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Seidel-Estrada index

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Abstract

Let G be a simple graph with n vertices and (0, 1)-adjacency matrix A. As usual, $S(G) = J - 2A - I$ denotes the Seidel matrix of the graph G. Suppose $\theta_1, \theta_2, \ldots, \theta_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix and the Seidel matrix of G, respectively. The Estrada index of the graph G is defined as $\sum_{i=1}^{n} e^{\theta_i}$. We define and investigate the Seidel-Estrada index, $SEE = SEE(G) = \sum_{i=1}^{n} e^{\lambda_i}$. In this paper the basic properties of the Seidel-Estrada index are investigated. Moreover, some lower and upper bounds for the Seidel-Estrada index in terms of the number of vertices are obtained. In addition, some relations between SEE and the Seidel energy $E_s(G)$ are presented.

MSC: 05C50; 05C90

Keywords: eigenvalue; Seidel matrix; Seidel-Estrada index

1 Introduction

Throughout this paper, let G be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A(G) = [a_{ij}]$ of G is a binary matrix of order n such that $a_{ij} = 1$ if the vertex v_i is adjacent to the vertex v_j, and 0 otherwise. The Seidel matrix $S(G) = [s_{ij}]$ is equal to $\mathbb{J}_n - 2A(G) - \mathbb{I}_n$, where the symbol $\mathbb{J}_n$ denotes the square matrix of order n all of whose entries are equal to 1. Since $A(G)$ and $S(G)$ are real symmetric matrices, their eigenvalues must be real. The eigenvalues of G are referred to as the eigenvalues of $A(G)$, denoted by $\theta_1(A(G)), \theta_2(A(G)), \ldots, \theta_n(A(G))$ and similarly, $\lambda_1(S(G)) \geq \lambda_2(S(G)) \geq \cdots \geq \lambda_n(S(G))$, the Seidel eigenvalues of G. For simplicity, we write λ_i instead of $\lambda_i(S(G))$. The sequence of n Seidel eigenvalues is called the Seidel spectrum of G (for short S-spec(G)). We now present an example of pairs of graphs on n vertices with the same Seidel spectrum such that one of them is a connected graph and the other one is not.

Example 1 Here we address two examples from non-isomorphic graphs which are cospectral:

$$\text{(i)} \quad \text{S-spec}(K_{p,q}) = \text{S-spec}(\overline{K}_n) \quad \text{if } p + q = n,$$

$$\text{(ii)} \quad \text{S-spec}(K_{n/2} \cup K_{n/2}) = \text{S-spec}(K_n) \quad (n \text{ is even}).$$

In our recent studies on Seidel eigenvalues it has been shown that a lower and upper bound exists for the sum of powers of the absolute eigenvalues of the Seidel matrix, suggesting a common core architecture similar to the cases of adjacency and signless Laplacian matrix [1]. The reader can find more information related to the eigenvalues of the adjacency matrix and the spectrum of G in [2]. The Estrada index of a graph G is defined as

$$EE(G) = \sum_{i=1}^{n} e^{\theta_i(A(G))}. \tag{1}$$

This graph-spectrum-based structural descriptor was first proposed by Estrada in 2000; see [3–5]. Already, de la Peña *et al.* [6] proposed to call it the Estrada index, a name that in the meantime has been commonly accepted. Several kinds of Estrada indices were discussed in [7–13] and the references therein. For the recent work of the mathematical properties on the Estrada and signless Laplacian Estrada indices, see [6, 14]. In this review, we summarize some indirect evidence to support the concept of a Seidel matrix. Similarly, we define the Seidel-Estrada index for the graph G in full analogy with equation (1) as

$$SEE(G) = \sum_{i=1}^{n} e^{\lambda_i}. \tag{2}$$

For details on the theory of the Estrada index and several lower and upper bounds, see [6, 11, 13, 15]. Ayyaswamy *et al.* [14] gave a lower bound for a signless Laplacian of the graph using the numbers of vertices and edges. A conference matrix is a square matrix C of order n with zero diagonal and ±1 off the diagonal, such that $CC^T = (n-1)I$. If C is symmetric, then C is the Seidel matrix of a graph and this graph is called a *conference graph*; see [1, 2]. The aim of this paper is to find the upper and lower bounds for the Seidel-Estrada index of the graph G. The rest of the paper is organized as follows: In Section 2, we give some definitions and obtain some upper and lower bounds for the Seidel-Estrada index. In Section 3, we present a relation between the Seidel-Estrada index and the Seidel energy of a graph G, and we prove several results on the Seidel-Estrada index.

