &\lesssim \left\| \left(2u'\varphi' + u\varphi''\right)C_\varphi\right\|_{e,H^\infty_{v_{1+n}}\to H^\infty_{v_1}} + \left\| u''C_\varphi\right\|_{e,H^\infty_{v_n}\to H^\infty_{v_1}} \\ &\quad + \left\| u\varphi'^2 C_\varphi\right\|_{e,H^\infty_{v_{n+2}}\to H^\infty_{v_1}} \\ &\lesssim \max\{M_1, M_2, M_3\}.\end{aligned}$$

The lower estimate. From Theorem 2.1, and Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} \gtrsim E &= \|(2u'\varphi' + u\varphi'')C_\varphi\|_{e,H^\infty_{v_{1+n}}\to H^\infty_{v_1}}\\ &= \limsup_{j\to\infty}\frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{1+n}}}\\ &\approx \limsup_{j\to\infty} j^{\alpha+n}\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_1},\\ \|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} \gtrsim F &= \|u''C_\varphi\|_{e,H^\infty_{v_n}\to H^\infty_{v_1}} = \limsup_{j\to\infty}\frac{\|u''\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_n}}\\ &\approx \limsup_{j\to\infty} j^{\alpha+n-1}\|u''\varphi^{j-1}\|_{v_1},\end{aligned}$$

and

$$\begin{aligned}\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} \gtrsim G &= \|u\varphi'^2C_\varphi\|_{e,H^\infty_{v_{n+2}}\to H^\infty_{v_1}} = \limsup_{j\to\infty}\frac{\|u\varphi'^2\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{n+2}}}\\ &\approx \limsup_{j\to\infty} j^{n+2}\|u\varphi'^2\varphi^{j-1}\|_{v_1}.\end{aligned}$$

Therefore $\|D^n_{\varphi,u}\|_{e,\mathcal{B}\to\mathcal{Z}} \gtrsim \max\{M_1, M_2, M_3\}$. This completes the proof. □

From Theorem 3.2, we immediately get the following result.

Theorem 3.3 *Let n be a positive integer, $u \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$ such that $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is bounded. Then $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is compact if and only if*

$$\limsup_{j\to\infty} j^{1+n}\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_1} = 0, \qquad \limsup_{j\to\infty} j^{n}\|u''\varphi^{j-1}\|_{v_1} = 0,$$

and

$$\limsup_{j\to\infty} j^{n+2}\|u(\varphi')^2\varphi^{j-1}\|_{v_1} = 0.$$

4 Conclusion

The boundedness and compactness of $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ were characterized in [12] and [16]. In this paper, we give a new characterization for the boundedness and compactness of $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$. Moreover, using the method in [21], we completely characterize the essential norm of $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The four authors contributed equally to the writing of this paper. They read and approved the final version of the manuscript.

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References

1. Hibschweiler, R, Portnoy, N: Composition followed by differentiation between Bergman and Hardy spaces. Rocky Mt. J. Math. **35**, 843-855 (2005)
2. Li, S, Stević, S: Composition followed by differentiation between Bloch type spaces. J. Comput. Anal. Appl. **9**, 195-205 (2007)
3. Li, S, Stević, S: Composition followed by differentiation between H^∞ and α-Bloch spaces. Houst. J. Math. **35**, 327-340 (2009)
4. Li, S, Stević, S: Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces. Appl. Math. Comput. **217**, 3144-3154 (2010)
5. Stević, S: Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H_μ^∞. Appl. Math. Comput. **207**, 225-229 (2009)
6. Stević, S: Products of composition and differentiation operators on the weighted Bergman space. Bull. Belg. Math. Soc. Simon Stevin **16**, 623-635 (2009)
7. Stević, S: Composition followed by differentiation from H^∞ and the Bloch space to nth weighted-type spaces on the unit disk. Appl. Math. Comput. **216**, 3450-3458 (2010)
8. Stević, S: Characterizations of composition followed by differentiation between Bloch-type spaces. Appl. Math. Comput. **218**, 4312-4316 (2011)
9. Yang, W: Products of composition differentiation operators from $\mathcal{Q}_K(p,q)$ spaces to Bloch-type spaces. Abstr. Appl. Anal. **2009**, Article ID 741920 (2009)
10. Stević, S, Sharma, A: Iterated differentiation followed by composition from Bloch-type spaces to weighted BMOA spaces. Appl. Math. Comput. **218**, 3574-3580 (2011)
11. Wu, Y, Wulan, H: Products of differentiation and composition operators on the Bloch space. Collect. Math. **63**, 93-107 (2012)
12. Li, H, Fu, X: A new characterization of generalized weighted composition operators from the Bloch space into the Zygmund space. J. Funct. Spaces Appl. **2013**, Article ID 925901 (2013)
13. Li, S, Stević, S: Generalized weighted composition operators from α-Bloch spaces into weighted-type spaces. J. Inequal. Appl. **2015**, Article ID 265 (2015)
14. Stević, S: Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces. Appl. Math. Comput. **211**, 222-233 (2009)
15. Stević, S: Weighted differentiation composition operators from mixed-norm spaces to the nth weighted-type space on the unit disk. Abstr. Appl. Anal. **2010**, Article ID 246287 (2010)
16. Stević, S: Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weighted-type spaces on the unit disk. Appl. Math. Comput. **216**, 3634-3641 (2010)
17. Yang, W, Zhu, X: Generalized weighted composition operators from area Nevanlinna spaces to Bloch-type spaces. Taiwan. J. Math. **16**, 869-883 (2012)
18. Zhu, X: Products of differentiation, composition and multiplication from Bergman type spaces to Bers type space. Integral Transforms Spec. Funct. **18**, 223-231 (2007)
19. Zhu, X: Generalized weighted composition operators on weighted Bergman spaces. Numer. Funct. Anal. Optim. **30**, 881-893 (2009)
20. Zhu, X: Generalized weighted composition operators on Bloch-type spaces. J. Inequal. Appl. **2015**, Article ID 59 (2015)
21. Zhu, X: Essential norm of generalized weighted composition operators on Bloch-type spaces. Appl. Math. Comput. **274**, 133-142 (2016)
22. Li, S, Stević, S: Generalized composition operators on Zygmund spaces and Bloch type spaces. J. Math. Anal. Appl. **338**, 1282-1295 (2008)
23. Li, S, Stević, S: Products of composition and integral type operators from H^∞ to the Bloch space. Complex Var. Elliptic Equ. **53**(5), 463-474 (2008)
24. Li, S, Stević, S: Composition followed by differentiation from mixed-norm spaces to α-Bloch spaces. Sb. Math. **199**(12), 1847-1857 (2008)
25. Li, S, Stević, S: Products of integral-type operators and composition operators between Bloch-type spaces. J. Math. Anal. Appl. **349**, 596-610 (2009)
26. Stević, S: On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces. Nonlinear Anal. TMA **71**, 6323-6342 (2009)
27. Stević, S: Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces. Sib. Math. J. **50**, 726-736 (2009)
28. Stević, S: On some integral-type operators between a general space and Bloch-type spaces. Appl. Math. Comput. **218**, 2600-2618 (2011)
29. Stević, S, Sharma, A, Bhat, A: Products of multiplication composition and differentiation operators on weighted Bergman spaces. Appl. Math. Comput. **217**, 8115-8125 (2011)
30. Stević, S, Sharma, A, Bhat, A: Essential norm of products of multiplication composition and differentiation operators on weighted Bergman spaces. Appl. Math. Comput. **218**, 2386-2397 (2011)
31. Zhu, K: Operator Theory in Function Spaces, 2nd edn. Am. Math. Soc., Providence (2007)
32. Cowen, C, Maccluer, B: Composition Operators on Spaces of Analytic Functions. CRC Press, Boca Raton (1995)
33. Hyvärinen, O, Lindström, M: Estimates of essential norm of weighted composition operators between Bloch-type spaces. J. Math. Anal. Appl. **393**, 38-44 (2012)
34. Li, S, Stević, S: Weighted composition operators from Bergman-type spaces into Bloch spaces. Proc. Indian Acad. Sci. Math. Sci. **117**(3), 371-385 (2007)

