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Some properties of a kind of generalized Teodorescu operator in Clifford analysis

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Abstract
First, a kind of generalized Teodorescu operator which is related to the k-regular functions are defined. Then the boundedness and the Hölder continuity of the generalized Teodorescu operator are discussed. Finally, the stability and error estimate of the generalized Teodorescu operator are given when the boundary surface of the integral domain Ω is perturbed.

Keywords: Clifford analysis; k-regular functions; generalized Teodorescu operator; Hölder continuity; stability

1 Introduction

The Clifford algebra $\mathcal{A}_n(R)$ was established by Clifford [1] in 1878, which is the extension of complex numbers, quaternions, and exterior algebras. It is an associative and incommutable algebra structure. And it possesses both theoretical and applicable value to many fields, such as quantum mechanics, quantum field theory [2], projective geometry, computer graphics [3–5], neural network theory [6, 7], and so on. Clifford analysis is an important branch of modern analysis, which studies functions defined on R^n with their values in Clifford algebra space $\mathcal{A}_n(R)$. Clifford analysis is an important tool of modern mathematics and physics. In addition, Clifford analysis is now a widely studied field [8]: function theory, harmonic analysis, potential theory, partial differential equations, differential geometry. Since 1970, on the basis of the Dirac operator, Brackx *et al.* [9–12] *etc.* put forward the regular function, which is an extension of the holomorphic function in higher dimensional space. Thus they lay a theoretical foundation for Clifford analysis. The k-regular function is a natural generalization of the regular functions. In 1976, Brackx [13] was first to introduce k-regular functions of the real quaternion and gave the Cauchy integral formula and Taylor expansion. In 1977, Delanghe and Brackx [14] studied k-regular functions which were defined on R^n with values in Clifford algebra space $\mathcal{A}_n(R)$ and obtained the corresponding Cauchy integral formula. Recently, many scholars such as Begehr *et al.* [15], Li *et al.* [16, 17], Zeng and Yang [18] *etc.* have studied some properties and corresponding boundary value problems of k-regular functions.

Teodorescu operator is the generalized solution of non-homogeneous Dirac equation. Therefore it plays a key role in studying the integral expression of non-homogeneous Dirac equation and many boundary value problems. In addition, it also have important applications in many subjects, such as physics, chemistry, engineering technology, *etc.* Due to its good properties and important applications, it has been a much studied and significant topic. Vekua [19] first discussed some properties of the Teodorescu operator detailedly, and Hile [20] studied some properties of the Teodorescu operator in R^n. Then Gilbert *et al.* [21] and Meng [22] studied its many properties in high dimensional complex space. Gürlebeck and Sprössig [23], and Yang [24] discussed its properties and the corresponding boundary value problems in the real quaternion analysis. Wang and Gong [25] discussed the stabilities of the singular integral operators when the boundary curve of integral domain is perturbed. Recently, Brackx *et al.* [26] studied some properties of the Teodorescu operator which is related to the Hermitian regular functions. Wang *et al.* [27, 28] showed some properties of the Teodorescu operator and corresponding boundary value problems.

In this paper, we define a kind of generalized Teodorescu operator which is related to the k-regular functions in real Clifford analysis. First, we discuss the boundedness and Hölder continuity for the generalized Teodorescu operator in a nonempty open bounded connected domain in R^n. Second, we discuss the stability and give the error estimate of the generalized Teodorescu operator when the boundary surface of the integral domain is perturbed. These results make the theory of Clifford analysis more perfect and also lay a theoretical foundation for studying the properties of singularity integral operator in Clifford analysis.

2 Preliminaries

Let $e_1, \dots, e_n$ be an orthogonal basis of the Euclidean space R^n, and let $\mathcal{A}_n(R)$ be the Clifford algebra with basis $\{e_A : e_A = e_{\alpha_1} \cdots e_{\alpha_h}\}$, where $A = \{\alpha_1, \dots, \alpha_h\} \subseteq \{1, \dots, n\}$, $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_h \le n$ and $e_\emptyset = e_0 = 1$. The associative and noncommutative multiplication of the basis in $\mathcal{A}_n(R)$ is governed by the rules:

$$\begin{cases} e_i^2 = -1, & i = 1, 2, \dots, n, \\ e_i e_j = -e_j e_i, & 1 \le i, j \le n, i \ne j, \\ e_0 e_i = e_i, & i = 1, 2, \dots, n. \end{cases}$$

Hence the real Clifford algebra is composed of elements having the type $a = \sum_A x_A e_A$, $x_A \in R$. The norm for an element $a \in \mathcal{A}_n(R)$ is taken to be $|a| = \sqrt{\sum_A |x_A|^2}$ and satisfies $|\bar{a}| = |a|$, $|a + b| \le |a| + |b|$, $|ab| \le 2^n |a||b|$.