2 Estimates of the Seidel-Estrada index

Here we give some new lower and upper bounds on Seidel-Estrada index. For convenience, we give some notation and properties which will be used in the following proofs of our results. Let $S_k = S_k(G) = \sum_{i=1}^{n}(\lambda_i)^k$, and $S^k(G) = \sum_{i=1}^{n}|\lambda_i|^k$. From the Taylor expansion of e^x, it is easy to see that the Seidel-Estrada index and $S_k(G)$ of G are related by

$$SEE(G) = \sum_{k=0}^{\infty} \frac{S_k(G)}{k!}. \tag{3}$$

It is easy to see that any graph G of order $n \geq 2$ has $SEE(G) > n$. (If equality holds, then $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. By Lemma 2.1(ii), we can get a contradiction.)

Lemma 2.1 [1] *For any graph G with n vertices, we have*

(i) $S_1(G) = \sum_{i=1}^{n} \lambda_i = \operatorname{trace}\big(S(G)\big) = 0,$

(ii) $S^2(G) = S_2(G) = \sum_{i=1}^{n} \lambda_i^2 = \operatorname{trace}\big(S^2(G)\big) = (n-1)^2 + (n-1) = n(n-1),$

(iii) $S_3(G) = \sum_{i=1}^{n} \lambda_i^3 \leq S^3(G) \leq (n-1)^3 + (n-1),$

(iv) $S^3(G) = \sum_{i=1}^{n} |\lambda_i|^3 \geq n\sqrt{(n-1)^3},$

(v) $S_k(G) \leq S^k(G) = \sum_{i=1}^{n} |\lambda_i|^k \leq (n-1)^k + (n-1), \quad k = 3, 4, \ldots,$

(vi) $S^k(G) = \sum_{i=1}^{n} |\lambda_i|^k \geq n\sqrt{(n-1)^k}, \quad k = 3, 4, \ldots.$

Lemma 2.2 [16] *Let B be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let B_k be its leading $k \times k$ submatrix of B. Then, for $i = 1, 2, \ldots, k$,*

$$\lambda_{n-i+1}(B) \leq \lambda_{k-i+1}(B_k) \leq \lambda_{k-i+1}(B), \tag{4}$$

where $\lambda_i(B)$ is the ith greatest eigenvalue of B.

Lemma 2.3 *Let G be a graph of order $n \geq 2$. Then $\lambda_1 \geq 1$.*

Proof Since $n \geq 2$, therefore $\overline{K}_2$ or K_2 must be an induced subgraph of G. Since $\lambda_1(K_2) = \lambda_1(\overline{K}_2) = 1$, by Lemma 2.2, we get the required result. □

Theorem 2.4 *Let G be a simple graph with $n \geq 2$ and $\det S(G) \neq 0$. Then the Seidel-Estrada index of G is bounded by*

$$\sqrt{n(3n-2)} < SEE(G) < n - 1 + e^{\sqrt{n(n-1)}}. \tag{5}$$

Proof (a) To prove this theorem, we apply a technique similar to the proof of Theorem 1 in [6]. At first we prove that the left inequality of (5):

From (2), we get

$$SEE^2(G) = \sum_{i=1}^{n} e^{2\lambda_i} + 2\sum_{i<j} e^{\lambda_i} e^{\lambda_j}. \tag{6}$$

In view by the inequality between the geometric and arithmetic mean, we get

$$\begin{aligned} 2\sum_{i<j} e^{\lambda_i} e^{\lambda_j} &\geq n(n-1)\left(\prod_{i<j} e^{\lambda_i} e^{\lambda_j}\right)^{\frac{2}{n(n-1)}} = n(n-1)\left[\left(\prod_{i=1}^{n} e^{\lambda_i}\right)^{n-1}\right]^{\frac{2}{n(n-1)}} \\ &= n(n-1)\left(e^{S_1(G)}\right)^{\frac{2}{n}} = n(n-1). \end{aligned} \tag{7}$$

