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Generalized hierarchical minimax theorems for set-valued mappings

Haijun Wang*

*Correspondence:
wanghjshx@126.com
Department of Mathematics,
Taiyuan Normal University, Taiyuan,
030619, P.R. China

Abstract
In this paper, we discuss generalized hierarchical minimax theorems with four set-valued mappings and we propose some scalar hierarchical minimax theorems and generalized hierarchical minimax theorems in topological spaces. Some examples are given to illustrate our results.

Keywords: minimax theorems; set-valued mappings; cone-convexities

1 Introduction

It is well known that minimax theorems are important in the areas of game theory, and mathematical economical and optimization theory (see [1–5]). Within recent years, many generalizations of minimax theorems have been successfully obtained. On the one hand, the minimax theorem of two functions has been studied based on the two-person non-zero-sum games (see [6, 7]); on the other hand, with the development of vector optimization, there are many authors paying their attention to minimax problems of vector-valued mappings (see [8–10]).

Since Kuroiwa [11] investigated minimax problems of set-valued mappings in 1996, many authors have devoted their efforts to the study of the minimax problems for set-valued mappings. Li *et al.* [12] proved some minimax theorems for set-valued by using section theorem and separation theorem. Some other minimax theorems for set-valued mappings can be found in [13–16]. Zhang *et al.* [17] established some minimax theorems for two set-valued mappings, which improved the corresponding results in [12, 13]. Lin *et al.* [18, 19] investigated some bilevel minimax theorems and hierarchical minimax theorems for set-valued mappings by using nonlinear scalarization function.

Recently, Balaj [20] proposed some minimax theorems for four real-valued functions by using some new alternative principles. Inspired by [17–20] we shall study some generalized hierarchical minimax theorems for set-valued mappings. The imposed conditions involve four set-valued mappings. In the second section, we introduce some notions and preliminary results. In the third section, we prove the hierarchical minimax theorem for scalar set-valued mappings. In the fourth section, we show some hierarchical minimax theorems for set-valued mappings in Hausdorff topological vector spaces by using the results obtained in the previous section.

2 Preliminary

In this section, we recall some notations and some known facts.

Let X, Y be two nonempty sets in two local convex Hausdorff topological vector spaces, respectively, Z be a local convex Hausdorff topological vector space, $S \subset Z$ be a closed convex pointed cone with $\operatorname{int} S \neq \emptyset$, and let Z^* denote the topological dual space of Z. A set-valued mapping $F : X \to 2^Z$ are associated with other two mappings $F^- : Z \to 2^X$, the inverse of F and $F^* : Z \to 2^X$ the dual of F, defined as $F^-(z) = \{x \in X : z \in F(x)\}$ and $F^*(z) = X \setminus F^-(z)$.

Definition 2.1 ([21]) Let $A \subset Z$ be a nonempty subset.

(i) A point $z \in A$ is called a minimal point of A if $A \cap (z - S) = \{z\}$, and $\operatorname{Min} A$ denotes the set of all minimal points of A.

(ii) A point $z \in A$ is called a weakly minimal point of A if $A \cap (z - \operatorname{int} S) = \emptyset$, and $\operatorname{Min}_w A$ denotes the set of all weakly minimal points of A.

(iii) A point $z \in A$ is called a maximal point of A if $A \cap (z + S) = \{z\}$, and $\operatorname{Max} A$ denotes the set of all maximal points of A.

(iv) A point $z \in A$ is called a weakly maximal point of A if $A \cap (z + \operatorname{int} S) = \emptyset$, and $\operatorname{Max}_w A$ denotes the set of all weakly maximal points of A.

For a nonempty compact subset $A \subset Z$, it follows from [12] that $\emptyset \neq \operatorname{Min} A \subset \operatorname{Min}_w A$; $A \subset \operatorname{Min} A + S$ and $\emptyset \neq \operatorname{Max} A \subset \operatorname{Max}_w A$; $A \subset \operatorname{Max} A - S$. We note that, when $Z = R$, $\operatorname{Min} A$ and $\operatorname{Max} A$ are equivalent to $\operatorname{Min}_w A$ and $\operatorname{Max}_w A$, respectively.

Definition 2.2 ([22]) Let $F : X \to 2^Z$ be a set-valued mapping with nonempty values.

(i) F is said to be upper semicontinuous (shortly, u.s.c.) at $x_0 \in X$, if for any neighborhood $N(F(x_0))$ of $F(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that $F(x) \subset N(F(x_0))$, $\forall x \in N(x_0)$. F is u.s.c. on X if F is u.s.c. at any $x \in X$.

(ii) F is said to be lower semicontinuous (shortly, l.s.c.) at $x_0 \in X$, if for any open neighborhood N in Z satisfying $F(x_0) \cap N \neq \emptyset$, there exists a neighborhood $N(x_0)$ of x_0 such that $F(x) \cap N \neq \emptyset$, $\forall x \in N(x_0)$. F is l.s.c. on X if F is l.s.c. at any $x \in X$.

