

RESEARCH

Open Access



# On higher-order adjacent derivative of perturbation map in parametric vector optimization

Le Thanh Tung\*

\*Correspondence:  
lttung@ctu.edu.vn  
Department of Mathematics,  
College of Natural Sciences, Can  
Tho University, Cantho, Vietnam

## Abstract

This paper deals with higher-order sensitivity analysis in terms of the higher-order adjacent derivative for nonsmooth vector optimization. The relations between the higher-order adjacent derivative of the minima/the proper minima/the weak minima of a multifunction and its profile map are given. Then the relationships between the higher-order adjacent derivative of the perturbation map/the proper perturbation map/the weak perturbation map, and the higher-order adjacent derivative of a feasible map in objective space are considered. Finally, the formulas for estimating the higher-order adjacent derivative of the perturbation map, the proper perturbation map, the weak perturbation map via the adjacent derivative of the constraint map, and the higher-order Fréchet derivative of the objective map are also obtained.

**MSC:** 90C46; 49J52; 46G05; 90C26; 90C29

**Keywords:** higher-order adjacent derivative; parameterized vector optimization problem; perturbation map; proper perturbation map; weak perturbation map; higher-order sensitivity analysis

## 1 Introduction

Sensitivity analysis provides quantitative information as regards the solution map of a parameterized multiobjective optimization problem. A number of interesting results have been obtained in sensitivity analysis for multiobjective optimization problems. One of the first results was given by Tanino in [1, 2]. By using the first-order contingent derivative, some results concerning the behavior of perturbation maps were obtained. The TP-derivative was presented in [3] and used to weaken some assumptions in [1, 4]. References [5–8] investigated the perturbation map in nonsmooth convex problems. In [9–11], the Clarke derivatives were used for analyzing the sensitivity. The concept of the proto-differentiability of a multifunction, in which the contingent cone coincides with the adjacent cone at a point to its graph, was presented by Rockafellar in [12]. In [13, 14], the important results on the proto-differentiability of the efficient solution maps were obtained for generalized equations, a general model including optimization problems. Some developments were obtained in [15, 16]. A second-order sensitivity analysis via the second-order contingent derivatives were considered in [17, 18]. In [19], the second-order proto-differentiability of a multifunction was proposed to discuss the second-order sensitivity

properties for generalized perturbation maps. The second-order radial-asymptotic derivative, introduced in [20], was used in qualification conditions to consider the second-order proto-differentiability of the efficient solution map and the efficient frontier map of a parameterized vector optimization problem in [21]. Some results in higher-order sensitivity analysis using a higher-order adjacent derivative in [22] and a higher-order contingent derivative in [23] of perturbation maps in a parameterized vector optimization were given. Using higher-order variational sets, presented in [24], some results in higher-order sensitivity analysis were obtained in [25].

Unlike higher-order contingent derivatives based on encounter information, the  $m$ th-order variations of a map, based on the different rates of change of the point under consideration in the domain space and the range space of a map, were proposed to obtain the open mapping principle in [26] and consider Hölder metric regularity of set-valued maps in [27]. Another kind of  $m$ th-order derivatives, presented in [28], was used to establish the optimality condition for isolated local minima of nonsmooth functions and modified to characterize weak sharp minima in [29]. The  $m$ th-order derivatives in [28] were generalized to set-valued maps in [30–32] to establish higher-order optimality conditions. In [31], the higher-order sensitivity was considered by using the  $m$ th-order contingent-type derivatives. In [33], the lower Studniarski derivative of a perturbation map in vector optimization was considered.

To the best of our knowledge, there is no paper dealing with the sensitivity of the  $m$ th-order adjacent derivatives of perturbation maps of parameterized vector optimization problems. Moreover, the proper perturbation maps and the case that the objective function is higher-order Fréchet differentiable in constraint vector optimization have not been considered yet. Motivated by the above observations, in this paper, by making use of the  $m$ th-order adjacent derivatives of set-valued maps which were introduced in [31], we investigate quantitatively the perturbation map, the proper perturbation map, and the weak perturbation map of parameterized vector optimization problems. The paper is organized as follows. Section 2 contains preliminary facts we need in the paper. In Section 3, the relations between the  $m$ th-order adjacent derivatives of a set-valued map and those of its profile map are discussed. The obtained results are employed in Section 4 to investigate the relationships between the  $m$ th-order adjacent derivatives of the perturbation map/the proper perturbation map/the weak perturbation map and the  $m$ th-order adjacent derivative of the feasible map in the objective space. In Section 4, the formulas for estimating the  $m$ th-order adjacent derivatives of the perturbation map, the proper perturbation map, and the weak perturbation map via the adjacent derivative of constraint map and the  $m$ th-order Fréchet derivative of the objective map are also given.

## 2 Preliminaries

In this paper, if not otherwise stated, let  $X$ ,  $Y$ , and  $Z$  be normed spaces, and  $C \subseteq Y$  be a pointed closed convex cone.  $\mathcal{U}(x_0)$  is used for the set of neighborhoods of  $x_0$ .  $\mathbb{R}$ ,  $\mathbb{R}_+$ , and  $\mathbb{N}$  stand for the set of the real numbers, nonnegative real numbers, and natural numbers, respectively (shortly, resp.). For  $M \subseteq X$ ,  $\text{int}M$ ,  $\text{cl}M$ ,  $\text{bd}M$  denote its interior, closure, and boundary, resp. A convex set  $B \subseteq Y$  is called a base of  $C$  iff  $0 \notin \text{cl}B$  and  $C = \{tb \mid t \in \mathbb{R}_+, b \in B\}$ . Clearly  $C$  has a compact base  $B$  if and only if  $C \cap \text{bd}B$  is compact. If  $Y$  is a finite dimensional space, then  $C$  has a compact base. For  $F : X \rightrightarrows Y$ , the domain, graph,

and epigraph of  $F$  are defined by, resp.,

$$\begin{aligned} \text{dom } F &:= \{x \in X \mid F(x) \neq \emptyset\}, & \text{gr } F &:= \{(x, y) \in X \times Y \mid y \in F(x)\}, \\ \text{epi } F &:= \{(x, y) \in X \times Y \mid y \in F(x) + C\}. \end{aligned}$$

The profile map of  $F$  is  $F + C$  (defined by  $(F + C)(x) := F(x) + C$ ). We recall some concepts of optimality/efficiency in vector optimization as follows, for  $a_0 \in A \subseteq Y$ .

- (i)  $a_0$  is called a local (Pareto) minimal/efficient point of  $A$  (with respect to  $C$ ), and denoted by  $a_0 \in \text{Min}_C A$ , iff there exists  $U \in \mathcal{U}(a_0)$  such that

$$(A \cap U - a_0) \cap (-C \setminus \{0\}) = \emptyset.$$

- (ii) Supposing that  $\text{int } C \neq \emptyset$ ,  $a_0$  is said to be a local weak minimal/efficient point of  $A$ , denoted by  $a \in \text{WMin}_C A$ , iff there exists  $U \in \mathcal{U}(a_0)$  such that

$$(A \cap U - a_0) \cap (-\text{int } C) = \emptyset.$$

- (iii) Assuming that  $C$  is pointed,  $a_0$  is termed a proper minimal/efficient point of  $A$ , denoted by  $a_0 \in \text{PrMin}_C A$ , iff there exists a convex cone  $K \subsetneq Y$  with  $C \setminus \{0\} \subseteq \text{int } K$  and  $U \in \mathcal{U}(a_0)$  such that

$$(A \cap U - a_0) \cap (-K) = \{0\}.$$

If  $U = Y$ , the word ‘local’ is omitted, *i.e.*, we have the corresponding global notions. For a subset  $A \subseteq Y$ ,  $A$  is said to have the domination property iff  $A \subseteq \text{Min}_C A + C$  and  $A$  is said to have the proper domination property iff  $A \subseteq \text{PrMin}_C A + C$ . Similarly, when  $\text{int } C \neq \emptyset$ ,  $A$  has the weak domination property iff  $A \subseteq \text{WMin}_C A + \text{int } C \cup \{0\}$ .

Recall now the four kinds of higher-order derivatives which we are most concerned with in the sequel. Let  $F : X \rightrightarrows Y$ ,  $u \in X$ ,  $m \in \mathbb{N}$ , and  $(x_0, y_0) \in \text{gr } F$ .