By using the power series expansion, and Lemma 2.1, we get

$$\begin{aligned} \sum_{i=1}^{n} e^{2\lambda_i} &= \sum_{i=1}^{n}\sum_{k=0}^{\infty} \frac{(2\lambda_i)^k}{k!} = \sum_{i=1}^{n} \frac{(2\lambda_i)^0}{0!} + \sum_{i=1}^{n} \frac{(2\lambda_i)^1}{1!} + \sum_{i=1}^{n} \frac{(2\lambda_i)^2}{2!} + \sum_{i=1}^{n}\sum_{k\geq 3} \frac{(2\lambda_i)^k}{k!} \\ &= S_0 + 2S_1 + 2S_2 + \sum_{i=1}^{n}\sum_{k\geq 3} \frac{(2\lambda_i)^k}{k!} = n + 0 + 2n(n-1) + \sum_{i=1}^{n}\sum_{k\geq 3} \frac{(2\lambda_i)^k}{k!}. \end{aligned}$$

Since $\sum_{k\geq 3}\frac{(2\lambda_i)^k}{k!}\geq 8\sum_{k\geq 3}\frac{(\lambda_i)^k}{k!}$, we shall use a multiplier $\gamma\in[0,8]$, so as to arrive at

$$\begin{aligned}\sum_{i=1}^{n}e^{2\lambda_i} &\geq n+2n(n-1)+\gamma\sum_{i=1}^{n}\sum_{k\geq 3}\frac{(\lambda_i)^k}{k!}\\ &= n+2n(n-1)-\gamma n-\frac{1}{2}\gamma n(n-1)+\gamma\sum_{i=1}^{n}\sum_{k\geq 0}\frac{(\lambda_i)^k}{k!}\\ &= n+2n(n-1)-\gamma n-\frac{1}{2}\gamma n(n-1)+\gamma SEE(G). \end{aligned}\tag{8}$$

By substituting (7) and (8) back into (6) and solving for $SEE(G)$, we obtain

$$SEE(G)\geq\frac{\gamma}{2}+\sqrt{\frac{\gamma^2}{4}+n^2(3-\gamma/2)-n(2+\gamma/2)}.\tag{9}$$

Now, we consider a function

$$f(x)=\frac{x}{2}+\sqrt{\frac{x^2}{4}+n^2\left(3-\frac{x}{2}\right)-n\left(2+\frac{x}{2}\right)}.\tag{10}$$

We have $f'(x)<0$ for $x\geq 0$. Thus $f(x)$ is a monotonically decreasing function for $x>0$. Consequently, the best lower bound for $SEE(G)$ is attained $\gamma=0$. Setting $\gamma=0$ in (9), we arrive at the first half of Theorem 2.4:

$$SEE(G)\geq\sqrt{n(3n-2)}.$$

Now, we have to prove that the lower bound is strict. For this purpose, we assume that the left equality holds in (5). Then we have

$$e^{\lambda_i+\lambda_j}=e^{\lambda_k+\lambda_\ell},\quad\text{for any } i,j,k,\ell\in\{1,2,\dots,n\},$$

that is,

$$e^{\lambda_1+\lambda_2}=e^{\lambda_1+\lambda_3}=\cdots=e^{\lambda_1+\lambda_n}=e^{\lambda_2+\lambda_3},$$

and hence

$$\lambda_1=\lambda_2=\cdots=\lambda_n.$$

By Lemma 2.3 and the trace of $S(G)$, we can get a contradiction. Thus the left equality in (5) is strict.

(b) Let us prove now the right inequality.

Since $f(x)=e^x$ monotonically increases in the interval $(-\infty,\infty)$, we starting with equation (2), we get

$$\begin{aligned}SEE(G) &= n+\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{(\lambda_i)^k}{k!}\leq n+\sum_{i=1}^{n}\sum_{k\geq 1}\frac{(|\lambda_i|)^k}{k!}\\ &= n+\sum_{k\geq 1}\sum_{i=1}^{n}\frac{[(\lambda_i)^2]^{\frac{k}{2}}}{k!}\end{aligned}$$

$$\begin{aligned}&\leq n+\sum_{k\geq 1}\frac{1}{k!}\big(S_2(G)\big)^{\frac{k}{2}}=n+\sum_{k\geq 1}\frac{(\sqrt{n(n-1)})^k}{k!}\\&=n-1+\sum_{k=0}^{\infty}\frac{(\sqrt{n(n-1)})^k}{k!}=n-1+e^{\sqrt{n(n-1)}}.\end{aligned}\tag{11}$$

Suppose that the right equality holds in (5). Then the equality holds in (11). Thus we have $\lambda_i=|\lambda_i|$, $i=1,2,\ldots,n$. Since $\lambda_1\geq 1$ and by Lemma 2.1(i), again we get a contradiction. Hence the right inequality in (5) is strict. □