(iii) F is said to be continuous at $x_0 \in X$, if F is both u.s.c. and l.s.c. at x_0. F is continuous on X if F is continuous at any $x \in X$.

(iv) F is said to be closed if the graph of F is closed subset of $X \times Z$.

Definition 2.3 ([17]) Let X be a nonempty subset of a topological vector space, $F : X \to 2^Z$ be a set-valued mapping.

(i) F is said to be S-concave (respectively, S-convex) on X, if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F\bigl(\lambda x_1 + (1 - \lambda)x_2\bigr) - S$$
$$\bigl(\text{respectively, } F\bigl(\lambda x_1 + (1 - \lambda)x_2\bigr) \subset \lambda F(x_1) + (1 - \lambda)F(x_2) - S\bigr);$$

(ii) F is said to be properly S-quasiconcave (respectively, properly S-quasiconvex) on X, if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$,

$$\text{either } F(x_1) \subset F\bigl(\lambda x_1 + (1 - \lambda)x_2\bigr) - S \quad \text{or} \quad F(x_2) \subset F\bigl(\lambda x_1 + (1 - \lambda)x_2\bigr) - S$$
$$\bigl(\text{respectively, either } F\bigl(\lambda x_1 + (1 - \lambda)x_2\bigr) \subset F(x_1) - S \quad \text{or}$$
$$F\bigl(\lambda x_1 + (1 - \lambda)x_2\bigr) \subset F(x_2) - S\bigr);$$

(iii) F is said to be naturally S-quasiconcave (respectively, naturally S-quasiconvex) on X, if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$

$$\begin{aligned} &\operatorname{co}\bigl(F(x_1) \cup F(x_2)\bigr) \subset F\bigl(\lambda x_1 + (1-\lambda)x_2\bigr) - S \\ &\bigl(\text{respectively, } F\bigl(\lambda x_1 + (1-\lambda)x_2\bigr) \subset \operatorname{co}\bigl(F(x_1) \cup F(x_2)\bigr) - S\bigr). \end{aligned}$$

Remark 2.1

(1) Obviously, any S-concave (S-convex) mapping F is naturally S-quasiconcave (naturally S-quasiconvex); any properly S-quasiconcave (properly S-quasiconvex) mapping F is naturally S-quasiconcave (naturally S-quasiconvex).

(2) One should note that the S-concave (respectively, S-convex, properly S-quasiconcave, properly S-quasiconvex, naturally S-quasiconcave, naturally S-quasiconvex) mapping is defined as above S-concave (respectively, above S-convex, above properly S-quasiconcave, above properly S-quasiconvex, above naturally S-quasiconcave, above naturally S-quasiconvex) mapping in [18, 19].

Lemma 2.1 ([22]) *Let $F : X \to 2^Z$ be a set-valued mapping. If X is compact and F is u.s.c. with compact values, then $F(X) = \bigcup_{x\in X} F(x)$ is compact.*

Lemma 2.2 ([17]) *Let $F : X \to 2^Z$ be a continuous set-valued mapping with compact values. Then the set-valued mapping*

$$\Gamma(x) = \operatorname{Max}_w F(x)$$

is nonempty closed and upper semicontinuous.

In the sequel we need the following alternative theorem which is a variant form of Balaj [20].

Lemma 2.3 ([20]) *Let X, Y be two nonempty compact convex subsets in two local convex Hausdorff topological vector spaces. The set-valued mappings $\mathcal{F}_i : X \to Z$, $i = 1, 2, 3, 4$, satisfy the following conditions:*

(i) *for each $x \in X$, $\operatorname{co} \mathcal{F}_1(x) \subset \mathcal{F}_2(x) \subset \mathcal{F}_3(x)$;*

(ii) *$\mathcal{F}_3(\operatorname{co} A) \subset \mathcal{F}_4(A)$ for any finite subset $A \subset X$;*

(iii) *$\mathcal{F}_1$ and $\mathcal{F}_4^*$ are u.s.c.;*

(iv) *$\mathcal{F}_2$ and $\mathcal{F}_3^*$ have compact values.*

Then at least one of the following assertions holds:

(a) *There exists $x_0 \in X$ such that $\mathcal{F}_1(x_0) = \emptyset$.*

(b) *$\bigcap_{x\in X} \mathcal{F}_4(x) \neq \emptyset$.*

3 Hierarchical minimax theorems for scalar set-valued mappings

In this section, we first establish the following hierarchical minimax theorems for scalar set-valued mappings.

Theorem 3.1 *Let X, Y be two nonempty compact convex subsets of local convex Hausdorff topological vector spaces, respectively. Let $F_i : X \times Y \to 2^R$, $i = 1, 2, 3, 4$ be set-valued mappings such that $F_i(x, y) \subset F_{i+1}(x, y) - R_+$. Assume that*

(i) $(x,y) \to F_1(x,y)$ *is u.s.c. with nonempty closed values, and* $(x,y) \to F_4(x,y)$ *is l.s.c.*

(ii) $y \to F_2(x,y)$ *is naturally* R_+*-quasiconcave on* Y *for each* $x \in X$, *and* $x \to F_3(x,y)$ *is naturally* R_+*-quasiconvex on* X *for each* $y \in Y$.