- (i) ([31]) The  $m$ th-order radial-contingent derivative of  $F$  at  $(x_0, y_0)$  is defined by

$$D_S^m F(x_0, y_0)(u) := \left\{ v \in Y \mid \exists t_n > 0, \exists (u_n, v_n) \rightarrow (u, v) : t_n u_n \rightarrow 0, \right. \\ \left. y_0 + t_n^m v_n \in F(x_0 + t_n u_n) \right\}.$$

- (ii) ([32]) The  $m$ th-order contingent-type derivative of  $F$  at  $(x_0, y_0)$  is defined by

$$D^m F(x_0, y_0)(u) := \left\{ v \in Y \mid \exists t_n \downarrow 0, \exists (u_n, v_n) \rightarrow (u, v), y_0 + t_n^m v_n \in F(x_0 + t_n u_n) \right\}.$$

- (iii) ([31]) The  $m$ th-order adjacent derivative of  $F$  at  $(x_0, y_0)$  is defined by

$$D^{hm} F(x_0, y_0)(u) := \left\{ v \in Y \mid \forall t_n \downarrow 0, \exists (u_n, v_n) \rightarrow (u, v), y_0 + t_n^m v_n \in F(x_0 + t_n u_n) \right\}.$$

- (iv) ([32]) The  $m$ th-order lower Studniarski derivative of  $F$  at  $(x_0, y_0)$  is defined by

$$D^{lm} F(x_0, y_0)(u) := \left\{ v \in Y \mid \forall t_n \downarrow 0, \forall u_n \rightarrow u, \exists v_n \rightarrow v, y_0 + t_n^m v_n \in F(x_0 + t_n u_n) \right\}.$$

**Remark 2.1**  $D^{lm}F(x_0, y_0)(u) \subseteq D^{bm}F(x_0, y_0)(u) \subseteq D^mF(x_0, y_0)(u), \forall u \in X$ .

The reverse conclusions in Remark 2.1 may not hold. The following examples show the cases.

**Example 2.1** Let  $\mathcal{I} = \{\frac{1}{n} : n \in \mathbb{N}\}$  and  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  be defined by

$$F(x) = \begin{cases} \{0\}, & \text{if } x \leq 0, \\ \{-x^2\}, & \text{if } x \in \mathcal{I}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, for  $(x_0, y_0) = (0, 0) \in \text{gr } F$ , we can check that

$$D^2F(x_0, y_0)(2) = \{-4\}.$$

Taking  $t_n = \frac{2}{n}$ , then  $t_n u_n = \frac{2u_n}{n} \notin \mathcal{I}$  for all  $u_n \rightarrow 2$ . Indeed, suppose to the contrary that there exists a subsequence  $\{\frac{1}{k}\} \subseteq \mathcal{I}, k \geq n$ , such that  $t_n u_n = \frac{1}{k}$ . Then  $u_n = \frac{1}{2} \cdot \frac{n}{k} \leq \frac{1}{2}$ , i.e.,  $u_n \not\rightarrow 2$ , a contradiction. Hence, for the above  $t_n, F(x_0 + t_n u_n) = \emptyset$ . Consequently,

$$D^{b2}F(x_0, y_0)(2) = \emptyset.$$

Hence,

$$D^2F(x_0, y_0)(2) \not\subseteq D^{b2}F(x_0, y_0)(2).$$

**Example 2.2** Let  $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$  be defined by

$$F(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq x^4, y_2 \leq 0\}, & \text{if } x < 0, \\ \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 \geq 2x^4\}, & \text{if } x \geq 0. \end{cases}$$

Then, for  $(x_0, y_0) = (0, (0, 0))$ ,

$$D^{b2}F(x_0, y_0)(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 \leq 0\}, & \text{if } x < 0, \\ \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 \in \mathbb{R}\}, & \text{if } x = 0, \\ \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 \geq 0\}, & \text{if } x > 0, \end{cases}$$

and

$$D^{l2}F(x_0, y_0)(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 \leq 0\}, & \text{if } x < 0, \\ \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 = 0\}, & \text{if } x = 0, \\ \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 \geq 0\}, & \text{if } x > 0. \end{cases}$$

Hence,

$$D^{b2}F(x_0, y_0)(0) \not\subseteq D^{l2}F(\bar{x}, \bar{y})(0).$$

**Definition 2.1** (see [31]) For  $u \in X, F : X \rightrightarrows Y$  is called  $m$ th-order  $u$ -directionally contingent compact at  $(x_0, y_0) \in \text{gr } F$  iff, for any  $t_n \downarrow 0, (u_n, v_n) \in X \times Y$  such that  $u_n \rightarrow u$ , and  $y_0 + t_n^m v_n \in F(x_0 + t_n u_n)$  for all  $n$ , there exists a convergent subsequence of  $\{v_n\}$ .

**Definition 2.2** Let  $F : X \rightrightarrows Y$  be a set-valued map,  $x_0 \in X$ ,  $m \in \mathbb{N} \setminus \{0\}$ , and  $\alpha > 0$ .

- (i)  $F$  is said locally Hölder continuous of order  $\alpha$  at  $(x_0, y_0) \in \text{gr } F$  if there exist  $\lambda > 0$ , and  $U \in \mathcal{U}(x_0)$  such that

$$F(x_2) \subset F(x_1) + \lambda \|x_1 - x_2\|^\alpha B_Y, \quad \forall x_1, x_2 \in U,$$

where  $B_Y$  stands for the closed unit ball in  $Y$ .

- (ii)  $F$  is said locally pseudo-Hölder calm of order  $m$  (see [34]) at  $(x_0, y_0) \in \text{gr } F$  if there exist a real number  $\lambda > 0$ ,  $\exists U \in \mathcal{U}(x_0)$ , and  $\exists V \in \mathcal{U}(y_0)$  such that

$$F(x) \cap V \subset \{y_0\} + \lambda \|x - x_0\|^m B_Y, \quad \forall x \in U.$$

When  $m = 1$ , the word ‘Hölder’ is replaced by ‘Lipschitz’. If  $V = Y$ , then ‘locally pseudo-Hölder calm’ is replaced by ‘locally Hölder calm’. In [13],  $F$  is called upper locally Lipschitz at  $x_0 \in \text{dom } F$  if there exist a real number  $\lambda > 0$  and  $U \in \mathcal{U}(x_0)$  such that  $F(x) \subset F(x_0) + \lambda \|x - x_0\| B_Y, \forall x \in U$ . It is easy to see that if  $F$  is upper locally Lipschitz at  $x_0$  and  $F(x_0) = \{y_0\}$  then  $F$  is locally Lipschitz calm at  $(x_0, y_0)$ .

**Proposition 2.1** (see [31]) *Let  $F : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gr } F$ , and  $Y$  be finite dimensional space. If  $D_S^m F(x_0, y_0)(0) = \{0\}$ , then  $F$  is  $m$ th-order  $u$ -directionally contingent compact at  $(x_0, y_0)$  for all  $u \in X$ .*

**Proposition 2.2** *Let  $F : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gr } F$ , and  $Y$  be finite dimensional space. If  $F$  is locally Hölder calm of order  $m$  at  $(x_0, y_0) \in \text{gr } F$ , then  $D_S^m F(x_0, y_0)(0) = \{0\}$ .*

*Proof* Consider an arbitrary  $y \in D_S^m F(x_0, y_0)(0)$ . Then there exist  $y_n \rightarrow y$ ,  $x_n \rightarrow 0$ , and  $t_n > 0$  such that  $y_0 + t_n^m y_n \in F(x_0 + t_n x_n)$  and  $t_n x_n \rightarrow 0$ . Since  $F$  is locally Hölder calm of order  $m$  at  $(x_0, y_0)$ , we derive that, for  $n$  large enough, there exists  $\lambda > 0$  such that

$$y_0 + t_n^m y_n \in y_0 + \lambda \|t_n x_n\|^m B_Y.$$

Consequently,

$$y_n \in \lambda \|x_n\|^m B_Y.$$

It follows from the above equation that  $x_n \rightarrow 0$ , and  $y_n \rightarrow y$  that one has  $y = 0$ . □

**Corollary 2.1** *Let  $F : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gr } F$ , and  $Y$  be finite dimensional space. If  $F$  is locally Hölder calm of order  $m$  at  $(x_0, y_0) \in \text{gr } F$ , then  $F$  is  $m$ th-order  $u$ -directionally contingent compact at  $(x_0, y_0)$  for all  $u \in X$ .*

*Proof* It follows from Proposition 2.1 and Proposition 2.2 that the conclusion is obtained. □

**Definition 2.3** (see [31]) *Let  $f : X \rightarrow Y$  be a vector-valued map.  $f$  is said to be  $m$ th-order Fréchet differentiable at  $x_0 \in X$  iff there exists a linear continuous operator  $d^m f(x_0) : X \times$*

$\dots \times X \rightarrow Y$  ( $m$  times  $X$ ), such that

$$f(x) = f(x_0) + d^m f(x_0)(x - x_0, \dots, x - x_0) + o(\|x - x_0\|^m) \quad (m \text{ times } x - x_0),$$

where  $o(\|x - x_0\|^m)$  satisfies  $o(\|x - x_0\|^m)/\|x - x_0\|^m \rightarrow 0$  when  $x \rightarrow x_0$ .  $d^m f(x_0)$  is called the  $m$ th-order Fréchet derivative.  $f$  is said  $m$ th-order Fréchet differentiable on  $X$  if  $f$  is  $m$ th-order Fréchet differentiable at any  $x \in X$ . If  $d^m f(\cdot)$  is continuous at  $x_0$  then  $f$  is said to be  $m$ th-order continuously Fréchet differentiable at  $x_0$ .