In addition, we suppose $\Omega \subset R^n$ ($n \ge 2$) is a bounded connected domain and the boundary $\partial\Omega$ is a differentiable, oriented, and compact Liapunov surface. The function f, which is defined in Ω with values in $\mathcal{A}_n(R)$, can be expressed as $f(x) = \sum_A f_A(x) e_A$, where the $f_A(x)$ are real-valued functions. In this paper, let $f(x) \in C^{(r)}(\Omega, \mathcal{A}_n(R)) = \{f \mid f : \Omega \to \mathcal{A}_n(R), f(x) = \sum_A f_A(x) e_A\}$, where $f_A(x)$ has continuous r times differentials. The Dirac operator is defined as follows:

$$Df = \sum_{i=1}^n e_i \frac{\partial f}{\partial x_i} = \sum_{i=1}^n \sum_A e_i e_A \frac{\partial f_A}{\partial x_i}, \qquad fD = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i = \sum_{i=1}^n \sum_A e_A e_i \frac{\partial f_A}{\partial x_i}.$$

If $Df(x) = 0$ ($f(x)D = 0$) ($x \in \Omega$), then f is called left(right) regular function. Furthermore, if $D^k f(x) = 0$ ($f(x)D^k = 0$) ($x \in \Omega$), where $r \ge k$, $k < n$. Then f is called left (right) k-regular function. Usually left regular (k-regular) is called regular (k-regular) for short. And obviously, when $k = 1$, a k-regular function is a regular function. $f(x) \in$

$L^p(\Omega)$ means $D^j f(x) \in L^p(\Omega)$, $j = 0, 1, \dots, k-1$, and $L_p[f(x), \Omega] = \sum_{j=0}^{k-1} L_p[D^j f(x), \Omega]$. Denote $B(\rho_0) = \{\omega \mid \omega = \sum_{i=1}^{n} \omega_i(x) e_i, \omega_i \in C^1(\partial\Omega), \omega_n(x) \geq 0, \|\omega\|_{\partial\Omega} < \rho_0\}$, where $\|\omega\|_{\partial\Omega} = \max_{x\in\partial\Omega} \sqrt{\sum_{i=1}^{n} \sqrt{\sum_{j=1}^{n} |\frac{\partial(\omega_i(x))}{\partial x_j}|^2}} + \max_{x\in\partial\Omega} |\omega(x)|$.

In the following, we define a kind of generalized singular Teodorescu operator and give some necessary lemmas. Then we discuss some properties of the generalized singular Teodorescu operator.

Definition 2.1 Let Ω be as stated above and $f(x) \in C^{(r)}(\Omega, \mathcal{A}_n(R))$, $D^j f(x) \in L^p(\Omega)$ ($j = 0, 1, \dots, k-1$), we define a kind of generalized singular Teodorescu operator as follows:

$$\big(T_\Omega[f]\big)(y) = \sum_{j=0}^{k-1} (-1)^j \int_\Omega \frac{A_{j+1}}{\omega_n} \frac{D^j f(x)(\bar{x} - \bar{y})^{j+1}}{|x-y|^{n+\alpha}}\, dx,$$

where $0 < \alpha < 1$ is a positive constant and ω_n is the area of the unit sphere in R^n, A_j is a constant as stated in [15], it is irrelevant to the x, y.

Remark 2.1 When $\alpha = 0$, $n = 2$, $k = 1$, the singular integral operator is a normal Teodorescu operator.

Remark 2.2 When $y \in \Omega^- = R^n - \overline{\Omega}$, $T[f]$ is a normal generalized integral. When $y \in \overline{\Omega}$, $T[f]$ is a kind of generalized singular integral.