(iii) $y \to F_2(x,y)$ *is closed for all* $x \in X$, *and* $x \to F_3(x,y)$ *is l.s.c. for all* $y \in Y$.

Then either there is $x_0 \in X$ *such that* $F_1(x_0,y) \subset (-\infty,\alpha)$ *for all* $y \in Y$ *or there is* $y_0 \in Y$ *such that* $F_4(x,y_0) \cap [\beta,+\infty) \neq \emptyset$ *for all* $x \in X$.

Furthermore, assume that the sets $\bigcup_{y\in Y} F_1(x,y)$ *and* $\bigcup_{x\in X} F_4(x,y)$ *are compact for all* $y \in Y$ *and* $x \in X$, *respectively. Assume the following condition holds*:

(iv) *for each* $w \in Y$, *there exists* $x_w \in X$ *such that*

$$\operatorname{Max} F_4(x_w,w) \leq \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min} \bigcup_{x\in X} F_4(x,y). \tag{1}$$

Then

$$\operatorname{Min} \bigcup_{x\in X} \operatorname{Max} \bigcup_{y\in Y} F_1(x,y) \leq \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min} \bigcup_{x\in X} F_4(x,y). \tag{2}$$

Proof For any real numbers $\alpha,\beta \in R$ with $\alpha > \beta$, we define the mappings $\mathcal{F}_i : X \to 2^Y$, $i = 1,2,3,4$ by

$$\mathcal{F}_1(x) = \{y \in Y : \exists f \in F_1(x,y), f \geq \alpha\}, \qquad \mathcal{F}_2(x) = \{y \in Y : \exists f \in F_2(x,y), f \geq \alpha\},$$
$$\mathcal{F}_3(x) = \{y \in Y : \exists f \in F_3(x,y), f > \beta\}, \qquad \mathcal{F}_4(x) = \{y \in Y : \exists f \in F_4(x,y), f > \alpha\}.$$

Then we can see that $\mathcal{F}_1(x) \subset \mathcal{F}_2(x) \subset \mathcal{F}_3(x) \subset \mathcal{F}_4(x)$, $\forall x \in X$. For any $x \in X$, if $y \in \mathcal{F}_1(x)$, there exists $f_1 \in F_1(x,y)$ such that $f_1 \geq \alpha$. Since $F_1(x,y) \subset F_2(x,y) - R_+$, there are $f_2 \in F_2(x,y)$ and $r \in R_+$ such that $f_2 = f_1 + r \geq \alpha$. Then $y \in \mathcal{F}_2(x)$, and so $\mathcal{F}_1(x) \subset \mathcal{F}_2(x)$. Noticing that $\alpha > \beta$, one can show $\mathcal{F}_2(x) \subset \mathcal{F}_3(x) \subset \mathcal{F}_4(x)$ by using similar deduction.

For any $x \in X$, we see that $\mathcal{F}_2(x)$ is convex valued. In fact, for any $y_1, y_1' \in \mathcal{F}_2(x)$, there exist $f_1 \in F_2(x,y_1)$ and $f_1' \in F_2(x,y_1')$ such that $f_1 \geq \alpha$ and $f_1' \geq \alpha$. Since $y \to F_2(x,y)$ is naturally R_+-quasiconcave, we have $\lambda f_1 + (1-\lambda)f_1' \in \lambda F_2(x,y_1) + (1-\lambda)F_2(x,y_1') \subset \operatorname{co}(F_2(x,y_1) \cup F_2(x,y_1')) \subset F_2(x,\lambda y_1 + (1-\lambda)y_1') - R_+$, $\forall \lambda \in [0,1]$. Then there exist $f \in F_2(x,\lambda y_1 + (1-\lambda)y_1')$ and $r \in R_+$ such that $f \in \lambda f_1 + (1-\lambda)f_1' + r \geq \alpha$. Therefore, $\lambda y_1 + (1-\lambda)y_1' \in \mathcal{F}_2(x)$, *i.e.* $\mathcal{F}_2(x)$ is convex valued. Thus $\operatorname{co}\mathcal{F}_1(x) \subset \operatorname{co}\mathcal{F}_2(x) = \mathcal{F}_2(x)$, $\forall x \in X$.