**Remark 2.2** (see [31]) For  $f : X \rightarrow Y$  and  $x_0, u \in X$ , if there exists  $d^m f(x_0)$ , then

$$\begin{aligned} \{d^m f(x_0)(u, u, \dots, u)\} &= D^{lm} f(x_0, f(x_0))(u) = D^{bm} f(x_0, f(x_0))(u) \\ &= D^m f(x_0, f(x_0))(u) = D_S^m f(x_0, f(x_0))(u). \end{aligned}$$

### 3 Higher-order adjacent derivatives of set-valued maps

In this section, the relations between higher-order adjacent derivative of a set-valued map and those of its profile map are discussed. Such relations for various kinds of efficient points of these derivatives are also investigated.

**Proposition 3.1** Let  $(x_0, y_0) \in \text{gr } F$ . Then, for any  $u \in X$ ,

$$D^{bm} F(x_0, y_0)(u) + C \subseteq D^{bm}(F + C)(x_0, y_0)(u). \tag{1}$$

*Proof* Let  $z = v + c$  for some  $v \in D^{bm} F(x_0, y_0)(u)$  and  $c \in C$ . Then, for all  $t_n \downarrow 0$ , there exists  $(u_n, v_n) \rightarrow (u, v)$  such that  $y_0 + t_n^m v_n \in F(x_0 + t_n u_n)$  for all  $n$ . Setting  $\bar{v}_n := v_n + c$ , one has  $\bar{v}_n \rightarrow v + c$  and, for all  $n$ ,

$$y_0 + t_n^m \bar{v}_n = y_0 + t_n^m (v_n + c) \in F(x_0 + t_n u_n) + C = (F + C)(x_0 + t_n u_n).$$

So,  $z = v + c \in D^{bm}(F + C)(x_0, y_0)(u)$ . □

Note that the opposite inclusion of (1) may not hold. The following example illustrates the case.

**Example 3.1** Let  $C = \mathbb{R}_+$ ,  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  be defined by

$$F(x) = \begin{cases} \{-1, x^3\}, & \text{if } x \geq 0, \\ \{0\}, & \text{if } x < 0. \end{cases}$$

Let  $(x_0, y_0) = (0, 0) \in \text{gr } F$ . Then

$$D^{b3} F(0, 0)(u) = \begin{cases} \{u^3\}, & \text{if } u \geq 0, \\ \{0\}, & \text{if } u < 0, \end{cases} \quad D^{b3}(F + C)(0, 0)(u) = \begin{cases} \mathbb{R}, & \text{if } u \geq 0, \\ \mathbb{R}_+, & \text{if } u < 0. \end{cases}$$

Hence, for all  $u$ ,

$$D^{b3}(F + C)(x_0, y_0)(u) \not\subseteq D^{b3} F(x_0, y_0)(u) + C.$$

**Proposition 3.2** *Suppose that either of the following conditions is satisfied:*

- (i) *for any  $u \in X$ ,  $F$  is  $m$ th-order  $u$ -directionally contingent compact at  $(x_0, y_0)$ ;*
- (ii)  *$C$  has a compact base and  $D_S^m F(x_0, y_0)(0) \cap (-C) = \{0\}$ ;*
- (iii)  *$C$  has a compact base and  $D^{bm}(F + C)(x_0, y_0)(u)$  has domination property.*

*Then, for all  $u \in X$ ,*

$$D^{bm}F(x_0, y_0)(u) + C = D^{bm}(F + C)(x_0, y_0)(u).$$

*Proof* It follows from Proposition 3.1 that we only need to show the reverse inclusion of (1).

(i) Let  $v \in D^{bm}(F + C)(x_0, y_0)(u)$ . If  $(u, v) = (0, 0)$ , we have  $0 \in D^{bm}F(x_0, y_0)(0) + C$ . For  $(u, v) \neq (0, 0)$ , for all  $t_n \downarrow 0$ , there exists  $(u_n, v_n) \rightarrow (u, v)$  such that  $y_0 + t_n^m v_n \in (F + C)(x_0 + t_n u_n)$ ,  $\forall n$ . Hence, there exists  $c_n \in C$  such that  $y_0 + t_n^m(v_n - c_n/t_n^m) \in F(x_0 + t_n u_n)$ . By the  $u$ -directionally contingent compactness of  $F$  at  $(x_0, y_0)$ , we can assume that  $v_n - c_n/t_n^m \rightarrow \bar{v} \in D^{bm}F(x_0, y_0)(u)$ . Since  $c_n/t_n^m = v_n - (v_n - c_n/t_n^m) \rightarrow v - \bar{v}$  and  $C$  is a closed convex cone, one gets  $v - \bar{v} \in C$ . Hence,  $v \in \bar{v} + C \subseteq D^{bm}F(x_0, y_0)(u) + C$ .

(ii) Let  $u \in X$  and  $v \in D^{bm}(F + C)(x_0, y_0)(u)$  be arbitrary. As in (i), we need to consider only  $(u, v) \neq (0, 0)$ . We see that, for all  $t_n \downarrow 0$ , there exist  $(u_n, v_n) \rightarrow (u, v)$ , and  $c_n \in C$  such that  $y_0 + t_n^m v_n \in F(x_0 + t_n u_n) + c_n$  for all  $n$ . If there exists  $n_0$  such that  $c_n = 0$  for all  $n > n_0$ , then  $v \in D^{bm}F(x_0, y_0)(u) + 0 \subseteq D^{bm}F(x_0, y_0)(u) + C$ . Now, assume that  $c_n \neq 0$  and  $c_n/\|c_n\| \rightarrow \bar{c}$  for some  $\bar{c} \in C$  with norm one. There are only two cases for  $s_n := \sqrt[m]{\|c_n\|} > 0$ .

*Case 1:*  $s_n/t_n \rightarrow +\infty$ . Then  $s_n[(t_n/s_n)u_n] = t_n u_n \rightarrow 0$ . Since

$$y_0 + (s_n)^m [(t_n/s_n)^m v_n - c_n/s_n^m] \in F(x_0 + s_n [(t_n/s_n)u_n]),$$

$(t_n/s_n)^m v_n - c_n/s_n^m \rightarrow -\bar{c}$ , and  $(t_n/s_n)u_n \rightarrow 0$ , one has  $-\bar{c} \in D_S^m F(x_0, y_0)(0)$ , an impossibility.

*Case 2:*  $\{s_n/t_n\}$  is bounded, and assume  $s_n/t_n \rightarrow \alpha \geq 0$ . Then, since

$$y_0 + (t_n)^m [v_n - (s_n/t_n)^m (c_n/s_n^m)] \in F(x_0 + t_n u_n),$$

$v_n - (s_n/t_n)^m (c_n/s_n^m) \rightarrow v - \alpha^m \bar{c}$ , and  $u_n \rightarrow u$ , one gets  $v - \alpha^m \bar{c} \in D^{bm}F(x_0, y_0)(u)$ , and hence  $v \in D^{bm}F(x_0, y_0)(u) + C$ .

(iii) Since  $D^{bm}(F + C)(x_0, y_0)(u)$  has the domination property, for any  $u \in X$ ,

$$D^{bm}(F + C)(x_0, y_0)(u) \subseteq \text{Min}_C D^{bm}(F + C)(x_0, y_0)(u) + C. \tag{2}$$

We claim that, for any  $u \in X$ ,

$$\text{Min}_C D^{bm}(F + C)(x_0, y_0)(u) \subseteq D^{bm}F(x_0, y_0)(u). \tag{3}$$

Indeed, let  $v \in \text{Min}_C D^{bm}(F + C)(x_0, y_0)(u)$ . Then, for all  $t_n \downarrow 0$ , there exist  $(u_n, v_n) \rightarrow (u, v)$  and  $c_n \in C$  such that, for all  $n$ ,  $y_0 + t_n^m(v_n - c_n) \in F(x_0 + t_n u_n)$ . Since  $C$  has a compact base, we may assume that  $c_n = \alpha_n b_n$  with  $\alpha_n > 0$  and  $b_n \rightarrow b \neq 0$ . We now show that  $\alpha_n \rightarrow 0$ . Suppose to the contrary that  $\alpha_n \not\rightarrow 0$ . Then there exists  $\epsilon > 0$  such that  $\alpha_n \geq \epsilon$  for all  $n$ . Setting  $\bar{c}_n = (\epsilon/\alpha_n)c_n$ . Then, for any  $n$ ,  $c_n - \bar{c}_n = (1 - \epsilon/\alpha_n)c_n \in C$  and

$$y_0 + t_n^m(v_n - \bar{c}_n) = y_0 + t_n^m(v_n - c_n) + t_n^m(c_n - \bar{c}_n) \in F(x_0 + t_n u_n) + C = (F + C)(x_0 + t_n u_n).$$

