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Solving nonlinear optimization problems with bipolar fuzzy relational equation constraints

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Abstract
This paper considers the problem of minimizing a nonlinear objective function subject to a system of bipolar fuzzy relational equations with max-T_L composition, where T_L is the Łukasiewicz triangular norm. It shows that the feasible domain, *i.e.*, the solution set of a system of bipolar fuzzy relational equations, can be reformulated as a system of 0-1 mixed integer inequalities. Consequently, such a type of optimization problems can be handled within the framework of 0-1 mixed integer optimization requiring no particular solving techniques.

Keywords: fuzzy relational equations; nonlinear optimization; mixed integer optimization

1 Introduction

Fuzzy relational equations have been intensively investigated as an important tool for fuzzy modeling and approximate reasoning (see, *e.g.*, [1, 2]). Among various types of fuzzy relational equations, those with max-T compositions are most fundamental and have been widely applied where $T : [0,1]^2 \to [0,1]$ is a continuous triangular norm. A system of fuzzy relational equations with max-T equations, max-T equations for short, can be formulated as

$$A \circ \mathbf{x} = \mathbf{b}, \tag{1}$$

where $A = (a_{ij})_{mn}$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ are all defined over $[0,1]$. More specifically, a system of max-T equations $A \circ \mathbf{x} = \mathbf{b}$ stands for

$$\max_{j\in N} T(a_{ij}, x_j) = b_i, \quad i \in M, \tag{2}$$

where $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$, respectively. In the context of fuzzy relational equations, the commonly used triangular norms include the *minimum* $T_M(x, y) = \min(x, y)$, the *product* $T_P(x, y) = xy$, and the *Łukasiewicz t-norm* $T_L(x, y) = \max(x + y - 1, 0)$ among which T_P and T_L are Archimedean.

For a system of max-T equations $A \circ \mathbf{x} = \mathbf{b}$, it is well known that the solution set, denoted by $S(A, \mathbf{b})$, is nonempty if and only if its principal solution $\hat{\mathbf{x}}$ is indeed a solution which can be constructed and verified in polynomial time. Moreover, when $S(A,\mathbf{b})$ is nonempty, the principal solution $\hat{\mathbf{x}}$ becomes the maximum solution and $S(A,\mathbf{b})$ can be determined by the maximum solution and a finite number of minimal solutions. However, to obtain all the minimal solutions to $A \circ \mathbf{x} = \mathbf{b}$ is in general a computationally difficult task because the number of minimal solutions could be exponentially large with respect to the input size. For a comprehensive discussion of fuzzy relational equations, the reader may refer to the monograph by Peeva and Kyosev [2] and the surveys by De Baets [3] and Li and Fang [4].

When a particular solution to a system of max-T equations is desired, the associated optimization problem is of concern. The problem of minimizing a linear objective function subject to a system of max-T equations has been intensively investigated with respect to various composition operations. It turns out that such an optimization problem can be transformed into the set covering problem which is known to be NP-hard (see, *e.g.*, [5–14]). The linear fractional optimization constrained by a system of max-T equations was also studied by Wu *et al.* [15] and Li and Fang [16] with respect to an Archimedean triangular norm.

The problem of minimizing a general nonlinear objective function subject to a system of max-T equations has been tackled by Lu and Fang [17], Khorram and Hassanzadeh [18], and Hassanzadeh *et al.* [19] using the genetic algorithm. It was pointed out by Li *et al.* [20] that such an optimization problem can be in general reformulated into a 0-1 mixed integer nonlinear optimization problem so that the traditional solving techniques, *e.g.*, the branch-and-bound method, may apply. However, when the objective function is max-separable and monotone, the associated optimization problem can be solved in polynomial time (see, *e.g.*, [21–23] and references therein).