Let $y \in \mathcal{F}_3(\operatorname{co}A)$ for a finite subset $A \subset X$. Without loss of generality, we suppose that $y \in \mathcal{F}_3(\lambda x_1 + (1-\lambda)x_2)$ for some $x_1,x_2 \in A$ and $\lambda \in [0,1]$. Then there exists $f \in F_3(\lambda x_1 + (1-\lambda)x_2, y)$ such that $f > \beta$. Since $x \to F_3(x,y)$ is naturally R_+-quasiconvex for each $y \in Y$, there exists $f' \in \operatorname{co}(F_3(x_1,y) \cup F_3(x_2,y))$ such that $f \in f' - R_+$. Therefore, there exist $\mu \in [0,1]$ and $f_1,f_2 \in F_3(x_1,y) \cup F_3(x_2,y)$ and $r \in R_+$ such that $f = f' - r = \mu f_1 + (1-\mu)f_2 - r > \beta$. Then at least one of the assertions $f_1 > \beta$ and $f_2 > \beta$ holds. Hence, $y \in (\mathcal{F}_3(x_1) \cup \mathcal{F}_3(x_2)) \subset \mathcal{F}_3(A)$. Therefore, $\mathcal{F}_3(\operatorname{co}A) \subset \mathcal{F}_3(A) \subset \mathcal{F}_4(A)$.

For any sequence $(x_n,y_n) \in \operatorname{graph}\mathcal{F}_1 = \{(x,y) : \exists f \in F_1(x,y), f \geq \alpha\}$ with $(x_n,y_n) \to (x,y)$, there exist $f_n \in F_1(x_n,y_n)$ such that $f_n \geq \alpha$. We can take subsequence $\{f_{n_k}\}$ such that $\lim_{k\to\infty} f_{n_k} = \liminf_{n\to\infty} f_n = f_0$. Then $f_0 \geq \alpha$. Since F_1 is u.s.c. with closed values, Then F_1 is closed. Thus $f_0 \in F(x_0,y_0)$, and so $(x_0,y_0) \in \operatorname{graph}\mathcal{F}_1$. This implies that $\mathcal{F}_1$ is closed. From compactness of Y it follows that $\mathcal{F}_1$ is upper semicontinuous.

Now, we show that graph $\mathcal{F}_4^* = \{(x,y) : \forall f \in F_4(x,y), f \leq \beta\}$ is closed. Let $(x_n,y_n) \in$ graph $\mathcal{F}_4^*$ with $(x_n,y_n) \to (x_0,y_0)$. From lower semicontinuity of F_4, it follows that for any $f_0 \in F_4(x_0,y_0)$, there exists $f_n \in F_4(x_n,y_n)$ such that $f_n \to f_0$. Then $f_0 \leq \beta$. Therefore graph $\mathcal{F}_4^*$ is closed. Noticing the compactness of Y, we see that $\mathcal{F}_4^*$ is upper semicontinuous.

Since $y \to F_2(x,y)$ is closed for all $x \in X$, $\mathcal{F}_2$ is closed valued. In fact, for any sequence $y_n \subset \mathcal{F}_2(x)$ with $y_n \to y_0$, there exists $f_n \in F_2(x,y_n)$ such that $f_n \geq \alpha$. We can take subsequence $\{f_{n_k}\}$ such that $\lim_{k\to\infty} f_{n_k} = \liminf_{n\to\infty} f_n = f_0$. Then $f_0 \geq \alpha$. It follows from the closedness of $F(x,\cdot)$ that $f_0 \in F(x,y_0)$, and so $\mathcal{F}_2$ has closed values. Next, we claim that $\mathcal{F}_3^*$ has closed values. For any sequence $x_n \subset \mathcal{F}_3^*(y)$ that converges to some point $x_0 \in X$, we see that $y \notin \mathcal{F}_3(x_n)$. Then $f \leq \beta$ for any $f \in F_3(x_n,y)$. Since $x \to F_3(x,y)$ is lower semicontinuous for all $y \in Y$, for any $y_0 \in F(x_0,y)$ there exists $f_n \in F_3(x_n,y)$ such that $f_n \to f_0$. Then $f_0 \leq \beta$ and hence $x_0 \in \mathcal{F}_3^*(y)$. This proves that $\mathcal{F}_3^*$ has closed values. It follows from the compactness of X and Y that both $\mathcal{F}_2$ and $\mathcal{F}_3^*$ have compact values.

Then from Lemma 2.3, it follows that either there is $x_0 \in X$ such that $\mathcal{F}_1(x_0) = \emptyset$, or $\bigcap_{x\in X} \mathcal{F}_4(x) \neq \emptyset$. That is, for any real numbers $\alpha, \beta \in R$ with $\alpha > \beta$, either there is $x_0 \in X$ such that $F_1(x_0,y) \subset (-\infty,\alpha)$ for all $y \in Y$ or there is $y_0 \in Y$ such that $F_4(x,y_0) \cap [\beta,+\infty) \neq \emptyset$ for all $x \in X$.