Lemma 2.1 ([2]) *Let $\Omega \subset R^n$ ($n \geq 2$) be as stated above. When $\mu < n$, for any $y \in R^n$, we have $\int_\Omega |x-y|^{-\mu}\, dx \leq M_3$, where M_3 is a constant number which only depends on μ, Ω.*

Lemma 2.2 ([16]) *For any $x, y_1, y_2 \in R^n$, when $j \geq 0$, we have*

$$\left| \frac{(x-y_1)^{j+1}}{|x-y_1|^n} - \frac{(x-y_2)^{j+1}}{|x-y_2|^n} \right| \leq \left[\sum_{i=1}^{n-1} \frac{(|x| + |y_2|)^j}{|x-y_1|^i |x-y_2|^{n-i}} + \sum_{i=1}^{j} \frac{(|x| + |y_2|)^{j-i}}{|x-y_1|^{n-i}} \right] |y_1 - y_2|.$$

Remark 2.3 When $j = 0$, the second part $\sum_{i=1}^{j} \frac{(|x|+|y_2|)^{j-i}}{|x-y_1|^{n-i}}$ is vanishing.

Lemma 2.3 ([29]) *If $\sigma_1, \sigma_2 > 0$, $0 \leq \alpha \leq 1$, then we have $|\sigma_1^\alpha - \sigma_2^\alpha| \leq |\sigma_1 - \sigma_2|^\alpha$.*

Lemma 2.4 ([21]) *Suppose Ω is a bounded domain in R^n ($n \geq 2$), and let α', β' satisfy $0 < \alpha', \beta' < n$, $\alpha' + \beta' > n$. Then, for all $x_1, x_2 \in R^n$ and $x_1 \neq x_2$, we have*

$$\int_\Omega |t - x_1|^{-\alpha'} |t - x_2|^{-\beta'}\, dt \leq M_1\big(\alpha', \beta'\big) |x_1 - x_2|^{n-\alpha'-\beta'}.$$

Lemma 2.5 ([25]) *Let Ω be as stated above. Suppose the area of $\partial\Omega$ is $A_{\partial\Omega}$ and $\partial\Omega_\omega = \{t + \omega(t) \mid t \in \partial\Omega, \omega(t) \in B(\rho_0)\}$. Suppose Ω_ω is the inside domain surrounded by $\partial\Omega_\omega$ and $\Omega_\omega^- = R^n - \Omega_\omega$, $E_1 = \Omega_\omega \cap \Omega^-$, $E_2 = \Omega_\omega^- \cap \Omega$. Then we have $A_{E_1 \cup E_2} \leq M_2 \|\omega\|_{\partial\Omega}$.*

3 Some properties of generalized Teodorescu operator

Theorem 3.1 *Let $\Omega \subset R^n$ ($n \geq 2$) be as stated above, $f(x) \in C^{(r)}(\Omega, \mathcal{A}_n(R))$, $D^j f(x) \in L^p(\Omega)$ ($j = 0, 1, \dots, k-1$), $p > n/(1-\alpha)$. Then $T_\Omega[f](y)$ is uniformly bounded on R^n, and we have*

$$\big| T_\Omega[f](y) \big| \leq M_4(n, p, \alpha, \Omega) L_p[f, \Omega].$$

Proof By the Hölder inequality, we have

$$\begin{aligned}\left|T_\Omega[f](y)\right| &= \left|\sum_{j=0}^{k-1}(-1)^j\int_\Omega \frac{A_{j+1}}{\omega_n}\frac{D^jf(x)(\bar{x}-\bar{y})^{j+1}}{|x-y|^{n+\alpha}}\,dx\right| \\ &\le J_1\sum_{j=0}^{k-1}\left[\int_\Omega \left|D^jf(x)\right|^p|dx|\right]^{\frac{1}{p}}\left[\int_\Omega \frac{|dx|}{|x-y|^{(n+\alpha-j-1)q}}\right]^{\frac{1}{q}} \\ &\le J_2L_p[f,\Omega]\sum_{j=0}^{k-1}\left[\int_\Omega \frac{|dx|}{|x-y|^{(n+\alpha-j-1)q}}\right]^{\frac{1}{q}}.\end{aligned}$$

Again from $p > n/(1-\alpha)$, $1/p + 1/q = 1$, we have $1 < q < n/(n+\alpha-1)$. Thus

$$(n+\alpha-j-1)q \le (n+\alpha-1)q < n, \quad j = 0,1,\ldots,k-1.$$

Hence for all $j = 0,1,\ldots,k-1$, the integral $\int_\Omega |x-y|^{-(n+\alpha-j-1)q}|dx|$ is convergent. Thus it is bounded. And from Lemma 2.1, we can know its boundary is independent of y. Thus for all $y \in R^n$, we have

$$\left|T_\Omega[f](y)\right| \le M_4(n,p,\alpha,\Omega)L_p[f,\Omega]. \qquad \square$$