Recently, bipolar max-T equations have been considered in the literature as a generalization of usual max-T equations. A system of bipolar max-T equations is formulated as

$$\max_{j\in N} \max\left\{T\left(a_{ij}^{+}, x_j\right), T\left(a_{ij}^{-}, \neg x_j\right)\right\} = b_i, \quad i \in M, \tag{3}$$

where $\neg x_j$ is the logical negation of x_j, *i.e.*, $\neg x_j = 1 - x_j$, $j \in N$. In the matrix form, it can be denoted as

$$A^{+} \circ \mathbf{x} \vee A^{-} \circ \neg\mathbf{x} = \mathbf{b}, \tag{4}$$

with $A^{+} = (a_{ij}^{+})_{mn}$, $A^{-} = (a_{ij}^{-})_{mn}$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ all being defined over $[0,1]$. It is clear that $A^{+} \circ \mathbf{x} \vee A^{-} \circ \neg\mathbf{x} = \mathbf{b}$ would degenerate into $A^{-} \circ \neg\mathbf{x} = \mathbf{b}$ or $A^{+} \circ \mathbf{x} = \mathbf{b}$, respectively, when A^{+} or A^{-} is the zero matrix. Therefore, a system of bipolar max-T equations can be viewed as a combination of two systems of max-T equations containing both independent variables and their logical negations.

The bipolar max-T_M equations and the associated linear optimization problem were first investigated in Freson *et al.* [24]. By resolving each single equation of a system of bipolar max-T_M equations $A^{+} \circ \mathbf{x} \vee A^{-} \circ \neg\mathbf{x} = \mathbf{b}$, Freson *et al.* [24] figured out analytically that the whole solution set $S(A^{+}, A^{-}, \mathbf{b})$ is the union of some interval-valued solutions, each of which is determined by a pair of maximal and minimal solutions. As a direct consequence, the bipolar max-T_M equation constrained linear optimization problem can be solved by examining these maximal and minimal solutions. An alternative reformulation for bipolar max-T_M equations was developed by Li and Jin [25] so that the associated linear optimization problem can be treated by integer optimization methods.

Besides, the bipolar max-T_L equation constrained linear optimization problem was studied by Li and Liu [26] using an analogous approach developed by Li and Jin [25, 27]. It turns out that the Archimedean property of T_L leads to a somewhat simpler structure for bipolar max-T_L equations, so that the associated linear optimization problem can be reformulated as a 0-1 integer linear optimization problem. This motivates us to extend this approach to nonlinear optimization scenarios.

In this paper, we aim to tackle the problem of minimizing a nonlinear objective function subject to a system of bipolar max-T_L equations, *i.e.*,

$$\begin{cases} \min Z = f(\mathbf{x}) \\ \text{subject to:} \\ \quad A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}. \end{cases} \tag{5}$$

Due to the simultaneous appearance of $\mathbf{x}$ and $\neg\mathbf{x}$, it was shown by Li and Jin [25] and Li and Liu [26] that determining whether $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ has a solution or not is an NP-complete problem because it can be viewed as a disguised form of the Boolean satisfiability problem. This implies that the optimization problem under consideration is inevitably NP-hard. Following the ideas of Li and Jin [25, 27] and Li and Liu [26], we demonstrate that $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ can be expressed equivalently as a system of 0-1 mixed integer linear inequalities. Consequently, the bipolar max-T_L equation constrained nonlinear optimization problem can be handled by those traditional techniques developed for solving 0-1 mixed integer optimization problems.

The rest of this paper is organized as follows. In Section 2, the reformulation of a system of bipolar max-T_L equations is presented. The bipolar max-T_L equation constrained optimization problem is discussed in Section 3, and the conclusions are presented in Section 4.

2 Bipolar max-T_L equations and their reformulation

In this section, we reveal the critical features of a system of bipolar max-T_L equations and develop its equivalent representation.

For a system of bipolar max-T_L equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$, it is said to be consistent if its solution set $S(A^+, A^-, \mathbf{b})$ is nonempty. Otherwise, it is said to be inconsistent. Due to the non-interactivity property of the *maximum* operation, *i.e.*, $\max(a, b) \in \{a, b\}$, if there is a vector $\mathbf{x} \in S(A^+, A^-, \mathbf{b})$, we have

$$T_L(a^+_{ij}, x_j) \le b_i, \qquad T_L(a^-_{ij}, \neg x_j) \le b_i, \quad \forall i \in M, j \in N. \tag{6}$$

Moreover, in order to fulfil the equality requirements, there must exist an index $j_i \in N$ for each $i \in M$ such that either $T_L(a^+_{ij_i}, x_{j_i}) = b_i$ or $T_L(a^-_{ij_i}, \neg x_{j_i}) = b_i$ holds. Consequently, we may first focus on the inequalities of the forms $T_L(a^+,x) \le b$ and $T_L(a^-,\neg x) \le b$ for any $a^+,a^-,b \in [0,1]$.