Since  $v_n - \bar{c}_n = v_n - (\epsilon/\alpha_n)c_n = v_n - \epsilon b_n \rightarrow v - \epsilon b$ , we have  $v - \epsilon b \in D^{bm}(F + C)(x_0, y_0)(u)$  and  $v - (v - \epsilon b) = \epsilon b \in C \setminus \{0\}$ , which contradicts  $v \in \text{Min}_C D^{bm}(F + C)(x_0, y_0)(u)$ . Therefore,  $\alpha_n \rightarrow 0$  and  $v_n - c_n = v_n - \alpha_n b_n \rightarrow v$ , i.e.,  $v \in D^{bm}F(x_0, y_0)(u)$ . Thus, (3) holds. It follows from (2) and (3) that

$$D^{bm}(F + C)(x_0, y_0)(u) \subseteq D^{bm}F(x_0, y_0)(u) + C, \quad \forall u \in X.$$

The proof is complete. □

**Proposition 3.3** *Suppose that either of the following conditions holds:*

- (i) *for any  $u \in X$ ,  $F$  is  $m$ th-order  $u$ -directionally contingent compact at  $(x_0, y_0)$ ;*
- (ii)  *$C$  has a compact base and  $D_S^m F(x_0, y_0)(0) \cap (-C) = \{0\}$ ;*
- (iii)  *$C$  has a compact base and  $D^{bm}(F + C)(x_0, y_0)(u)$  has domination property.*

*Then, for all  $u \in X$ ,*

$$\text{Min}_C D^{bm}F(x_0, y_0)(u) = \text{Min}_C D^{bm}(F + C)(x_0, y_0)(u).$$

*Proof* Similarly to the proof of Proposition 4.3 in [31], we obtain the conclusion. □

Since the following propositions are proven similarly, the proofs are omitted.

**Proposition 3.4** *Suppose that either of the following conditions holds:*

- (i) *for any  $u \in X$ ,  $F$  is  $m$ th-order  $u$ -directionally contingent compact at  $(x_0, y_0)$ ;*
- (ii)  *$C$  has a compact base and  $D_S^m F(x_0, y_0)(0) \cap (-C) = \{0\}$ ;*
- (iii)  *$C$  has a compact base and  $D^{bm}(F + C)(x_0, y_0)(u)$  has proper domination property.*

*Then, for all  $u \in X$ ,*

$$\text{PrMin}_C D^{bm}F(x_0, y_0)(u) = \text{PrMin}_C D^{bm}(F + C)(x_0, y_0)(u).$$

**Proposition 3.5** *Assume that  $\text{int } C \neq \emptyset$  and  $\tilde{K}$  is a closed convex cone with  $\tilde{K} \subseteq \text{int } C \cup \{0\}$ . Suppose further that either of the following conditions holds:*

- (i) *for any  $u \in X$ ,  $F$  is  $m$ th-order  $u$ -directionally contingent compact at  $(x_0, y_0)$ ;*
- (ii)  *$C$  has a compact base and  $D_S^m F(x_0, y_0)(0) \cap (-\tilde{K}) = \{0\}$ ;*
- (iii)  *$C$  has a compact base and  $D^{bm}(F + \tilde{K})(x_0, y_0)(u)$  has weak domination property.*

*Then, for all  $u \in X$ ,*

$$\text{WMin}_C D^{bm}F(x_0, y_0)(u) = \text{WMin}_C D^{bm}(F + \tilde{K})(x_0, y_0)(u).$$

The following example illustrates that we cannot replace  $\tilde{K}$  by  $C$  in the conclusion of Proposition 3.5.

**Example 3.2** Let  $X = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $(x_0, y_0) = (0, (0, 0))$ ,  $C = \mathbb{R}_+^2$ , and  $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$  be defined by  $F(x) = \{(x^2, x^4)\}$ . Then we can check that  $D_S^2 F(x_0, y_0)(u) = D^{2b} F(x_0, y_0)(u) = \{(u^2, 0)\}$  for any  $u \in \mathbb{R}$ , and

$$D^{2b}(F + C)(x_0, y_0)(u) = \{(y_1, y_2) \in \mathbb{R}_+^2 : y_1 \geq u^2, y_2 \geq 0\}.$$



Hence,  $\text{WMin}_C D^{2b}F(x_0, y_0)(u) = \{(u^2, 0)\}$  and

$$\begin{aligned} & \text{WMin}_C D^{2b}(F + C)(x_0, y_0)(u) \\ &= \{(y_1, y_2) \in \mathbb{R}_+^2 \mid y_1 \geq u^2, y_2 = 0\} \cup \{(y_1, y_2) \in \mathbb{R}_+^2 \mid y_1 = u^2, y_2 \geq 0\}. \end{aligned}$$

Since  $D^2_5F(x_0, y_0)(0) \cap (-C) = \{0\}$  and  $C$  has a compact base  $B = \{(y_1, y_2) \in \mathbb{R}^2, y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\}$ , the assumption (ii) is fulfilled. We can check that

$$\text{WMin}_C D^{2b}(F + C)(x_0, y_0)(u) \not\subseteq \text{WMin}_C D^{2b}F(x_0, y_0)(u).$$

#### 4 Higher-order adjacent derivatives of perturbations maps

In this section, we consider the following parameterized vector optimization problem:

$$\text{Min}_K f(x, u) = (f_1(x, u), f_2(x, u), \dots, f_q(x, u)), \quad \text{s.t. } x \in X(u) \subseteq \mathbb{R}^l,$$

where  $x$  is a  $l$ -dimensional decision variable,  $u$  is a  $p$ -dimensional parameter,  $f_i$  is a real valued objective function on  $\mathbb{R}^l \times \mathbb{R}^p$  for  $i = 1, 2, \dots, q$ ,  $X$  is a set-valued map from  $\mathbb{R}^p$  to  $\mathbb{R}^l$ , which defines a feasible decision set, and  $K$  is a nonempty pointed closed convex ordering cone in  $\mathbb{R}^q$ . Let  $F(u)$  be the value at  $u$  of the feasible set map in the objective space, *i.e.*,

$$F(u) := \{y \in \mathbb{R}^q \mid y = f(x, u) \text{ for some } x \in X(u)\}.$$

We define the perturbation/frontier map  $\mathcal{F}$ , the weak perturbation/frontier map  $\mathcal{W}$ , and the proper perturbation/frontier map  $\mathcal{P}$  of the considered problem as follows:

$$\mathcal{F}(u) := \text{Min}_K F(u), \quad \mathcal{W}(u) := \text{WMin}_K F(u), \quad \mathcal{P}(u) := \text{PrMin}_K F(u).$$

For  $u_0 \in \mathbb{R}^p$  and a closed convex cone  $\tilde{K} \subseteq \mathbb{R}^q$ ,

- (i)  $F$  is said to be  $K$ -dominated by  $\mathcal{F}$  near  $u_0$  iff  $F(u) \subseteq \mathcal{F}(u) + K$ , for all  $u$  in some  $U \in \mathcal{U}(u_0)$ .
- (ii)  $F$  is said to be  $K$ -dominated by  $\mathcal{P}$  near  $u_0$  iff  $F(u) \subseteq \mathcal{P}(u) + K$ , for all  $u$  in some  $U \in \mathcal{U}(u_0)$ .
- (iii)  $F$  is said to be  $\tilde{K}$ -dominated by  $\mathcal{W}$  near  $u_0$  iff  $F(u) \subseteq \mathcal{W}(u) + \tilde{K}$ , for all  $u$  in some  $U \in \mathcal{U}(u_0)$ .

**Remark 4.1** Since  $\mathcal{F}(u) \subseteq F(u)$ , the  $K$ -dominatedness of  $F$  by  $\mathcal{F}$  near  $u_0$  implies that, for all  $u \in U$ ,  $\mathcal{F}(u) + K = F(u) + K$ . Hence, if  $F$  is  $K$ -dominated by  $\mathcal{F}$  near  $u_0$ , then for any  $u_0 \in \mathbb{R}^p$ ,  $y_0 \in \mathcal{F}(u_0)$ , and  $u \in U$ ,

$$D^{bm}(\mathcal{F} + K)(u_0, y_0)(u) = D^{bm}(F + K)(u_0, y_0)(u).$$

Similar assertions are true for  $\mathcal{P}$  and  $\mathcal{W}$  as follows.