Theorem 3.2 *Let Ω be as stated above, $f(x) \in C^{(r)}(\Omega,\mathcal{A}_n(R))$, $D^jf(x) \in L^p(\Omega)$ $(j = 0,1,\ldots,k-1)$, $p > n/(1-\alpha)$. Then we have the following results*:

(1) *If $1/2 \le \alpha < 1$, then, for any $y_1, y_2 \in \Omega$, we can obtain*

$$\left|T_\Omega[f](y_1) - T_\Omega[f](y_2)\right| \le M_5(n,p,\alpha,\Omega)L_p[f,\Omega]|y_1-y_2|^\beta,$$

where $\beta = 1-\alpha-n/p$, $0 < \beta < 1$.

(2) *If $0 < \alpha < 1/2$, let p_1 be a constant and satisfies $p_1 < p$, $n/(1-\alpha) < p_1 < n/(1-2\alpha)$, then, for any $y_1, y_2 \in \Omega$, we have*

$$\left|T_\Omega[f](y_1) - T_\Omega[f](y_2)\right| \le M_6(n,p,\alpha,\Omega)L_{p_1}[f,\Omega]|y_1-y_2|^\gamma,$$

where $\gamma = 1-\alpha-n/p_1$, $0 < \gamma < 1$.

Proof (1) From Definition 2.1, we get

$$\begin{aligned}&\left|T_\Omega[f](y_1) - T_\Omega[f](y_2)\right| \\ &\quad= \left|\sum_{j=0}^{k-1}(-1)^j\int_\Omega \frac{A_{j+1}}{\omega_n}\frac{D^jf(x)(\bar{x}-\bar{y}_1)^{j+1}}{|x-y_1|^{n+\alpha}}\,dx \right.\\ &\qquad\left. - \sum_{j=0}^{k-1}(-1)^j\int_\Omega \frac{A_{j+1}}{\omega_n}\frac{D^jf(x)(\bar{x}-\bar{y}_2)^{j+1}}{|x-y_2|^{n+\alpha}}\,dx\right| \\ &\quad\le J_3\sum_{j=0}^{k-1}\int_\Omega \left|D^jf(x)\right|\left|\frac{(\bar{x}-\bar{y}_1)^{j+1}}{|x-y_1|^{n+\alpha}} - \frac{(\bar{x}-\bar{y}_2)^{j+1}}{|x-y_2|^{n+\alpha}}\right||dx|. \qquad (3.1)\end{aligned}$$

Again, from Lemma 2.2, Lemma 2.3, and Ω being a bounded domain, we obtain

$$\begin{aligned}
&\left|\frac{(\bar{x}-\bar{y}_1)^{j+1}}{|x-y_1|^{n+\alpha}}-\frac{(\bar{x}-\bar{y}_2)^{j+1}}{|x-y_2|^{n+\alpha}}\right| \\
&\leq \frac{1}{|x-y_1|^{\alpha}}\left[\sum_{i=1}^{n-1}\frac{(|x|+|y_2|)^j}{|x-y_1|^i|x-y_2|^{n-i}}+\sum_{i=1}^{j}\frac{(|x|+|y_2|)^{j-i}}{|x-y_1|^{n-i}}\right]|y_1-y_2| \\
&\quad+\frac{1}{|x-y_2|^{n-j-1}}\left|\frac{1}{|x-y_1|}-\frac{1}{|x-y_2|}\right|^{\alpha} \\
&\leq \frac{1}{|x-y_1|^{\alpha}}\sum_{i=1}^{n-1}\frac{(|x|+|y_2|)^j}{|x-y_1|^i|x-y_2|^{n-i}}|y_1-y_2| \\
&\quad+\frac{1}{|x-y_1|^{\alpha}}\sum_{i=1}^{j}\frac{(|x|+|y_2|)^{j-i}}{|x-y_1|^{n-i}}|y_1-y_2|+\frac{1}{|x-y_2|^{n-j-1}}\frac{|y_1-y_2|^{\alpha}}{|x-y_1|^{\alpha}|x-y_2|^{\alpha}} \\
&\leq J_4\frac{1}{|x-y_1|^{\alpha}}\sum_{i=1}^{n-1}\frac{1}{|x-y_1|^i|x-y_2|^{n-i}}|y_1-y_2| \\
&\quad+J_5\frac{1}{|x-y_1|^{\alpha}}\sum_{i=1}^{j}\frac{1}{|x-y_1|^{n-i}}|y_1-y_2|+\frac{|y_1-y_2|^{\alpha}}{|x-y_1|^{\alpha}|x-y_2|^{n+\alpha-j-1}}.
\end{aligned} \tag{3.2}$$