Lemma 1 *For any $a^+,b \in [0,1]$, $T_L(a^+,x) \le b$ if and only if $x \le S_L(\neg a^+,b)$ where $S_L : [0,1]^2 \to [0,1]$ is the Łukasiewicz t-conorm defined as $S_L(x,y) = \min(x+y,1)$. Analogously, $T_L(a^-,\neg x) \le b$ if and only if $x \ge T_L(a^-,\neg b)$ for any $a^-,b \in [0,1]$.*

Lemma 1 can be verified directly. Moreover, if $b \in (0,1]$, $T_L(a^+,x) = b$ if and only if $a^+ \ge b$ and $x = S_L(\neg a^+,b)$ while $T_L(a^-,\neg x) = b$ if and only if $a^- \ge b$ and $x = T_L(a^-,\neg b)$. The only exception occurs when $b = 0$, in which case, $T_L(a^+,x) = 0$ implies $0 \le x \le S_L(\neg a^+,0) = \neg a^+$ and $T_L(a^-,\neg x) = 0$ implies $a^- = T_L(a^-,1) \le x \le 1$, respectively.

By Lemma 1, the lower and upper bound information of the solutions to a system of max-T_L equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ can be retrieved. Let $\check{\mathbf{x}} = (\check{x}_1,\check{x}_2,\ldots,\check{x}_n)^T$ such that

$$\check{x}_j = \max_{i\in M} T_L\bigl(a^-_{ij},\neg b_i\bigr), \quad \forall j \in N, \tag{7}$$

and $\hat{\mathbf{x}} = (\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n)^T$ such that

$$\hat{x}_j = \min_{i\in M} S_L\bigl(\neg a^+_{ij},b_i\bigr), \quad \forall j \in N. \tag{8}$$

It is clear that if $S(A^+,A^-,\mathbf{b}) \neq \emptyset$, then $\check{\mathbf{x}} \le \hat{\mathbf{x}}$. It also holds that $\check{\mathbf{x}} \le \mathbf{x} \le \hat{\mathbf{x}}$ for any $\mathbf{x} \in S(A^+,A^-,\mathbf{b})$. In other words, $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are the lower and upper bounds of the solutions to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$, respectively. However, as indicated by Li and Liu [26], $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ themselves are not necessarily solutions to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$. Even if both $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are solutions, it does not mean that $S(A^+,A^-,\mathbf{b}) = \{\mathbf{x} | \check{\mathbf{x}} \le \mathbf{x} \le \hat{\mathbf{x}}\}$.

Note that for the elements of $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ if $\check{x}_j = \hat{x}_j$ for some $j \in N$, the value of x_j is fixed in any possible solution. In such a case, the variable x_j can be removed in further analysis as well as those equations where either $T_L(a^+_{ij},\hat{x}_j) = b_i$ or $T_L(a^-_{ij},\neg\check{x}_j) = b_i$ holds. Consequently, $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ can be reduced to a system of bipolar max-T_L equations with fewer variables such that the lower and upper bounds are strictly different. Hereafter, we assume without loss of generality that $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are strictly different for a system of bipolar max-T_L equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ under consideration, *i.e.*, $\check{x}_j < \hat{x}_j$ for all $j \in N$.

Moreover, as indicated by Li and Liu [26], the equations with a zero right hand side play no role once $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ have been obtained. Therefore, we may assume as well that the right hand side vector $\mathbf{b}$ is strictly positive for a system of bipolar max-T_L equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ under consideration.

When the equality requirements of $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ are concerned, we need take a close look on $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ because, according to Lemma 1, all the critical information for the equality requirements is preserved in $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$. Let $Q^+ = (q^+_{ij})_{mn}$ be a 0-1 matrix such that

$$q^+_{ij} = \begin{cases} 1, & \text{if } T_L(a^+_{ij},\hat{x}_j) = b_i, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall i \in M, j \in N. \tag{9}$$

It is clear that Q^+ records the information of those equations where the equalities hold at the upper bound $\hat{\mathbf{x}}$. Analogously, the counterpart corresponding to $\check{\mathbf{x}}$ is offered by the 0-1 matrix $Q^- = (q^-_{ij})_{mn}$ such that

$$q^-_{ij} = \begin{cases} 1, & \text{if } T_L(a^-_{ij}, \neg\check{x}_j) = b_i, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall i \in M, j \in N. \tag{10}$$

The matrices Q^+ and Q^- are called the characteristic matrices of $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ by Li and Liu [26]. It turns out that a system of bipolar max-T_L equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ can be equivalently represented in terms of its lower and upper bounds $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ and its characteristic matrices Q^+ and Q^-.