Furthermore, the compactness of $\bigcup_{x\in X} F_4(x,y)$ implies that $\operatorname{Min}\bigcup_{x\in X} F_4(x,y)$ is nonempty for all $y \in Y$. Since $(x,y) \to F_4(x,y)$ is lower semicontinuous, it follows that $y \to \bigcup_{x\in X} F_4(x,y)$ is lower semicontinuous. By the compactness of Y and the proof of Lemma 3.2 [12], the set $\bigcup_{y\in Y} \operatorname{Min}\bigcup_{x\in X} F_4(x,y)$ is nonempty and compact, and so $\operatorname{Max}\bigcup_{y\in Y} \operatorname{Min}\bigcup_{x\in X} F_4(x,y) \neq \emptyset$. Set any real numbers $\alpha, \beta \in R$ with $\alpha > \beta > \operatorname{Max}\bigcup_{y\in Y} \operatorname{Min}\bigcup_{x\in X} F_4(x,y)$. From (iv), we see that, for each $w \in Y$, there exists $x_w \in X$ such that $F_4(x_w,w) \cap [\beta,+\infty) = \emptyset$. Therefore, there is $x_0 \in X$ such that $F_1(x_0,y) \subset (-\infty,\alpha)$ for all $y \in Y$. Hence

$$\operatorname{Min}\bigcup_{x\in X} \operatorname{Max}\bigcup_{y\in Y} F_1(x,y) \leq \operatorname{Max}\bigcup_{y\in Y} F_1(x_0,y) \leq \alpha.$$

By the arbitrariness of α and β, (2) holds. □

Example 3.1 Let $X = Y = [0,1] \subset R$. Define four mappings $F_i : X \times Y \to 2^R$, $i = 1,2,3,4$, as

$$F_1(x,y) = [x^2 - 1 + y, x]; \qquad F_2(x,y) = \left[x^2 - \frac{1}{2} + y, x + \frac{1}{2}\right];$$
$$F_3(x,y) = [x^2 + y, x^2 + 1]; \qquad F_4(x,y) = [x^2 + y, x + 1].$$

We can see that $F_i(x,y) \subset F_{i+1}(x,y) - R_+$ for all $(x,y) \in X \times Y$ and conditions (i)-(iii) of Theorem 3.1 hold. It is obvious that $\bigcup_{x\in X} F_1(x,y)$ and $\bigcup_{y\in Y} F_4(x,y)$ are compact for all $y \in Y$ and $x \in X$, respectively. Now, we show condition (iv) of Theorem 3.1 is true. One can calculate that $\operatorname{Min}\bigcup_{x\in X} F_4(x,y) = \{y\}$, $\forall y \in Y$, and $\operatorname{Max}\bigcup_{y\in Y} \operatorname{Min}\bigcup_{x\in X} F_4(x,y) = 1$. Taking $x = 0$, we have

$$\operatorname{Max} F_4(0,y) \leq \operatorname{Max}\bigcup_{y\in Y} \operatorname{Min}\bigcup_{x\in X} F_4(x,y), \quad \forall y \in Y.$$

Then all of the conditions of Theorem 3.1 valid. So, the conclusion of Theorem 3.1 holds. In fact, $\operatorname{Min}\bigcup_{x\in X}\operatorname{Max}\bigcup_{y\in Y}F_1(x,y) = 0 < 1 = \operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}\bigcup_{x\in X}F_4(x,y)$.

When $F_1(x,y) = F_2(x,y) = F(x,y)$ and $F_3(x,y) = F_4(x,y) = G(x,y)$ in Theorem 3.1, we state the special case of Theorem 3.1 as follows.

Theorem 3.2 *Let X, Y be two nonempty compact convex subsets of local convex Hausdorff topological vector spaces, respectively. The set-valued mappings $F, G : X \times Y \to 2^R$ with $F(x,y) \subset G(x,y) - R_+$. Assume that*

(i) *$(x,y) \to F(x,y)$ is u.s.c. with nonempty closed values, and $(x,y) \to G(x,y)$ is l.s.c.*

(ii) *$y \to F(x,y)$ is naturally R_+-quasiconcave on Y for each $x \in X$, and $x \to G(x,y)$ is naturally R_+-quasiconvex on X for each $y \in Y$.*

Then either there is $x_0 \in X$ such that $F(x_0,y) \subset (-\infty,\alpha)$ for all $y \in Y$ or there is $y_0 \in Y$ such that $G(x,y_0) \cap [\beta,+\infty) \neq \emptyset$ for all $x \in X$.

Furthermore, assume that the sets $\bigcup_{y\in Y}F(x,y)$ and $\bigcup_{x\in X}G(x,y)$ are compact for all $y \in Y$ and $x \in X$, respectively. Assume the following condition holds:

(iii) *for each $w \in Y$, there exists $x_w \in X$ such that*

$$\operatorname{Max} G(x_w,w) \leq \max\bigcup_{y\in Y}\operatorname{Min}\bigcup_{x\in X}F_4(x,y).$$

Then

$$\operatorname{Min}\bigcup_{x\in X}\operatorname{Max}\bigcup_{y\in Y}F(x,y) \leq \operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}\bigcup_{x\in X}G(x,y).$$

Proof Since F is u.s.c. with nonempty closed values, it follows that $y \to F(x,y)$ is closed for all $x \in X$ by Proposition 7 in [22], p. 110. From Theorem 3.1, it is easy to show the conclusion holds. □

Remark 3.1 It is obvious that $F(x,y) \subset G(x,y)$ implies $F(x,y) \subset G(x,y) - R_+$. So Theorem 3.2 generalizes Theorem 2.1 in [18].