Theorem 2.5 *Let G be a conference graph. Then the Seidel-Estrada index of G is equal to*

$$SEE(G)=nch(\sqrt{n-1}),\tag{12}$$

where $ch(x)$ is the hyperbolic cosine of x defined as follows:

$$ch(x)=\frac{e^x+e^{-x}}{2}.$$

Proof Since G is a conference graph, the Seidel matrix of a graph is symmetric and $SS^T=(n-1)I$, thus each Seidel eigenvalue equals $\lambda_i=\sqrt{n-1}$ or $\lambda_i=-\sqrt{n-1}$. Let the number of positive eigenvalues of Seidel matrix $S(G)$ be n_+. Hence, $\lambda_i=\pm\sqrt{n-1}$ and $S_1=\sum_{i=1}^{n}\lambda_i=\sum_{i=1}^{n_+}\sqrt{n-1}+\sum_{i=n_++1}^{n}-\sqrt{n-1}=0$, then $n_+=\frac{n}{2}$. Therefore

$$\begin{aligned}SEE(G)&=\sum_{i=1}^{n}e^{\lambda_i}=\sum_{i=1}^{n_+}e^{\lambda_i}+\sum_{i=n_++1}^{n}e^{\lambda_i}=\sum_{i=1}^{n_+}e^{\sqrt{n-1}}+\sum_{i=n_++1}^{n}e^{-\sqrt{n-1}}\\&=\sum_{i=1}^{n_+}\big(e^{\sqrt{n-1}}+e^{-\sqrt{n-1}}\big)=2\sum_{i=1}^{\frac{n}{2}}ch(\sqrt{n-1})=nch(\sqrt{n-1}).\end{aligned}$$

□

3 Relation between Seidel-Estrada index and Seidel energy

Let G be a simple graph of order n, and its Seidel eigenvalues will be denoted by $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$. The Seidel energy $E_s(G)$ of graph G is defined by $E_s(G)=E_s=\sum_{i=1}^{n}|\lambda_i|$ [1]. Since

$$\sum_{i=1}^{n}\lambda_i=0,$$

we have

$$E_s(G)=2\sum_{i=1}^{n_+}\lambda_i=-2\sum_{i=n_++1}^{n}\lambda_i.\tag{13}$$

In this section, we investigate the relation between the Seidel-Estrada index and the Seidel energy.

Theorem 3.1 *The Seidel-Estrada index $SEE(G)$ and the Seidel energy $E_s(G)$ satisfy the following inequality*:

$$\frac{e}{2}E_s(G)+(n-n_+)e^{-\frac{E_s(G)}{2(n-n_+)}}\leq SEE(G)\leq n-1+e^{E_s(G)}\tag{14}$$

with left equality holding if and only if $G\cong K_n$.

Proof (a) At first, we prove the left inequality of (14).

For $G \cong K_n$, $SEE(G) = (n-1)e + e^{-n+1}$ and hence the left equality holds in (14). Otherwise, we have to prove that the lower bound is strict for $G \ncong K_n$. We have $e^x \geq ex$ with equality holding if and only if $x = 1$. By the arithmetic-geometric mean inequality, we get

$$\sum_{i=n_+ +1}^{n} e^{\lambda_i} \geq (n-n_+)\left(\prod_{i=n_+ +1}^{n} e^{\lambda_i}\right)^{\frac{1}{n-n_+}} = (n-n_+)\left(e^{\sum_{i=n_+ +1}^{n}\lambda_i}\right)^{\frac{1}{n-n_+}} = (n-n_+)e^{-\frac{E_S(G)}{2(n-n_+)}}.$$

Using the above result, we have

$$\begin{aligned} SEE(G) &= \sum_{i=1}^{n} e^{\lambda_i} = \sum_{\lambda_i>0} e^{\lambda_i} + \sum_{\lambda_i\leq 0} e^{\lambda_i} \\ &\geq \sum_{i=1}^{n_+} e\lambda_i + (n-n_+)e^{-\frac{E_S(G)}{2(n-n_+)}} \\ &= \frac{e}{2}E_s(G) + (n-n_+)e^{-\frac{E_S(G)}{2(n-n_+)}}. \end{aligned}$$

Suppose that the left equality holds in (14) for $G \ncong K_n$. Then we must have

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n_+} = 1 \quad \text{and} \quad \lambda_{n_+ +1} = \lambda_{n_+ +2} = \cdots = \lambda_n.$$

Since $G \ncong K_n$, we see that $K_{1,2}$ is an induced subgraph G or $K_2 \cup K_2$ is an induced subgraph of G. We have $\lambda_1(G) \geq \lambda_1(K_{1,2}) = 2$ and $\lambda_1(G) \geq \lambda_1(K_2 \cup K_2) = 3$. In both cases, we get a contradiction.