- (i)  $D^{bm}(\mathcal{P} + K)(u_0, y_0)(u) = D^{bm}(F + K)(u_0, y_0)(u)$  for any  $u_0 \in \mathbb{R}^p$ ,  $y_0 \in \mathcal{P}(u_0)$ , and  $u \in U$  if  $F$  is  $K$ -dominated by  $\mathcal{P}$  near  $u_0$ .
- (ii)  $D^{bm}(\mathcal{W} + \tilde{K})(u_0, y_0)(u) = D^{bm}(F + \tilde{K})(u_0, y_0)(u)$  for any  $u_0 \in \mathbb{R}^p$ ,  $y_0 \in \mathcal{W}(u_0)$ , and  $u \in U$  if  $F$  is  $\tilde{K}$ -dominated by  $\mathcal{W}$  near  $u_0$ .

### 4.1 Higher-order adjacent derivatives of perturbation maps without constraints

Now, the relations between the higher-order adjacent derivative of feasible map and the higher-order adjacent derivative of perturbation/ weak perturbation maps are investigated in this subsection.

**Proposition 4.1** *Assume that  $F$  is  $m$ th-order  $u$ -directionally compact at  $(u_0, y_0)$  for any  $u \in \mathbb{R}^p$ .*

(i) *If  $F$  is  $K$ -dominated by  $\mathcal{F}$  near  $u_0$ , then, for  $u$  near  $u_0$ ,*

$$\text{Min}_K D^{bm} F(u_0, y_0)(u) \subseteq D^{bm} \mathcal{F}(u_0, y_0)(u).$$

(ii) *If  $F$  is  $K$ -dominated by  $\mathcal{P}$  near  $u_0$ , then, for  $u$  near  $u_0$ ,*

$$\text{PrMin}_K D^{bm} F(u_0, y_0)(u) \subseteq D^{bm} \mathcal{P}(u_0, y_0)(u).$$

(iii) *If  $\text{int } K \neq \emptyset$ , there is a closed convex cone  $\tilde{K}$  satisfying  $\tilde{K} \subseteq \text{int } K \cup \{0\}$ , and  $F$  is  $\tilde{K}$ -dominated by  $\mathcal{W}$  near  $u_0$ , then, for  $u$  near  $u_0$ ,*

$$\text{WMin}_K D^{bm} F(u_0, y_0)(u) \subseteq D^{bm} \mathcal{W}(u_0, y_0)(u).$$

*Proof* Since the proof is similar, we prove only assertion (iii). Observe that, being a pointed closed convex cone in  $\mathbb{R}^q$ ,  $K$  clearly has a compact base and hence so does  $\tilde{K}$ . Moreover,  $F$  is  $m$ th-order  $u$ -directionally compact at  $(u_0, y_0)$  implies that  $\mathcal{W}$  is  $m$ th-order  $u$ -directionally compact at  $(u_0, y_0)$ . Therefore, one has

$$\begin{aligned} \text{WMin}_K D^{bm} F(u_0, y_0)(u) &= \text{WMin}_K D^{bm} (F + \tilde{K})(u_0, y_0)(u) \\ &= \text{WMin}_K D^{bm} (\mathcal{W} + \tilde{K})(u_0, y_0)(u) \\ &= \text{WMin}_K D^{bm} \mathcal{W}(u_0, y_0)(u) \\ &\subseteq D^{bm} \mathcal{W}(u_0, y_0)(u). \end{aligned}$$

Here the first and third equalities are due to Propositions 3.5, and the second one follows from Remark 4.1. □

Now, we investigate the reverse conclusion in Proposition 4.1.

**Proposition 4.2** *Suppose that the following conditions are satisfied:*

- (i)  *$F$  is locally Hölder continuous of order  $m$  at  $u_0$ ;*
- (ii)  *$F$  is  $K$ -dominated by  $\mathcal{F}$  near  $u_0$ ;*
- (iii) *there is a neighborhood  $U$  of  $u_0$  such that for any  $u \in U$ ,  $\mathcal{F}(u)$  is a single-point set.*

*Then, for any  $u \in \mathbb{R}^p$ ,*

$$D^{bm} \mathcal{F}(u_0, y_0)(u) \subseteq \text{Min}_K D^{bm} F(u_0, y_0)(u).$$

*Proof* Let  $u \in \mathbb{R}^p$  and  $v \in D^{bm} \mathcal{F}(u_0, y_0)(u)$ . Then, for any sequence  $t_n \downarrow 0$ , there exists  $(u_n, v_n) \rightarrow (u, v)$  such that

$$y_0 + t_n^m v_n \in \mathcal{F}(u_0 + t_n u_n) \subseteq F(u_0 + t_n u_n), \quad \forall n. \tag{4}$$

Suppose to the contrary that  $v \notin \text{Min}_K D^{bm}F(u_0, y_0)(u)$ . Then, there exists  $\bar{v} \in D^{bm}F(u_0, y_0)(u)$  such that  $v - \bar{v} \in K \setminus \{0\}$ . Hence, for the preceding  $t_n$ , there exists  $(\bar{u}_n, \bar{v}_n) \rightarrow (u, \bar{v})$  such that

$$y_0 + t_n^m \bar{v}_n \in F(u_0 + t_n \bar{u}_n), \quad \forall n. \tag{5}$$

Since  $F$  is  $K$ -dominated by  $\mathcal{F}$  near  $u_0$ , there exists  $U_1 \in \mathcal{U}(u_0)$  such that, for all  $u \in U_1$ ,

$$F(u) \subseteq \mathcal{F}(u) + K. \tag{6}$$

It follows from the locally Hölder continuity of order  $m$  of  $F$  that there exist  $U_2 \in \mathcal{U}(u_0)$  and  $L > 0$  such that, for all  $u_1, u_2 \in U_2$  and

$$F(u_1) \subseteq F(u_2) + L \|u_1 - u_2\|^m B_{\mathbb{R}^q}. \tag{7}$$

Naturally, since  $t_n \downarrow 0$ , there exists  $N > 0$  such that

$$u_0 + t_n \bar{u}_n, u_0 + t_n u_n \in U \cap U_1 \cap U_2, \quad \forall n > N. \tag{8}$$

Therefore, from (5), (7), (8), and (6), there exists  $b_n \in B_{\mathbb{R}^q}$  such that, for all  $n$  large enough,

$$y_0 + t_n^m (\bar{v}_n - L \|\bar{u}_n - u_n\|^m b_n) \in F(u_0 + t_n u_n) \subseteq \mathcal{F}(u_0 + t_n u_n) + K. \tag{9}$$

Thus, it follows from (4), (9), and assumption (iii) that

$$y_0 + t_n^m (\bar{v}_n - L \|\bar{u}_n - u_n\|^m b_n) - (y_0 + t_n^m v_n) = t_n^m (\bar{v}_n - L \|\bar{u}_n - u_n\|^m b_n - v_n) \in K. \tag{10}$$

Since  $\bar{v}_n - L \|\bar{u}_n - u_n\|^m b_n - v_n \rightarrow \bar{v} - v$  and  $K$  is a pointed closed convex cone, one has  $\bar{v} - v \in K$ , which contradicts  $v - \bar{v} \in K \setminus \{0\}$ . This completes the proof.  $\square$

The following example shows that the assumption (iii) in Proposition 4.2 cannot be dropped.

**Example 4.1** Let  $p = 1, q = 2, K = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 0, y_2 \geq 0\}, (u_0, y_0) = (0, (0, 0))$ , and  $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$  be defined by

$$F(u) = \begin{cases} K, & \text{if } u = 0, \\ K \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = u^2, y_2 \geq -\sqrt{1 + u^2}\}, & \text{if } u \neq 0. \end{cases}$$

Then

$$\mathcal{F}(u) = \begin{cases} \{(0, 0)\}, & \text{if } u = 0, \\ \{(0, 0), (u^2, -\sqrt{1 + u^2})\}, & \text{if } u \neq 0. \end{cases}$$

Hence, we can check that  $\mathcal{F}(u)$  is not a single-point set near  $u_0$ ,  $F$  is  $K$ -dominated by  $\mathcal{F}$  near  $u_0$ , and  $F$  is locally Hölder continuous of order 2 at  $u_0$ . By direct calculation, one has, for any  $u \in \mathbb{R}$ ,

$$D^{b2}F(u_0, y_0)(u) = K \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = u^2\},$$

$$D^{b^2} \mathcal{F}(u_0, y_0)(u) = \{(0, 0)\},$$

and then

$$\text{Min}_K D^{b^2} F(u_0, y_0)(0) = \emptyset.$$

Therefore,

$$D^{b^2} \mathcal{F}(u_0, y_0)(0) \not\subseteq \text{Min}_K D^{b^2} F(u_0, y_0)(0).$$

**Remark 4.2** Similar properties to Proposition 4.2 for higher-order contingent-type derivatives of perturbation maps have not yet been investigated in [31]. With some suitable modifications, we can obtain similar properties for higher-order contingent-type derivatives of perturbation maps in [31].