Theorem 1 *For a system of bipolar max-T_L equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in S(A^+, A^-, \mathbf{b})$ if and only if there exist two* 0-1 *vectors $\mathbf{u}^+$ and $\mathbf{u}^-$ such that $\mathbf{u}^+ + \mathbf{u}^- \le \mathbf{e}$, $Q^+\mathbf{u}^+ + Q^-\mathbf{u}^- \ge \mathbf{e}$, and*

$$V\mathbf{u}^+ + \check{\mathbf{x}} \le \mathbf{x} \le -V\mathbf{u}^- + \hat{\mathbf{x}}, \tag{11}$$

where $V = \operatorname{diag}(\hat{\mathbf{x}} - \check{\mathbf{x}})$ and $\mathbf{e}$ is the vector of all ones.

Proof If $\mathbf{x} \in S(A^+, A^-, \mathbf{b})$, denote $\mathbf{u}^+ = (u^+_1, u^+_2, \ldots, u^+_n)^T$ with

$$u^+_j = \begin{cases} 1, & \text{if } x_j = \hat{x}_j, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall j \in N, \tag{12}$$

and $\mathbf{u}^- = (u^-_1, u^-_2, \ldots, u^-_n)^T$ with

$$u^-_j = \begin{cases} 1, & \text{if } x_j = \check{x}_j, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall j \in N, \tag{13}$$

respectively. It can be verified that $\mathbf{u}^+ + \mathbf{u}^- \le \mathbf{e}$ and $V\mathbf{u}^+ + \check{\mathbf{x}} \le \mathbf{x} \le -V\mathbf{u}^- + \hat{\mathbf{x}}$. Furthermore, because for each $i \in M$, there exists an index $j_i \in N$ such that either $T_L(a^+_{ij_i}, x_{j_i}) = b_i$ or $T_L(a^-_{ij_i}, \neg x_{j_i}) = b_i$, it implies that either $x_{j_i} = \hat{x}_{j_i}$, $q^+_{ij_i} = 1$ or $x_{j_i} = \check{x}_{j_i}$, $q^-_{ij_i} = 1$. Therefore, $Q^+\mathbf{u}^+ + Q^-\mathbf{u}^- \ge \mathbf{e}$. Conversely, if there are two 0-1 vectors $\mathbf{u}^+$ and $\mathbf{u}^-$ such that $\mathbf{u}^+ + \mathbf{u}^- \le \mathbf{e}$ and $Q^+\mathbf{u}^+ + Q^-\mathbf{u}^- \ge \mathbf{e}$, a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ such that $V\mathbf{u}^+ + \check{\mathbf{x}} \le \mathbf{x} \le -V\mathbf{u}^- + \hat{\mathbf{x}}$ has the form

$$x_j = \begin{cases} \hat{x}_j, & \text{if } u^+_j = 1, \\ \check{x}_j, & \text{if } u^-_j = 1, \\ x_j, & \text{otherwise}, \end{cases} \qquad \forall j \in N. \tag{14}$$

Besides, for each $i \in M$ there exists an index $j_i \in N$ such that either $q^+_{ij_i}u^+_{j_i} = 1$ or $q^-_{ij_i}u^-_{j_i} = 1$, which indicates that either $T_L(a^+_{ij_i}, x_{j_i}) = b_i$ or $T_L(a^-_{ij_i}, \neg x_{j_i}) = b_i$. Consequently, $\mathbf{x}$ is a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$. □

Theorem 1 demonstrates that a system of bipolar max-T_L equations can be equivalently expressed as a system of 0-1 mixed integer linear inequalities involving an additional pair of 0-1 vectors. Although this alternative formulation has a relatively larger size, it eliminates the nonlinear structure in bipolar max-T_L equations and allows us to tackle the associated optimization problem by the usual optimization techniques. Besides, Theorem 1 implies that the solution set of a system of bipolar max-T_L equations is a union of some interval-valued vectors analogous to that of bipolar max-T_M equations. Unfortunately, determining all these interval-valued vectors requires an enumeration of all minimal solutions to a system of 0-1 integer linear inequalities, the number of which could be exponentially large.