(b) Upper bound:

Starting with equation (2), we get

$$\begin{aligned} SEE(G) &= n + \sum_{i=1}^{n}\sum_{k\geq 1}\frac{(\lambda_i)^k}{k!} \leq n + \sum_{i=1}^{n}\sum_{k\geq 1}\frac{|\lambda_i|^k}{k!} \\ &\leq n + \sum_{k\geq 1}\frac{1}{k!}\left(\sum_{i=1}^{n}|\lambda_i|\right)^k = n-1+\sum_{k=0}^{\infty}\frac{(E_s)^k}{k!} = n-1+e^{E_s(G)}. \end{aligned} \tag{15}$$

Suppose now that the right equality holds in (14). Then all the above inequalities must be equalities. From (15), we have $|\lambda_i| = \lambda_i$, for all i. By the trace of $S(G)$, we have $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, a contradiction by Lemma 2.3. This completes the proof of the theorem. □

Remark 3.2 From equation (15) and Lemma 2.1, we get

$$\begin{aligned} SEE(G) &\leq n + \sum_{i=1}^{n}\sum_{k\geq 1}\frac{|\lambda_i|^k}{k!} = n + E_s(G) + \sum_{i=1}^{n}\sum_{k\geq 2}\frac{|\lambda_i|^k}{k!} \\ &= n + E_s(G) + \sum_{i=1}^{n}\sum_{k\geq 2}\frac{[(\lambda_i)^2]^{\frac{k}{2}}}{k!} \leq n + E_s(G) + \sum_{k\geq 2}\frac{1}{k!}\left[\sum_{i=1}^{n}(\lambda_i)^2\right]^{\frac{k}{2}} \\ &= n + E_s(G) - 1 - \sqrt{n(n-1)} + \sum_{k\geq 0}\frac{(\sqrt{n(n-1)})^k}{k!}. \end{aligned}$$

Hence

$$SEE(G) - E_s(G) \leq n - 1 - \sqrt{n(n-1)} + e^{\sqrt{n(n-1)}}. \tag{16}$$

Equality does not hold because if the equality is to occur, then we have $|\lambda_i| = \lambda_i$, for all i. Hence by Lemma 2.1(i), again we get a contradiction. We also have

$$SEE(G) - E_s(G) < \sqrt{n-1}(\sqrt{n-1} - \sqrt{n}) + e^{\sqrt{n(n-1)}}$$

and $SEE(G) < n - 1 + e^{E_s}$; we also give an inequality between the $SEE(G)$ and $E_s(G)$.

Theorem 3.3 *Let G be a simple graph with n vertices. Then*

$$e^{|\lambda_1|} + e^{|\lambda_2|} + \cdots + e^{|\lambda_n|} \geq 1 + e^{\frac{2E_s}{n}} + (n-2)e^{\frac{E_s}{n}} \tag{17}$$

with equality holding if and only if $|\lambda_1| = \frac{2E_s}{n}, |\lambda_2| = |\lambda_3| = \cdots = |\lambda_{n-1}| = \frac{E_s}{n}, |\lambda_n| = 0$.

Proof We have the Seidel eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $|\lambda_1| > 0$, $|\lambda_n| \geq 0$. Then, by the arithmetic-geometric mean inequality, we get

$$\begin{aligned} e^{|\lambda_1|} + e^{|\lambda_2|} + \cdots + e^{|\lambda_n|} &\geq e^{|\lambda_1|} + e^{|\lambda_n|} + (n-2)\left(\prod_{i=2}^{n-1} e^{|\lambda_i|}\right)^{\frac{1}{n-2}} \\ &= e^{|\lambda_1|} + e^{|\lambda_n|} + (n-2)\left(e^{E_s - |\lambda_1| - |\lambda_n|}\right)^{\frac{1}{n-2}} \end{aligned} \tag{18}$$

as $E_s = \sum_{i=1}^{n} |\lambda_i|$. Now, we consider the function

$$f(x,y) = e^x + e^y + (n-2)e^{\frac{E_s - x - y}{n-2}}, \quad \text{for } x > 0, y \geq 0.$$