Thus, from the above, we get

$$\begin{aligned}
&\left|T_\Omega[f](y_1)-T_\Omega[f](y_2)\right| \\
&\leq J_6\sum_{j=0}^{k-1}\sum_{i=1}^{n-1}\int_\Omega\frac{1}{|x-y_1|^{i+\alpha}|x-y_2|^{n-i}}\left|D^jf(x)\right||dx||y_1-y_2| \\
&\quad+J_7\sum_{j=1}^{k-1}\sum_{i=1}^{j}\int_\Omega\frac{1}{|x-y_1|^{n+\alpha-i}}\left|D^jf(x)\right||dx||y_1-y_2| \\
&\quad+J_8\sum_{j=0}^{k-1}\int_\Omega\frac{1}{|x-y_1|^{\alpha}|x-y_2|^{n+\alpha-j-1}}\left|D^jf(x)\right||dx||y_1-y_2|^{\alpha} \\
&= I_1+I_2+I_3.
\end{aligned} \tag{3.3}$$

In the following, first we evaluate I_1. From the Hölder inequality, we have

$$\begin{aligned}
I_1 &\leq J_9L_p[f,\Omega]\sum_{i=1}^{n-1}\left[\int_\Omega|x-y_1|^{-(i+\alpha)q}|x-y_2|^{-(n-i)q}|dx|\right]^{\frac{1}{q}}\cdot|y_1-y_2| \\
&= J_9L_p[f,\Omega]|y_1-y_2|\sum_{i=1}^{n-1}I_{1i}^{\frac{1}{q}},
\end{aligned} \tag{3.4}$$

where $I_{1i}=\int_\Omega|x-y_1|^{-(i+\alpha)q}|x-y_2|^{-(n-i)q}|dx|$, $i=1,2,\ldots,n-1$.

In addition, when $p>n/(1-\alpha)$, we have $1<q<n/(n+\alpha-1)$. Thus let $\alpha'=(i+\alpha)q$, $\beta'=(n-i)q$, then

$$0<\alpha'=(i+\alpha)q<(i+\alpha)\frac{n}{n+\alpha-1}\leq\frac{(n-1+\alpha)n}{n+\alpha-1}=n,$$

$$0 < \beta' = (n-i)q < (n-i)\frac{n}{n+\alpha-1} \le \frac{n(n-1)}{n+\alpha-1} < n.$$

And $\alpha' + \beta' = (n+\alpha)q > (n+\alpha) > n$.

Thus from Lemma 2.4, we have

$$I_{1i} \le J_{10}|y_1 - y_2|^{n-\alpha'-\beta'} = J_{10}|y_1 - y_2|^{n-(n+\alpha)q}, \quad i = 1, 2, \ldots, n-1.$$

So by (3.4), we know

$$I_1 \le J_9 L_p[f, \Omega]|y_1 - y_2| \sum_{i=1}^{n-1} J_{10}^{\frac{1}{q}} |y_1 - y_2|^{\frac{n-(n+\alpha)q}{q}} \le J_{11} L_p[f, \Omega]|y_1 - y_2|^{\beta}, \tag{3.5}$$

where $\beta = 1 - \alpha - n/p$, and from $p > n/(1-\alpha)$, we have $0 < \beta = 1 - \alpha - n/p = 1 - \alpha - n/p < 1$.