Example 1 Consider the system of bipolar max-T_L equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ where

$$A^+ = \begin{pmatrix} 0.9 & 0.7 & 1.0 \\ 0.4 & 0.2 & 0.4 \end{pmatrix}, \qquad A^- = \begin{pmatrix} 1.0 & 0.8 & 0.6 \\ 0.7 & 0.9 & 0.2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}.$$

The lower and upper bounds of the solutions can be calculated, respectively, as

$$\check{\mathbf{x}} = (0.2, 0.3, 0)^T, \qquad \hat{\mathbf{x}} = (0.9, 1, 0.8)^T.$$

Subsequently, the two 0-1 characteristic matrices can be constructed, respectively, as

$$Q^+ = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

According to Theorem 1, the system of bipolar max-T_L equations under consideration can be reformulated as

$$\begin{pmatrix} 0.7 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.8 \end{pmatrix} \begin{pmatrix} u_1^+ \\ u_2^+ \\ u_3^+ \end{pmatrix} + \begin{pmatrix} 0.2 \\ 0.3 \\ 0 \end{pmatrix} \le \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\begin{pmatrix} -0.7 & 0 & 0 \\ 0 & -0.7 & 0 \\ 0 & 0 & -0.8 \end{pmatrix} \begin{pmatrix} u_1^- \\ u_2^- \\ u_3^- \end{pmatrix} + \begin{pmatrix} 0.9 \\ 1 \\ 0.8 \end{pmatrix} \ge \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1^+ \\ u_2^+ \\ u_3^+ \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1^- \\ u_2^- \\ u_3^- \end{pmatrix} \ge \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1^+ \\ u_2^+ \\ u_3^+ \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1^- \\ u_2^- \\ u_3^- \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where $\mathbf{u}^+ = (u_1^+, u_2^+, u_3^+)^T \in \{0,1\}^3$, $\mathbf{u}^- = (u_1^-, u_2^-, u_3^-)^T \in \{0,1\}^3$, and $\mathbf{x} = (x_1, x_2, x_3)^T \in [0,1]^3$. For this small size instance, the solution set can be figured out explicitly, which is the union of three interval solutions, *i.e.*,

$$S(A^+, A^-, \mathbf{b}) = \bigcup_{k=1,2,3} \{\mathbf{x} \in [0,1]^3 | \check{\mathbf{v}}^k \le \mathbf{x} \le \hat{\mathbf{v}}^k\},$$

with

$$\check{\mathbf{v}}^1 = \begin{pmatrix} 0.2 \\ 0.3 \\ 0 \end{pmatrix}, \qquad \check{\mathbf{v}}^2 = \begin{pmatrix} 0.9 \\ 0.3 \\ 0 \end{pmatrix}, \qquad \check{\mathbf{v}}^3 = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.8 \end{pmatrix},$$

$$\hat{\mathbf{v}}^1 = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.8 \end{pmatrix}, \qquad \hat{\mathbf{v}}^2 = \begin{pmatrix} 0.9 \\ 0.3 \\ 0.8 \end{pmatrix}, \qquad \hat{\mathbf{v}}^3 = \begin{pmatrix} 0.9 \\ 0.3 \\ 0.8 \end{pmatrix}.$$

3 Bipolar max-T_L equation constrained optimization

Based on the result in Section 2, the bipolar max-T_L equation constrained optimization problem

$$\begin{cases} \min Z = f(\mathbf{x}) \\ \text{subject to:} \\ \quad A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b} \end{cases} \tag{15}$$

is equivalent to

$$\begin{cases} \min Z = f(\mathbf{x}) \\ \text{subject to:} \\ \quad V\mathbf{u}^+ + \check{\mathbf{x}} \le \mathbf{x} \le -V\mathbf{u}^- + \hat{\mathbf{x}}, \\ \quad Q^+\mathbf{u}^+ + Q^-\mathbf{u}^- \ge \mathbf{e}, \\ \quad \mathbf{u}^+ + \mathbf{u}^- \le \mathbf{e}, \\ \quad \mathbf{u}^+, \mathbf{u}^- \in \{0,1\}^n, \qquad \mathbf{x} \in [0,1]^n \end{cases} \tag{16}$$

which is a general 0-1 mixed integer nonlinear optimization problem. Therefore, we may apply some well developed optimization techniques to handle the bipolar max-T_L equations constrained optimization problem based on this alternative formulation. Furthermore, as indicated by Li and Liu [26], such an optimization problem can be further reduced to a 0-1 integer linear optimization problem when it concerns a linear objective function.