35. Li, S, Stević, S: Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball. Taiwan. J. Math. **12**, 1625-1639 (2008)
36. Li, S, Stević, S: Weighted composition operators from Zygmund spaces into Bloch spaces. Appl. Math. Comput. **206**, 825-831 (2008)
37. MacCluer, B, Zhao, R: Essential norm of weighted composition operators between Bloch-type spaces. Rocky Mt. J. Math. **33**, 1437-1458 (2003)
38. Madigan, K, Matheson, A: Compact composition operators on the Bloch space. Trans. Am. Math. Soc. **347**, 2679-2687 (1995)
39. Manhas, J, Zhao, R: New estimates of essential norms of weighted composition operators between Bloch type spaces. J. Math. Anal. Appl. **389**, 32-47 (2012)
40. Montes-Rodriguez, A: The essential norm of composition operators on the Bloch space. Pac. J. Math. **188**, 339-351 (1999)
41. Ohno, S, Stroethoff, K, Zhao, R: Weighted composition operators between Bloch-type spaces. Rocky Mt. J. Math. **33**, 191-215 (2003)
42. Stević, S: Essential norms of weighted composition operators from the α-Bloch space to a weighted-type space on the unit ball. Abstr. Appl. Anal. **2008**, Article ID 279691 (2008)
43. Tjani, M: Compact composition operators on some Möbius invariant Banach space. PhD dissertation, Michigan State University (1996)
44. Wulan, H, Zheng, D, Zhu, K: Compact composition operators on BMOA and the Bloch space. Proc. Am. Math. Soc. **137**, 3861-3868 (2009)
45. Zhao, R: Essential norms of composition operators between Bloch type spaces. Proc. Am. Math. Soc. **138**, 2537-2546 (2010)
46. Choe, B, Koo, H, Smith, W: Composition operators on small spaces. Integral Equ. Oper. Theory **56**, 357-380 (2006)
47. Duren, P: Theory of H^p Spaces. Academic Press, New York (1970)
48. Esmaeili, K, Lindström, M: Weighted composition operators between Zygmund type spaces and their essential norms. Integral Equ. Oper. Theory **75**, 473-490 (2013)
49. Li, S, Stević, S: Volterra type operators on Zygmund spaces. J. Inequal. Appl. **2007**, Article ID 32124 (2007)
50. Yu, Y, Liu, Y: Weighted differentiation composition operators from H^∞ to Zygmund spaces. Integral Transforms Spec. Funct. **22**, 507-520 (2011)
51. Montes-Rodriguez, A: Weighed composition operators on weighted Banach spaces of analytic functions. J. Lond. Math. Soc. **61**, 872-884 (2000)
52. Hyvärinen, O, Kemppainen, M, Lindström, M, Rautio, A, Saukko, E: The essential norm of weighted composition operators on weighted Banach spaces of analytic functions. Integral Equ. Oper. Theory **72**, 151-157 (2012)