It is well known that both sets $\bigcup_{y\in Y}F(x,y)$ and $\bigcup_{x\in X}G(x,y)$ are compact for any $y \in Y$ and $x \in X$ whenever the mappings F and G are upper semicontinuous with nonempty compact values. Hence we can deduce the following result.

Corollary 3.1 *Let X, Y be two nonempty compact convex subsets of local convex Hausdorff topological vector spaces, respectively. The set-valued mappings $F, G : X \times Y \to 2^R$ come with nonempty compact values and $F(x,y) \subset G(x,y) - R_+$. Assume that*

(i) *$(x,y) \to F(x,y)$ is u.s.c., and $(x,y) \to G(x,y)$ is continuous.*

(ii) *$y \to F(x,y)$ is naturally R_+-quasiconcave on Y for each $x \in X$, and $x \to G(x,y)$ is naturally R_+-quasiconvex on X for each $y \in Y$.*

(iii) *For each $w \in Y$, there exists $x_w \in X$ such that*

$$\operatorname{Max} G(x_w,w) \leq \operatorname{Max}\bigcup_{y\in Y}\operatorname{Min}\bigcup_{x\in X}G(x,y).$$

Then

$$\mathrm{Min}\bigcup_{x\in X}\mathrm{Max}\bigcup_{y\in Y}F(x,y)\le \mathrm{Max}\bigcup_{y\in Y}\mathrm{Min}\bigcup_{x\in X}G(x,y).$$

Remark 3.2 Corollary 3.1 generalizes Theorem 2.1 in [17] and weakens the continuity of F_1 in Theorem 2.1 in [17]. It also generalizes Theorem 2.1 in [12] from one set-valued mapping to two set-valued mappings.

4 Generalized hierarchical minimax theorem

In this section, we will discuss some generalized hierarchical minimax theorems for set-valued mappings valued in a complete locally convex Hausdorff topological vector space.

Lemma 4.1 *Let X, Y be two nonempty compact convex subsets of local convex Hausdorff topological vector spaces, respectively. The set-valued mapping $F:X\times Y\to 2^Z$ comes with nonempty compact values. If $(x,y)\to F(x,y)$ is u.s.c., and $x\to F(x,y)$ is l.s.c. for each $y\in Y$, then the set-valued mapping*

$$A(x)=\mathrm{Max}_w\bigcup_{y\in Y}F(x,y)$$

is u.s.c. with nonempty compact values.

Proof Define a set-valued mapping $T:X\to 2^Z$ as

$$T(x)=\bigcup_{y\in Y}F(x,y).$$

It follows from Lemma 2.4 in [16] that T is continuous. By Lemma 2.1 and compactness of Y, T is compact-valued. Then, by Lemma 2.2, we see that A is nonempty closed and u.s.c. on X. By compactness of X, it follows that $A(x)$ is compact for each $x\in X$. □

Theorem 4.1 *Let X, Y be two nonempty compact convex subsets of local convex Hausdorff topological vector spaces, respectively, Z be a complete locally convex Hausdorff topological vector space. The set-valued mappings $F_i:X\times Y\to 2^Z$, $i=1,2,3,4$ come with nonempty compact values and $F_i(x,y)\subset F_{i+1}(x,y)-S$. Assume that*

(i) *$(x,y)\to F_1(x,y)$ is u.s.c., $x\to F_1(x,y)$ is l.s.c. for each $y\in Y$, and $(x,y)\to F_4(x,y)$ is continuous;*

(ii) *$y\to F_2(x,y)$ is naturally S-quasiconcave on Y for each $x\in X$, and $x\to F_3(x,y)$ is naturally S-quasiconvex on X for each $y\in Y$;*

(iii) *$y\to F_2(x,y)$ is u.s.c. for all $x\in X$, and $x\to F_3(x,y)$ is l.s.c. for all $y\in Y$;*

(iv) *for each $w\in Y$, there exists $x_w\in X$ such that*

$$\mathrm{Max}\bigcup_{y\in Y}\mathrm{Min}_w\bigcup_{x\in X}F_4(x,y)-F_4(x_w,w)\subset S;$$

(v) *for each $w\in Y$*

$$\mathrm{Max}\bigcup_{y\in Y}\mathrm{Min}_w\bigcup_{x\in X}F_4(x,y)\subset \mathrm{Min}_w\bigcup_{x\in X}F_4(x,w)+S.$$

Then

$$\operatorname{Max} \bigcup_{x\in X} \operatorname{Min}_w \bigcup_{y\in Y} F_4(x,y) \subset \operatorname{Min}\left(\operatorname{co} \bigcup_{y\in Y} \operatorname{Max}_w \bigcup_{x\in X} F_1(x,y)\right) + S. \tag{3}$$