**Proposition 4.3** *Suppose that the following conditions are satisfied:*

- (i) *F is locally Hölder continuous of order m at u<sub>0</sub>;*
- (ii) *F is K-dominated by P near u<sub>0</sub>;*
- (iii) *there is a neighborhood U of u<sub>0</sub> such that for any u ∈ U, P(u) is a single-point set.*

*Then, for any u ∈ ℝ<sup>p</sup>,*

$$D^{bm} \mathcal{P}(u_0, y_0)(u) \subseteq \text{PrMin}_K D^{bm} F(u_0, y_0)(u).$$

*Proof* The proof is similar to that of Proposition 4.2. □

**Proposition 4.4** *Assume that int K ≠ ∅. If F is locally Hölder continuous of order m at u<sub>0</sub>, then, for any u ∈ ℝ<sup>p</sup>,*

$$D^{bm} \mathcal{W}(u_0, y_0)(u) \subseteq \text{WMin}_K D^{bm} F(u_0, y_0)(u).$$

*Proof* Let u ∈ ℝ<sup>p</sup> and v ∈ D<sup>bm</sup>W(u<sub>0</sub>, y<sub>0</sub>)(u). Then, for any sequence t<sub>n</sub> ↓ 0, there exists (u<sub>n</sub>, v<sub>n</sub>) → (u, v) such that

$$y_0 + t_n^m v_n \in \mathcal{W}(u_0 + t_n u_n) \subseteq F(u_0 + t_n u_n), \quad \forall n. \tag{11}$$

Suppose to the contrary that v ∉ WMin<sub>K</sub> D<sup>bm</sup>F(u<sub>0</sub>, y<sub>0</sub>)(u). Then there exists  $\bar{v} \in D^{bm} F(u_0, y_0)(u)$  such that v −  $\bar{v} \in \text{int } K$ . Hence, for the preceding t<sub>n</sub>, there exists ( $\bar{u}_n, \bar{v}_n$ ) → (u,  $\bar{v}$ ) such that

$$y_0 + t_n^m \bar{v}_n \in F(u_0 + t_n \bar{u}_n), \quad \forall n. \tag{12}$$

It follows from the locally Hölder continuity of order m of F that there exist U ∈ U(u<sub>0</sub>) and L > 0 such that, for all u<sub>1</sub>, u<sub>2</sub> ∈ U, one has

$$F(u_1) \subseteq F(u_2) + L \|u_1 - u_2\|^m B_{\mathbb{R}^q}. \tag{13}$$

Naturally, since  $t_n \downarrow 0$ , there exists  $N > 0$  such that

$$u_0 + t_n \bar{u}_n, u_0 + t_n u_n \in U, \quad \forall n > N. \tag{14}$$

Therefore, from (12), (13), and (14), there exists  $b_n \in B_{\mathbb{R}^q}$  such that, for all  $n$  large enough,

$$y_0 + t_n^m (\bar{v}_n - L \|\bar{u}_n - u_n\|^m b_n) \in F(u_0 + t_n u_n). \tag{15}$$

It follows from  $v_n - (\bar{v}_n - L \|\bar{u}_n - u_n\|^m b_n) \rightarrow v - \bar{v}$  and  $v - \bar{v} \in \text{int} K$  that we have  $v_n - (\bar{v}_n - L \|\bar{u}_n - u_n\|^m b_n) \in \text{int} K$  for  $n$  large enough. Therefore, for  $n$  large enough,

$$y_0 + t_n^m v_n - (y_0 + t_n^m (\bar{v}_n - L \|\bar{u}_n - u_n\|^m b_n)) \in \text{int} K,$$

which contradicts with (11). The conclusion is obtained. □

### 4.2 Higher-order adjacent derivatives of perturbation maps with constraints

In this section, the formulas for estimating higher-order adjacent derivative of perturbation map/ weak perturbation map via adjacent derivative of constraint map together with higher-order Fréchet derivative of the objective function are established.

**Proposition 4.5** *Let  $u_0 \in \mathbb{R}^p, x_0 \in X(u_0), y_0 = f(x_0, u_0)$ . Assume that  $f$  is  $m$ th-order Fréchet continuously differentiable at  $(x_0, u_0)$ . Then, for any  $u \in \mathbb{R}^p$ ,*

$$\begin{aligned} & \{y \in \mathbb{R}^q \mid \exists x \in D^b X(x_0, u_0)(u), y = d^m f(x_0, u_0)((x, u), \dots, (x, u))\} \\ & \subseteq D^{bm} F(u_0, y_0)(u). \end{aligned} \tag{16}$$

Moreover, if  $X$  is (first-order)  $u$ -directionally compact at  $(u_0, x_0)$  for any  $u \in \mathbb{R}^p$ , then the reverse inclusion of (16) is also valid.

*Proof* Let  $y$  be as in the left hand side of (16). Then there exist  $u \in \mathbb{R}^p$  and  $x \in D^b X(x_0, u_0)(u)$  such that  $y = d^m f(x_0, u_0)((x, u), \dots, (x, u))$ . Since  $x \in D^b X(x_0, u_0)(u)$ , for all  $t_n \downarrow 0$ , there exists  $(u_n, x_n) \rightarrow (u, x)$  such that, for all  $n, x_0 + t_n x_n \in X(u_0 + t_n u_n)$ . Then

$$f(x_0 + t_n x_n, u_0 + t_n u_n) \in F(u_0 + t_n u_n), \quad \forall n. \tag{17}$$

It follows from the  $m$ th-order Fréchet continuously differentiability of  $f$  and (17) that we have

$$f(x_0, u_0) + t_n^m d^m f(x_0, u_0)((x_n, u_n), \dots, (x_n, u_n)) + o(t_n^m \|(x_n, u_n)\|^m) \in F(u_0 + t_n u_n).$$

Consequently,

$$y_0 + t_n^m \left( d^m f(x_0, u_0)((x_n, u_n), \dots, (x_n, u_n)) + \frac{o(t_n^m \|(x_n, u_n)\|^m)}{t_n^m} \right) \in F(u_0 + t_n u_n). \tag{18}$$

It follows from (18) and

$$d^m f(x_0, u_0)((x_n, u_n), \dots, (x_n, u_n)) + \frac{o(t_n^m \|(x_n, u_n)\|^m)}{t_n^m} \rightarrow d^m f(x_0, u_0)((x, u), \dots, (x, u))$$

when  $n \rightarrow \infty$ , one has  $y = d^m f(x_0, u_0)((x, u), \dots, (x, u)) \in D^{bm} F(u_0, y_0)(u)$ . Hence, (16) has been established.

Now, let  $y \in D^{bm} F(u_0, y_0)(u)$ . Then, for all  $t_n \downarrow 0$ , there exists  $(u_n, y_n) \rightarrow (u, y)$  such that  $y_0 + t_n^m y_n \in F(u_0 + t_n u_n)$ ,  $\forall n$ . Hence, there exists  $x_n \in X(u_0 + t_n u_n)$  such that

$$y_0 + t_n^m y_n = f(x_n, u_0 + t_n u_n), \quad \forall n. \tag{19}$$

Setting  $\tilde{x}_n := \frac{x_n - x_0}{t_n}$ , we have

$$x_0 + t_n \tilde{x}_n \in X(u_0 + t_n u_n) \tag{20}$$

and

$$y_0 + t_n^m y_n = f(x_0 + t_n \tilde{x}_n, u_0 + t_n u_n), \quad \forall n. \tag{21}$$

Since  $X$  is first-order  $u$ -directionally compact at  $(u_0, x_0)$ , for preceding  $t_n, u_n$ , and  $\tilde{x}_n$ , by taking subsequence if necessary, one gets  $\tilde{x}_n \rightarrow \tilde{x} \in D^b X(x_0, u_0)(u)$ . It follows from (21) and the  $m$ th-order Fréchet continuously differentiability of  $f$  at  $(x_0, u_0)$  that one has

$$y_0 + t_n^m y_n = f(x_0, u_0) + t_n^m d^m f(x_0, u_0)((\tilde{x}_n, u_n), \dots, (\tilde{x}_n, u_n)) + o(t_n^m \|(\tilde{x}_n, u_n)\|^m).$$

It implies that

$$y_n = d^m f(x_0, u_0)((\tilde{x}_n, u_n), \dots, (\tilde{x}_n, u_n)) + \frac{o(t_n^m \|(\tilde{x}_n, u_n)\|^m)}{t_n^m}.$$

Letting  $n \rightarrow \infty$ , we have

$$y = d^m f(x_0, u_0)((\tilde{x}, u), \dots, (\tilde{x}, u)).$$

The proof is complete. □

The following example illustrates that the assumption for the validity of the reverse inclusion of (16) in Proposition 4.5 cannot be omitted.

**Example 4.2** Let  $p = q = l = 1, m = 2, f(x, u) = x^4$ , and  $X : \mathbb{R} \rightrightarrows \mathbb{R}$  be defined by  $X(u) = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ . Then  $F(u) = \{y \in \mathbb{R} \mid 0 \leq y \leq 1\}$ .