Besides, because the feasible domain $S(A^+, A^-, \mathbf{b})$, when it is nonempty, is a union of several interval-valued solutions as illustrated in Example 1, such an optimization problem can also be viewed as a disjunctive optimization problem once $S(A^+, A^-, \mathbf{b})$ has been explicitly determined. Consequently, the bipolar max-T_L equation constrained optimization problem can be theoretically decomposed into a series of box-constrained optimization problems and be solved separately. This strategy works for small size problem instances, but it would inevitably suffer some computational obstacles for large size problem instances.

Example 2 Consider the system of bipolar max-T_L equations in Example 1 with a quadratic objective function

$$f(\mathbf{x}) = 2x_1^2 - (x_2 - x_3)^2.$$

According to Example 1, this instance can be reformulated as

$$\begin{cases} \min f(\mathbf{x}) = 2x_1^2 - x_2^2 - x_3^2 + 2x_2x_3 \\ \text{subject to:} \\ \quad \begin{pmatrix} -1 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0.8 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0.7 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u_1^+ \\ u_2^+ \\ u_3^+ \\ u_1^- \\ u_2^- \\ u_3^- \end{pmatrix} \leq \begin{pmatrix} -0.2 \\ -0.3 \\ 0 \\ 0.9 \\ 1 \\ 0.8 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \\ \quad x_1, x_2, x_3 \in [0,1], \qquad u_1^+, u_2^+, u_3^+, u_1^-, u_2^-, u_3^- \in \{0,1\}. \end{cases}$$

It can be resolved by some nonlinear optimization solver, *e.g.*, CPLEX, that it has an optimal solution

$$\mathbf{x}^* = (0.2, 0.3, 0.8)^T, \qquad \mathbf{u}^{+*} = (0,0,0)^T, \qquad \mathbf{u}^{-*} = (1,1,0)^T,$$

with the optimal objective value being $f(\mathbf{x}^*) = -0.17$. Note that the optimal 0-1 vectors $\mathbf{u}^{+*}$ and $\mathbf{u}^{-*}$ are usually not unique.

However, for this simple instance, the feasible domain can be readily determined and hence the considered optimization problem can be decomposed into three subproblems as

$$\begin{cases} \min Z_1(\mathbf{x}) = 0.08 - (0.3 - x_3)^2 \\ \text{subject to:} \\ \quad x_3 \in [0, 0.8], \end{cases}$$

$$\begin{cases} \min Z_2(\mathbf{x}) = 1.62 - (0.3 - x_3)^2 \\ \text{subject to:} \\ \quad x_3 \in [0, 0.8], \end{cases}$$

$$\begin{cases} \min Z_3(\mathbf{x}) = 2x_1^2 - 0.25 \\ \text{subject to:} \\ \quad x_1 \in [0.2, 0.9]. \end{cases}$$

By solving these three subproblems separately and comparing their results, the optimal solution to the original problem can be obtained:

$$\mathbf{x}^* = (0.2, 0.3, 0.8)^T$$

with the optimal objective value being $f(\mathbf{x}^*) = -0.17$.

In general, the problem of minimizing a quadratic objective function subject to a system of bipolar max-T_L equations can be reformulated into a 0-1 mixed integer quadratic optimization problem, which has been intensively investigated in the literature and can be numerically solved with the aid of some commercial software packages.

4 Conclusions

Following the ideas in Li and Jin [25, 27] and Li and Liu [26], it is demonstrated that a system of bipolar max-T_L equations can be represented equivalently by a system of 0-1 mixed integer linear inequalities. Consequently, the bipolar max-T_L equation constrained optimization problem can be handled within the framework of 0-1 mixed integer optimization.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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