Proof Let $L(x) := \operatorname{Max}_w \bigcup_{y\in Y} F_1(x,y)$. By Lemma 4.1, $L(x)$ is u.s.c. with nonempty compact values. From Lemma 2.1, it follows that $L(X) = \bigcup_{x\in X} L(x)$ is compact, and so is $\operatorname{co}(L(X))$. Then $\operatorname{co}(L(X)) + S$ is a closed set with nonempty interior. Suppose that $v \in Z$ and $v \notin \operatorname{co}(L(X)) + S$. By the separation theorem, there exist $\xi \in Z^*$ and $\alpha_1, \alpha_2 \in R$ such that

$$\xi(v) < \alpha_1 < \alpha_2 < \xi(u+s), \quad \forall u \in \operatorname{co}\bigl(L(X)\bigr), \forall s \in S. \tag{4}$$

By using a similar discussion to Theorem 3.1 in [17], we have $\xi \in S^*$ and $\xi(S) = R^+$. From assumptions (i) and (iii), it is easy to see that $(x,y) \to \xi(F_1(x,y))$ is u.s.c., $(x,y) \to \xi(F_4(x,y))$ is l.s.c., $y \to \xi(F_2(x,y))$ is closed for all $x \in X$, and $x \to \xi(F_3(x,y))$ is l.s.c. for all $y \in Y$. From condition (ii), applying Proposition 3.9 and Proposition 3.13 in [16], we see that $y \to \xi(F_2(x,y))$ is naturally R^+-quasiconcave on Y for each $x \in X$, and $x \to \xi(F_3(x,y))$ is naturally R^+-quasiconvex on X for each $y \in Y$. By the condition (iv), for each $w \in Y$, there exists $x_w \in X$ such that

$$\operatorname{Max} \xi\bigl(F_4(x_w, w)\bigr) \le \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min} \bigcup_{x\in X} \xi\bigl(F_4(x,y)\bigr).$$

Since F_1 and F_4 are u.s.c. and come with compact values, we see that $\bigcup_{x\in X} \xi(F_1(x,y))$ and $\bigcup_{y\in Y} \xi(F_4(x,y))$ are compact for all $y \in Y$ and $x \in X$, respectively. Then for set-valued mappings $\xi(F_i)$, $i = 1,2,3,4$, all conditions of Theorem 3.1 hold. Therefore we see that

$$\operatorname{Min} \bigcup_{x\in X} \operatorname{Max} \bigcup_{y\in Y} \xi\bigl(F_1(x,y)\bigr) \le \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min} \bigcup_{x\in X} \xi\bigl(F_4(x,y)\bigr). \tag{5}$$

Since Y is compact and F_1 has nonempty compact values, for any $x \in X$, there exist y_x and $f(x,y_x) \in F_1(x,y_x)$ with $f(x,y_x) \in L(x)$ such that

$$\xi\bigl(F_1(x,y_x)\bigr) = \operatorname{Max} \bigcup_{y\in Y} \xi\bigl(F_1(x,y)\bigr).$$

From (4), choosing $s = 0$ and $u = f(x,y_x)$, it follows that

$$\xi(v) < \xi\bigl(f(x,y_x)\bigr) = \operatorname{Max} \bigcup_{y\in Y} \xi\bigl(F_1(x,y)\bigr)$$

for all $x \in X$. Then

$$\xi(v) < \operatorname{Min} \bigcup_{x\in X} \operatorname{Max} \bigcup_{y\in Y} \xi\bigl(F_1(x,y)\bigr).$$

By (5),

$$\xi(v) < \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min} \bigcup_{x\in X} \xi\bigl(F_4(x,y)\bigr).$$

Since Y is compact, there exists $y' \in Y$ such that

$$\xi(v) < \operatorname{Min} \bigcup_{x\in X} \xi\big(F_4\big(x,y'\big)\big) = \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min} \bigcup_{x\in X} \xi\big(F_4(x,y)\big).$$

From $\xi(s) \geq 0$ for all $s \in S$, it follows that $v \notin \bigcup_{x\in X}(F_4(x,y')) + S$, and then

$$v \notin \operatorname{Min}_w \bigcup_{x\in X} \big(F_4\big(x,y'\big)\big) + S.$$

Combined with the assumption (v), we have

$$v \notin \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F_4(x,y).$$

That is, for any $v \in \operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F_4(x,y)$,

$$v \in \operatorname{co}\big(L(x)\big) + S.$$

Hence

$$\operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} \big(F_4(x,y)\big) \subset \operatorname{co}\big(L(x)\big) + S.$$

Since $\operatorname{co}(L(X)) = \operatorname{co}(\bigcup_{x\in X} L(x)) = \operatorname{co}(\bigcup_{x\in X} \operatorname{Max}_w \bigcup_{y\in Y} F_1(x,y))$ is compact, we have

$$\operatorname{co}\Big(\bigcup_{x\in X} \operatorname{Max}_w \bigcup_{y\in Y} F_1(x,y)\Big) \subset \operatorname{Min}\Big(\operatorname{co}\Big(\bigcup_{x\in X} \operatorname{Max}_w \bigcup_{y\in Y} F_1(x,y)\Big)\Big) + S.$$