Let  $(x_0, u_0) = (0, 0)$ . Then  $y_0 = f(x_0, u_0) = 0$ . We can check that  $X$  is not  $u$ -directionally compact at  $(u_0, x_0)$  for any  $u \in \mathbb{R}$ . Indeed, by taking  $t_n = \frac{1}{n}, u_n \rightarrow u$ , and  $x_n = \frac{1}{2}n$ , we have  $x_0 + t_n x_n = \frac{1}{2} \in F(u_0 + t_n u_n)$ , and  $x_n$  has no convergent subsequence.

By direct calculation, one has, for any  $u \in \mathbb{R}$ ,

$$D^b X(u_0, x_0)(u) = \mathbb{R}_+, \quad D^{b^2} F(u_0, y_0)(u) = \mathbb{R}_+,$$

$$d^2 f(x, u) = \begin{bmatrix} 12x^2 & 0 \\ 0 & 0 \end{bmatrix},$$

and then

$$d^2f(x_0, u_0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, for any  $u \in \mathbb{R}$ ,

$$\{y \in \mathbb{R} \mid x \in D^bX(x_0, u_0)(u), y = d^2f(x_0, u_0)((x, u), (x, u))\} = \{0\} \subsetneq D^{b^2}F(u_0, y_0)(u).$$

**Corollary 4.1** *Let  $u_0 \in \mathbb{R}^p, x_0 \in X(u_0), y_0 = f(x_0, u_0)$ , and  $\tilde{X} : \mathbb{R}^p \times \mathbb{R}^q \rightrightarrows \mathbb{R}^l$  be defined by  $\tilde{X}(u, y) := \{x \in X(u) : y = f(x, u)\}$ . Assume that  $f$  is  $m$ th-order Fréchet continuously differentiable at  $(x_0, u_0)$ . Let one of the following conditions be fulfilled:*

- (i)  $X$  is locally Lipschitz calm at  $(u_0, x_0) \in \text{gr } X$ ;
- (ii)  $D_S X(u_0, x_0)(0) = \{0\}$ ;
- (iii)  $\tilde{X}$  is  $(u, y)$ -directionally compact at  $((u_0, y_0), x_0)$  for any  $(u, y) \in \mathbb{R}^p \times \mathbb{R}^q$ ;
- (iv)  $\tilde{X}$  is locally Lipschitz calm at  $((u_0, y_0), x_0) \in \text{gr } \tilde{X}$ ;
- (v)  $D_S \tilde{X}((u_0, y_0), x_0)(0, 0) = \{0\}$ ;
- (vi)  $\tilde{X}$  is locally pseudo-Lipschitz at  $((u_0, y_0), x_0)$ .

Then, for any  $u \in \mathbb{R}^p$ ,

$$D^{bm}F(u_0, y_0)(u) = \{y \in \mathbb{R}^q \mid \exists x \in D^bX(x_0, u_0)(u), y = d^mf(x_0, u_0)((x, u), \dots, (x, u))\}.$$

*Proof* From Proposition 2.1, Corollary 2.1, Proposition 4.5 and the analysis similar the proof of Corollary 4.1 and Proposition 4.2 in [35], we obtain the conclusion. □

**Theorem 4.1** *Let  $u_0 \in \mathbb{R}^p, x_0 \in X(u_0), y_0 = f(x_0, u_0)$ . Suppose that the following conditions are satisfied:*

- (i)  $F$  is  $m$ th-order  $u$ -directionally compact at  $(u_0, y_0)$  for any  $u \in \mathbb{R}^p$ ;
- (ii)  $F$  is  $K$ -dominated by  $\mathcal{F}$  near  $u_0$ ;
- (iii)  $F$  is locally Hölder continuous of order  $m$  at  $u_0$ ;
- (iv) there exists  $U \in \mathcal{U}(u_0)$  such that for any  $u \in U, \mathcal{F}(u)$  is a single-point set;
- (v)  $f$  is  $m$ th-order Fréchet continuously differentiable at  $(x_0, u_0)$ ;
- (vi)  $X$  is first-order  $u$ -directionally compact at  $(u_0, x_0)$ .

Then, for any  $u \in \mathbb{R}^p$ ,

$$\begin{aligned} D^{bm}F(u_0, y_0)(u) &= \text{Min}_K D^{bm}F(u_0, y_0)(u) \\ &= \text{Min}_K \{y \in \mathbb{R}^q \mid \exists x \in D^bX(x_0, u_0)(u), y = d^mf(x_0, u_0)((x, u), \dots, (x, u))\}. \end{aligned}$$

*Proof* It follows from Proposition 4.1(i), Proposition 4.2, and Proposition 4.5 that the proof is complete. □

The result in Theorem 4.1 is illustrated in the following example.

**Example 4.3** Let  $p = q = l = 1, m = 2, K = \mathbb{R}_+, f(x, u) = x^2$ , and  $X : \mathbb{R} \rightrightarrows \mathbb{R}$  be defined by  $X(u) = \{x \in \mathbb{R} \mid u^2 \leq x \leq 2u^2\}$ . Then

$$F(u) = \{y \in \mathbb{R} \mid u^4 \leq y \leq 4u^4\},$$

$$\mathcal{F}(u) = \{u^4\}.$$

Let  $(x_0, u_0) = (0, 0)$ . Then  $y_0 = f(x_0, u_0) = 0$ . It is easy to see that the assumptions (ii) and (iv) in Theorem 4.1 are satisfied. Since  $D_S^2 F(u_0, y_0)(0) = \{0\}$ , from the Proposition 2.1, the assumption (i) Theorem 4.1 is fulfilled.

Moreover, we can check that  $F$  is locally Hölder continuous of order 2 at  $u_0$  and  $X$  is first-order  $u$ -directionally compact at  $(u_0, x_0)$ . Hence, the assumptions (iii) and (vi) in Theorem 4.1 are fulfilled.

By direct calculation, one has, for any  $u \in \mathbb{R}$ ,

$$\begin{aligned} D^b X(u_0, x_0)(u) &= \{0\}, & D^{b2} F(u_0, y_0)(u) &= \{0\}, \\ D^{b2} \mathcal{F}(u_0, y_0)(u) &= \{0\}, \\ d^2 f(x, u) &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

and then

$$d^2 f(x_0, u_0)((x, u), (x, u)) = 2x^2.$$

Thus, all the assumptions in Theorem 4.1 are satisfied. For any  $u \in \mathbb{R}$  and  $x \in D^b X(u_0, x_0)(u)$ , i.e.,  $x = 0$ , one has

$$d^2 f(x_0, u_0)((x, u), (x, u)) = 0.$$

Hence, for any  $u \in \mathbb{R}$ ,

$$\begin{aligned} D^{b2} \mathcal{F}(u_0, y_0)(u) &= \text{Min}_K D^{b2} F(u_0, y_0)(u) \\ &= \text{Min}_K \{y \in \mathbb{R} \mid x \in D^b X(x_0, u_0)(u), y = d^2 f(x_0, u_0)((x, u), (x, u))\}. \end{aligned}$$

**Theorem 4.2** *Let  $u_0 \in \mathbb{R}^p, x_0 \in X(u_0), y_0 = f(x_0, u_0)$ . Suppose that the following conditions are satisfied:*

- (i)  $F$  is  $m$ th-order  $u$ -directionally compact at  $(u_0, y_0)$  for any  $u \in \mathbb{R}^p$ ;
- (ii)  $F$  is  $K$ -dominated by  $\mathcal{P}$  near  $u_0$ ;
- (iii)  $F$  is locally Hölder continuous of order  $m$  at  $u_0$ ;
- (iv) there exists  $U \in \mathcal{U}(u_0)$  such that for any  $u \in U$ ,  $\mathcal{P}(u)$  is a single-point set;
- (v)  $f$  is  $m$ th-order Fréchet continuously differentiable at  $(x_0, u_0)$ ;
- (vi)  $X$  is first-order  $u$ -directionally compact at  $(u_0, x_0)$ .