We have

$$\begin{aligned} &\frac{\partial f}{\partial x} = f_x = e^x - e^{\frac{E_s - x - y}{n-2}}, \qquad \frac{\partial f}{\partial y} = f_y = e^y - e^{\frac{E_s - x - y}{n-2}} \\ &f_{xx} = e^x + \frac{1}{n-2}e^{\frac{E_s - x - y}{n-2}}, \qquad f_{yy} = e^y + \frac{1}{n-2}e^{\frac{E_s - x - y}{n-2}} \\ &f_{xy} = f_{yx} = \frac{1}{n-2}e^{\frac{E_s - x - y}{n-2}}. \end{aligned}$$

To find the minimum of the function of $f(x,y)$, we get

$$f_x = f_y = 0 \quad \Rightarrow \quad (n-1)x + y = E_s, \qquad x + (n-1)y = E_s \quad \Rightarrow \quad x + y = \frac{2E_s}{n}. \tag{19}$$

For $x + y = \frac{2E_s}{n}$, we have $f_{xx} > 0$ and

$$\begin{aligned} f_{xx}f_{yy} - f_{xy}^2 &= \left(e^x + \frac{1}{n-2}e^{\frac{E_s - x - y}{n-2}}\right)\left(e^y + \frac{1}{n-2}e^{\frac{E_s - x - y}{n-2}}\right) - \left(\frac{1}{n-2}e^{\frac{E_s - x - y}{n-2}}\right)^2 \\ &= e^{x+y} + \frac{1}{n-2}e^{\frac{E_s - x - y}{n-2}}\left(e^x + e^y\right) = e^{\frac{2E_s}{n}} + \frac{1}{n-2}e^{\frac{E_s}{n}}\left(e^x + e^{\frac{2E_s}{n} - x}\right) > 0. \end{aligned}$$

From the above, we conclude that $f(x,y)$ has a minimum value at $x+y=\frac{2E_S}{n}$ and the minimum value is $e^y+e^{\frac{2E_S}{n}-y}+(n-2)e^{\frac{E_S-\frac{2E_S}{n}}{n-2}}$. Now we can see easily that $g(y)=e^y+e^{\frac{2E_S}{n}-y}+(n-2)e^{\frac{E_S}{n}}$ is an increasing function for $y\geq 0$. Thus

$$e^{|\lambda_n|}+e^{\frac{2E_S}{n}-|\lambda_n|}+(n-2)e^{\frac{E_S}{n}}\geq e^0+e^{\frac{2E_S}{n}-0}+(n-2)e^{\frac{E_S}{n}}=1+e^{\frac{2E_S}{n}}+(n-2)e^{\frac{E_S}{n}}.$$

Hence we get the required result in (17).

Now suppose that equality holds in (17). Then all inequalities in the above argument must be equalities. From equality in (18) and $E_s=\sum_{i=1}^n|\lambda_i|$, we get $|\lambda_2|=|\lambda_3|=\cdots=|\lambda_{n-1}|=\frac{E_S}{n}$ as $|\lambda_1|+|\lambda_n|=\frac{2E_S}{n}$. Thus, $|\lambda_1|=\frac{2E_S}{n}$, $|\lambda_2|=|\lambda_3|=\cdots=|\lambda_{n-1}|=\frac{E_S}{n}$, $|\lambda_n|=0$.

Conversely, one can easily see that equality holds in (17) for a Seidel matrix of graph by $|\lambda_1|=\frac{2E_S}{n}$, $|\lambda_2|=|\lambda_3|=\cdots=|\lambda_{n-1}|=\frac{E_S}{n}$, $|\lambda_n|=0$. □

4 Conclusion

In this paper, we investigate the Seidel matrix and Seidel eigenvalues. Moreover, we defined the Seidel-Estrada index and Seidel energy, and computed the upper and lower bounds for the Seidel-Estrada index. We obtained a relation between the Seidel-Estrada index and the Seidel energy of a graph G, and we proved several theorems on the Seidel-Estrada index. The reader can use these results to calculate the Seidel energy and the Seidel-Estrada index.

Competing interests
The authors declare that they have no conflict of interest.

Authors' contributions
All authors contributed equally in drafting this manuscript and giving the main proofs. All authors read and approved the final manuscript.

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