In addition

$$\begin{aligned}
I_2 &\le J_{12} \sum_{j=1}^{k-1} \sum_{i=1}^{j} \left[\int_\Omega |D^j f(x)|^p |dx|\right]^{\frac{1}{p}} \left[\int_\Omega \frac{|dx|}{|x-y_1|^{(n+\alpha-i)q}}\right]^{\frac{1}{q}} |y_1 - y_2| \\
&= J_{12}|y_1 - y_2| \left\{ \left[\int_\Omega |Df(x)|^p |dx|\right]^{\frac{1}{p}} \left[\int_\Omega \frac{|dx|}{|x-y_1|^{(n+\alpha-1)q}}\right]^{\frac{1}{q}} \right. \\
&\quad + \sum_{i=1}^{2} \left[\int_\Omega |D^2 f(x)|^p |dx|\right]^{\frac{1}{p}} \left[\int_\Omega \frac{|dx|}{|x-y_1|^{(n+\alpha-i)q}}\right]^{\frac{1}{q}} \\
&\quad + \cdots \\
&\quad \left. + \sum_{i=1}^{k-1} \left[\int_\Omega |D^{k-1} f(x)|^p |dx|\right]^{\frac{1}{p}} \left[\int_\Omega \frac{|dx|}{|x-y_1|^{(n+\alpha-i)q}}\right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Again from $1 < q < n/(n+\alpha-1)$. Thus, for all $1 \le i \le k-1$ $(k < n)$, we have

$$(n+\alpha-i)q < \frac{(n+\alpha-i)n}{(n+\alpha-1)} \le n.$$

Thus, for all $i = 1, 2, \ldots, k-1$ $(k < n)$, the integral $\int_\Omega |x-y_1|^{-(n+\alpha-i)q}|dx|$ is convergent. So we obtain

$$I_2 \le J_{13} L_p[f, \Omega]|y_1 - y_2| \le J_{14} L_p[f, \Omega]|y_1 - y_2|^{\beta}. \tag{3.6}$$

And again from the Hölder inequality, we have

$$\begin{aligned}
I_3 &\le J_{15} L_p[f, \Omega]|y_1 - y_2|^{\alpha} \sum_{j=0}^{k-1} \left[\int_\Omega \frac{|dx|}{|x-y_1|^{\alpha q}|x-y_2|^{(n+\alpha-j-1)q}}\right]^{\frac{1}{q}} \\
&= J_{15} L_p[f, \Omega]|y_1 - y_2|^{\alpha} \sum_{j=0}^{k-1} I_{3j}^{\frac{1}{q}},
\end{aligned}$$

where $I_{3j} = \int_\Omega |x-y_1|^{-\alpha q}|x-y_2|^{-(n+\alpha-j-1)q}|dx|$.

When $j = 0$, $I_{30} = \int_\Omega |x - y_1|^{-\alpha q}|x - y_2|^{-(n+\alpha-1)q}|dx|$. Let $\alpha' = \alpha q$, $\beta' = (n + \alpha - 1)q$. From $1 < q < n/(n + \alpha - 1)$, we get

$$0 < \alpha' = \alpha q < \frac{\alpha n}{n + \alpha - 1} < n,$$
$$0 < \beta' = (n + \alpha - 1)q < n.$$

Again from $1/2 \le \alpha < 1$, we have

$$\alpha' + \beta' = (n + 2\alpha - 1)q > nq > n.$$

Thus by Lemma 2.4, we obtain

$$I_{30} \le J_{16}|y_1 - y_2|^{n-\alpha'-\beta'} = J_{16}|y_1 - y_2|^{n-(n+2\alpha-1)q}.$$

When $j \neq 0$, that is, $j = 1, 2, \ldots, k - 1$, $k < n$,

$$\alpha q + (n + \alpha - j - 1)q < \frac{(n + \alpha - 1 + \alpha - j)n}{n + \alpha - 1} = \left(1 + \frac{\alpha - j}{n + \alpha - 1}\right)n < n.$$

So I_{3j} ($j = 1, 2, \ldots, k - 1$, $k < n$) is convergent, thus there is no harm to suppose $I_{3j} \le J_{17}$. Therefore from the above, we obtain

$$\begin{aligned} I_3 &\le J_{15}L_p[f,\Omega]|y_1 - y_2|^\alpha \left[I_{30}^{\frac{1}{q}} + \sum_{j=1}^{k-1} I_{3j}^{\frac{1}{q}}\right] \\ &= J_{15}L_p[f,\Omega]|y_1 - y_2|^\alpha \left[J_{16}^{\frac{1}{q}}|y_1 - y_2|^{\frac{n-(n+2\alpha-1)q}{q}} + (k-1)J_{17}^{\frac{1}{q}}\right] \\ &\le J_{18}L_p[f,\Omega]|y_1 - y_2|^{\alpha+\frac{n-(n+2\alpha-1)q}{q}} + J_{19}L_p[f,\Omega]|y_1 - y_2|^\alpha \\ &= J_{18}L_p[f,\Omega]|y_1 - y_2|^{1-\alpha-\frac{n}{p}} + J_{19}L_p[f,\Omega]|y_1 - y_2|^{2\alpha-1-\frac{n}{p}}|y_1 - y_2|^{1-\alpha-\frac{n}{p}}. \end{aligned}$$