Then, for any  $u \in \mathbb{R}^p$ ,

$$\begin{aligned} D^{bm} \mathcal{P}(u_0, y_0)(u) &= \text{PrMin}_K D^{bm} F(u_0, y_0)(u) \\ &= \text{PrMin}_K \{y \in \mathbb{R}^q \mid \exists x \in D^b X(x_0, u_0)(u), y = d^m f(x_0, u_0)((x, u), \dots, (x, u))\}. \end{aligned}$$



*Proof* It follows from Proposition 4.1(ii), Proposition 4.3, and Proposition 4.5 that the conclusion is obtained.  $\square$

**Theorem 4.3** *Let  $u_0 \in \mathbb{R}^p, x_0 \in X(u_0), y_0 = f(x_0, u_0)$ . Suppose that the following conditions are satisfied:*

- (i)  *$F$  is  $m$ th-order  $u$ -directionally compact at  $(u_0, y_0)$  for any  $u \in \mathbb{R}^p$ ;*
- (ii)  *$\text{int} K \neq \emptyset$ , there is a closed convex cone  $\tilde{K}$  satisfying  $\tilde{K} \subseteq \text{int} K \cup \{0\}$  and  $F$  is  $\tilde{K}$ -dominated by  $\mathcal{W}$  near  $u_0$ ;*
- (iii)  *$F$  is locally Hölder continuous of order  $m$  at  $u_0$ ;*
- (iv)  *$f$  is  $m$ th-order Fréchet continuously differentiable at  $(x_0, u_0)$ ;*
- (v)  *$X$  is  $u$ -directionally compact at  $(u_0, x_0)$ .*

*Then, for any  $u \in \mathbb{R}^p$ ,*

$$\begin{aligned} & D^{bm}\mathcal{W}(u_0, y_0)(u) \\ &= \text{WMin}_K D^{bm}F(u_0, y_0)(u) \\ &= \text{WMin}_K \{y \in \mathbb{R}^q \mid \exists x \in D^bX(x_0, u_0)(u), y = d^m f(x_0, u_0)((x, u), \dots, (x, u))\}. \end{aligned}$$

*Proof* It follows from Proposition 4.1(iii), Proposition 4.4, and Proposition 4.5 that the conclusion is given.  $\square$

### 5 Conclusions

Although there are some similar properties between the contingent derivatives and the adjacent derivatives, the adjacent derivatives have some advantages and drawbacks in some cases. When using adjacent derivatives instead of contingent derivatives in sensitivity analysis, the proto-differentiability assumption, such as in Theorem 3.3 in [4], Theorem 4.3 in [17], Theorem 5.1 in [18], can be omitted. The drawbacks of using of the adjacent derivatives is that the adjacent derivatives can be empty set in some cases such as in Example 2.1. Hence, in the case that the adjacent derivatives are not empty and avoiding the proto-differentiability assumption, the adjacent derivatives can be used. Based on the above observation, the  $m$ th-order adjacent derivatives was employed to consider higher-order sensitivity analysis for nonsmooth vector optimization in this paper. First of all, we considered the relationships between the  $m$ th-order adjacent derivatives of the perturbation map/the proper perturbation map/the weak perturbation map, and the  $m$ th-order adjacent derivative of feasible map in objective space. Then the above relations were used to establish the formulas for estimating the  $m$ th-order adjacent derivatives of the perturbation map, the proper perturbation map, and the weak perturbation map via the adjacent derivative of constraint map and the  $m$ th-order Fréchet derivative of the objective map. Some examples are provided to ensure the need of the assumptions and illustrate the results. When  $m = 1$ , the results become the first-order sensitivity analysis using adjacent derivatives and also may be new.

#### Competing interests

The author declares that there is no conflict of interests.

#### Acknowledgements

The author would like to thank Professor Alexander Zalavski and the handling editors for the help in the processing of the paper. The author is very grateful to the anonymous referees for the useful suggestions and remarks which helped to improve the contents of this article.

Received: 31 December 2015 Accepted: 1 April 2016 Published online: 08 April 2016

## References

1. Tanino, T: Sensitivity analysis in multiobjective optimization. *J. Optim. Theory Appl.* **56**, 479-499 (1988)
2. Tanino, T: Stability and sensitivity analysis in convex vector optimization. *SIAM J. Control Optim.* **26**, 521-536 (1988)
3. Shi, DS: Contingent derivative of the perturbation map in multiobjective optimization. *J. Optim. Theory Appl.* **70**, 385-396 (1991)
4. Kuk, H, Tanino, T, Tanaka, M: Sensitivity analysis in vector optimization. *J. Optim. Theory Appl.* **89**, 713-730 (1996)
5. Balbás, A, Jiménez Guerra, P: Sensitivity analysis for convex multiobjective programming in abstract spaces. *J. Math. Anal. Appl.* **202**, 645-648 (1996)
6. Jiménez Guerra, P, Melguizo, MA, Muñoz, MJ: Sensitivity analysis in convex programming. *Comput. Math. Appl.* **58**, 1239-1246 (2009)
7. Kuk, H, Tanino, T, Tanaka, M: Sensitivity analysis in parameterized convex vector optimization. *J. Math. Anal. Appl.* **202**, 511-522 (1996)
8. Shi, DS: Sensitivity analysis in convex vector optimization. *J. Optim. Theory Appl.* **77**, 145-159 (1993)
9. Chuong, TD, Yao, J-C: Generalized Clarke epiderivatives of parametric vector optimization problems. *J. Optim. Theory Appl.* **146**, 77-94 (2010)
10. García, F, Melguizo Padial, MA: Sensitivity analysis in convex optimization through the circatangent derivative. *J. Optim. Theory Appl.* **165**, 420-438 (2015)
11. Jiménez Guerra, P, Melguizo Padial, MA: Sensitivity analysis in differential programming through the Clarke derivative. *Mediterr. J. Math.* **9**, 537-550 (2012)
12. Rockafellar, RT: Proto-differentiability of set-valued mappings and its applications in optimization. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **6**, 449-482 (1989)
13. Levy, AB, Rockafellar, RT: Sensitivity analysis of solutions to generalized equations. *Trans. Am. Math. Soc.* **345**, 661-671 (1994)
14. Levy, AB, Rockafellar, RT: Variational conditions and the proto-differentiation of partial subgradient mappings. *Nonlinear Anal.* **26**, 1951-1964 (1996)
15. Lee, GM, Huy, NQ: On proto-differentiability of generalized perturbation maps. *J. Math. Anal. Appl.* **324**, 1297-1309 (2006)
16. Lee, GM, Huy, NQ: On sensitivity analysis in vector optimization. *Taiwan. J. Math.* **11**, 945-958 (2007)
17. Wang, QL, Li, SJ: Second-order contingent derivative of the perturbation map in multiobjective optimization. *Fixed Point Theory Appl.* **2011**, Article ID 857520 (2011)
18. Wang, QL, Li, SJ: Sensitivity and stability for the second-order contingent derivative of the proper perturbation map in vector optimization. *Optim. Lett.* **6**, 731-748 (2012)
19. Li, SJ, Liao, CM: Second-order differentiability of generalized perturbation maps. *J. Glob. Optim.* **52**, 243-252 (2012)
20. Tung, LT: Second-order radial-asymptotic derivatives and applications in set-valued vector optimization. *Pac. J. Optim.* (accepted for publication)
21. Tung, LT: On second-order proto-differentiability of perturbation maps (submitted for publication)
22. Sun, XK, Li, SJ: Stability analysis for higher-order adjacent derivative in parametrized vector optimization. *J. Inequal. Appl.* **2010**, Article ID 510838 (2010)
23. Tung, LT: Higher-order contingent derivatives of perturbation maps in multiobjective optimization. *J. Nonlinear Funct. Anal.* **2015**, Article ID 19 (2015)
24. Khanh, PQ, Tuan, ND: Variational sets of multivalued mappings and a unified study of optimality conditions. *J. Optim. Theory Appl.* **139**, 45-67 (2008)
25. Anh, NLH, Khanh, PQ: Variational sets of perturbation maps and applications to sensitivity analysis for constrained vector optimization. *J. Optim. Theory Appl.* **158**, 363-384 (2013)
26. Frankowska, H: An open mapping principle for set-valued maps. *J. Math. Anal. Appl.* **127**, 172-180 (1987)
27. Frankowska, H, Quincampoix, M: Hölder metric regularity of set-valued maps. *Math. Program.* **132**, 333-345 (2012)
28. Studniarski, M: Necessary and sufficient conditions for isolated local minima of nonsmooth functions. *SIAM J. Control Optim.* **25**, 1044-1049 (1986)
29. Studniarski, M, Ward, D: Weak sharp minima: characterizations and sufficient condition. *SIAM J. Control Optim.* **38**, 219-236 (1999)
30. Anh, NLH, Khanh, PQ, Tung, LT: Higher-order radial derivatives and optimality conditions in nonsmooth vector optimization. *Nonlinear Anal. TMA* **74**, 7365-7379 (2011)
31. Diem, HTH, Khanh, PQ, Tung, LT: On higher-order sensitivity analysis in nonsmooth vector optimization. *J. Optim. Theory Appl.* **162**, 463-488 (2014)
32. Sun, XK, Li, SJ: Weak lower Studniarski derivative in set-valued optimization. *Pac. J. Optim.* **8**, 307-320 (2012)
33. Sun, XK, Li, SJ: Lower Studniarski derivative of the perturbation map in parameterized vector optimization. *Optim. Lett.* **5**, 601-614 (2011)
34. Anh, NLH, Khanh, PQ: Calculus and applications of Studniarski's derivatives to sensitivity and implicit function theorems. *Control Cybern.* **43**, 35-47 (2014)
35. Aubin, JP, Frankowska, H: *Set-Valued Analysis*. Birkhäuser, Boston (1990)