Therefore, (3) holds. □

Example 4.1 Let $X = Y = [0,1]$, $Z = R^2$, and $S = R^2_+$. Define set-valued mappings $F_i : X \times Y \to 2^Z$, $i = 1,2,3,4$, as

$$F_1(x,y) = \big[x^2 - 1 + y, x\big] \times \{-1\}, \qquad F_2(x,y) = \left[x^2 - \frac{1}{2} + y, x + \frac{1}{2}\right] \times \left\{x^2 - \frac{1}{2}\right\},$$

$$F_3(x,y) = \big[x^2 + y, x + 1\big] \times \left\{y + \frac{1}{2}\right\}, \qquad F_4(x,y) = \{x + 1\} \times [y + 1, 2].$$

For all $(x,y) \in X \times Y$, we can see that the $F_i(x,y)$, $i = 1,2,3,4$, are compact and

$$F_i(x,y) \subset F_{i+1}(x,y) - S.$$

It is easy to show that the conditions (i)-(iii) hold in Theorem 4.1. We explain conditions (iv) and (v) are valid. We can calculate that

$$\operatorname{Min}_w \bigcup_{x\in X} F_4(x,y) = \{1\} \times [y + 1, 2] \cup [1,2] \times \{y + 1\},$$

$$\operatorname{Max} \bigcup_{y\in Y} \operatorname{Min}_w \bigcup_{x\in X} F_4(x,y) = \big\{(2,2)\big\}.$$

For each $w \in Y$, let $x_w = 0$. Then

$$\mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} F_4(x, y) - F_4(x_w, w) = \{(2,2)\} - \{1\} \times [y+1, 2] \subset S.$$

The condition (iv) holds. We can see that

$$\begin{aligned} &\mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} F_4(x, y) \\ &\quad = \{(2,2)\} \subset \{1\} \times [y+1, 2] \cup [1,2] \times \{y+1\} + S \\ &\quad = \mathrm{Min}_w \bigcup_{x \in X} F_4(x, y) + S. \end{aligned}$$

Then all of the assumptions of Theorem 4.1 are valid. So, the conclusion of Theorem 4.1 holds. In fact,

$$\mathrm{Min}\left(\mathrm{co} \bigcup_{x \in X} \mathrm{Max}_w \bigcup_{y \in Y} F_1(x, y) \right) = \{(0, -1)\}.$$

Then

$$\begin{aligned} &\mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} F_4(x, y) \\ &\quad = \{(2,2)\} \subset \{(0,-1)\} + S \\ &\quad = \mathrm{Min}_w \mathrm{Min}\left(\mathrm{co}\left(\bigcup_{x \in X} \mathrm{Max}_w \bigcup_{y \in Y} F_1(x, y) \right) \right) + S. \end{aligned}$$

When $F_1(x, y) = F_2(x, y) = F(x, y)$ and $F_3(x, y) = F_4(x, y) = G(x, y)$ in Theorem 4.1, we state the special case of Theorem 4.1 as follows.

Corollary 4.1 *Let X, Y be two nonempty compact convex subsets of local convex Hausdorff topological vector spaces, respectively, Z be a complete locally convex Hausdorff topological space. The set-valued mappings $F, G : X \times Y \to 2^Z$ come with nonempty compact values and $F(x, y) \subset G(x, y) - S$. Assume that*

(i) *$(x, y) \to F(x, y)$ is u.s.c., $x \to F(x, y)$ is l.s.c. for each $y \in Y$, and $(x, y) \to G(x, y)$ is continuous;*

(ii) *$y \to F(x, y)$ is naturally S-quasiconcave on Y for each $x \in X$, and $x \to G(x, y)$ is naturally S-quasiconvex on X for each $y \in Y$;*

(iii) *for each $w \in Y$, there exists $x_w \in X$ such that*

$$\mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} G(x, y) - G(x_w, w) \subset S;$$

(iv) *for each $w \in Y$*

$$\mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w \bigcup_{x \in X} G(x, y) \subset \mathrm{Min}_w \bigcup_{x \in X} G(x, w) + S.$$

Then

$$\operatorname{Max} \bigcup_{x \in X} \operatorname{Min}_w \bigcup_{y \in Y} G(x,y) \subset \operatorname{Min}\left(\operatorname{co} \bigcup_{y \in Y} \operatorname{Max}_w \bigcup_{x \in X} F(x,y) \right) + S. \tag{6}$$

Remark 4.1

(1) Corollary 4.1 generalizes Theorem 3.1 in [17] and weakens the continuity if F.

(2) Corollary 4.1 also generalizes Theorem 3.1 in [18] since $F(x,y) \subset G(x,y)$ implies $F(x,y) \subset G(x,y) - S$.

5 Concluding remarks

We have proven some hierarchical minimax theorems for scalar set-valued mappings and generalized hierarchical minimax theorems for set-valued mappings valued in a complete locally convex Hausdorff topological vector space. The imposed conditions involved four set-valued mappings. The main tools to prove our results have been an alternative principle and separation theorems. Some examples have been provided to illustrate our results.

Competing interests
The author declares to have no competing interests.

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