Again when $1/2 \le \alpha < 1$, we know $\alpha - (1 - \alpha) - n/p = 2\alpha - 1 + n/p > 0$ and Ω is bounded. Thus we know that $|y_1 - y_2|^{2\alpha-1-n/p}$ is bounded. So we have

$$I_3 \le J_{20}L_p[f,\Omega]|y_1 - y_2|^{1-\alpha-n/p} = J_{20}L_p[f,\Omega]|y_1 - y_2|^\beta, \tag{3.7}$$

where $\beta = 1 - \alpha - n/p$.

Thus from (3.3), (3.5), (3.6), and (3.7), we can get

$$\left|T_\Omega[f](y_1) - T_\Omega[f](y_2)\right| \le (J_{11} + J_{14} + J_{20})L_p[f,\Omega]|y_1 - y_2|^\beta = J_{21}L_p[f,\Omega]|y_1 - y_2|^\beta.$$

In addition, from the above proof process, we know that J_{21} is only dependent on n, p, α, Ω. Taking $M_5 = J_{21}$, then, when $1/2 \le \alpha < 1$, for all $y_1, y_2 \in \Omega$, we can obtain

$$\left|T_\Omega[f](y_1) - T_\Omega[f](y_2)\right| \le M_5(n,p,\alpha,\Omega)L_p[f,\Omega]|y_1 - y_2|^\beta.$$

(2) When $0 < \alpha < 1/2$, because of $p_1 < p$, $f \in L^p(\Omega)$, we know $f \in L^{p_1}(\Omega)$. Let q_1 satisfies $1/p_1 + 1/q_1 = 1$. From $n/(1-\alpha) < p_1 < n/(1-2\alpha)$, thus we get $n/(n+2\alpha-1) < q_1 < n/(n+\alpha-1)$.

So similar to the text in front of the discussion of I_1, obviously, we can get $0 < \alpha', \beta' < n$ and $\alpha' + \beta' = (n+\alpha)q_1 > n$.

Thus from Lemma 2.4, we have

$$I_1 \le J'_{11} L_{p_1}[f,\Omega]|y_1 - y_2|^{\gamma}, \tag{3.8}$$

where $\gamma = 1 - \alpha - n/p_1$, $0 < \gamma < 1$.

Again completely analogous to the discussion in (1), we have

$$I_2 \le J'_{14} L_{p_1}[f,\Omega]|y_1 - y_2|^{\gamma}. \tag{3.9}$$

In addition, similar to the text in front of the discussion of I_3, we have

$$I_3 \le J'_{15} L_{p_1}[f,\Omega]|y_1 - y_2|^{\alpha} \sum_{j=0}^{k-1} I_{3j}^{\frac{1}{q_1}},$$

where $I_{3j} = \int_{\Omega} |x-y_1|^{-\alpha q_1} |x-y_2|^{-(n+\alpha-j-1)q_1} |dx|$.

When $j = 0$, $0 < \alpha' = \alpha q_1 < n$, $0 < \beta' = (n+\alpha-1)q_1 < n$. Thus

$$\alpha' + \beta' = (n+2\alpha-1)q_1 > (n+2\alpha-1)\frac{n}{n+2\alpha-1} = n.$$

Thus from Lemma 2.4, we can get

$$I_{30} \le J'_{16} |y_1 - y_2|^{n-(n+2\alpha-1)q_1}.$$

When $j \ne 0$, again completely analogous to the discussion in (1), we know I_{3j} is convergent. So there is no harm to suppose $I_{3j} \le J'_{17}$.

Thus, similarly, we have

$$I_3 \le J'_{20} L_{p_1}[f,\Omega]|y_1 - y_2|^{\gamma}. \tag{3.10}$$

Therefore from (3.3), (3.8), (3.9), and (3.10), we obtain

$$\bigl|T_{\Omega}[f](y_1) - T_{\Omega}[f](y_2)\bigr| \le J'_{21} L_{p_1}[f,\Omega]|y_1 - y_2|^{\gamma}.$$

Taking $M_6 = J'_{21}$, we can get

$$\bigl|T_{\Omega}[f](y_1) - T_{\Omega}[f](y_2)\bigr| \le M_6(n,p,\alpha,\Omega) L_{p_1}[f,\Omega]|y_1 - y_2|^{\gamma}. \qquad \square$$

4 The stability and error estimate of generalized Teodorescu operator

Theorem 4.1 *Let Ω be as stated above, $f(x) \in C^{(r)}(\Omega, \mathcal{A}_n(R))$, $D^j f(x) \in L^p(\Omega)$ $(j = 0,1,\ldots, k-1)$, $p > n/(1-\alpha)$, $y \in \Omega$, ρ_0 is a given constant, and $\omega \in B(\rho_0)$. Then we have*

$$\bigl|\bigl(T_{\Omega_\omega}[f]\bigr)(y) - \bigl(T_{\Omega}[f]\bigr)(y)\bigr| \le M_7(n,p,\alpha,\Omega) L_p[f,\Omega] \|\omega\|_{\partial\Omega}^{\gamma},$$

where $\gamma = (1-\alpha)/n - 1/p > 0$.

Proof First

$$
\begin{aligned}
&\left|\left(T_{\Omega_\omega}[f]\right)(y)-\left(T_\Omega[f]\right)(y)\right| \\
&=\left|\sum_{j=0}^{k-1}\int_{E_1}\frac{A_{j+1}}{\omega_n}\frac{D^jf(x)(\bar{x}-\bar{y})^{j+1}}{|x-y|^{n+\alpha}}\,dx-\sum_{j=0}^{k-1}\int_{E_2}\frac{A_{j+1}}{\omega_n}\frac{D^jf(x)(\bar{x}-\bar{y})^{j+1}}{|x-y|^{n+\alpha}}\,dx\right| \\
&\le\left|\sum_{j=0}^{k-1}\int_{E_1}\frac{A_{j+1}}{\omega_n}\frac{D^jf(x)(\bar{x}-\bar{y})^{j+1}}{|x-y|^{n+\alpha}}\,dx\right|+\left|\sum_{j=0}^{k-1}\int_{E_2}\frac{A_{j+1}}{\omega_n}\frac{D^jf(x)(\bar{x}-\bar{y})^{j+1}}{|x-y|^{n+\alpha}}\,dx\right| \\
&=I_1+I_2,
\end{aligned}
\tag{4.1}
$$

where E_1, E_2 are the domains as in Lemma 2.5.

Then from Ω, E_1 being a bounded domain, $|x-y|^{jq}$ is bounded. Again by the Hölder inequality, we have

$$
\begin{aligned}
I_1&\le J_{22}\sum_{j=0}^{k-1}\left[\int_{E_1}\left|D^jf(x)\right|^p|dx|\right]^{\frac{1}{p}}\left[\int_{E_1}\frac{|x-y|^{(j+1)q}}{|x-y|^{(n+\alpha)q}}|dx|\right]^{\frac{1}{q}} \\
&\le J_{23}L_p[f,\Omega]\sum_{j=0}^{k-1}\left[\int_{E_1}\frac{|x-y|^{jq}}{|x-y|^{(n+\alpha-1)q}}|dx|\right]^{\frac{1}{q}} \\
&\le J_{24}L_p[f,\Omega]\sum_{j=0}^{k-1}\left[\int_{E_1}\frac{1}{|x-y|^{(n+\alpha-1)q}}|dx|\right]^{\frac{1}{q}} \\
&\le J_{25}L_p[f,\Omega]\left[\int_{E_1}\frac{1}{|x-y|^{(n+\alpha-1)q}}|dx|\right]^{\frac{1}{q}}.
\end{aligned}
\tag{4.2}
$$

Taking $E_{11}=E_1\cap\{x\mid|x-y|\ge\|\omega\|_{\partial\Omega}^{\frac{1}{n}}\}$, $E_{12}=E_1\cap\{x\mid|x-y|<\|\omega\|_{\partial\Omega}^{\frac{1}{n}}\}$, then obviously $E_1=E_{11}\cup E_{12}$, $E_{11}\cap E_{12}=\emptyset$. Thus

$$
\begin{aligned}
&\left[\int_{E_1}\frac{1}{|x-y|^{(n+\alpha-1)q}}|dx|\right]^{\frac{1}{q}} \\
&=\left[\int_{E_{11}}\frac{1}{|x-y|^{(n+\alpha-1)q}}|dx|+\int_{E_{12}}\frac{1}{|x-y|^{(n+\alpha-1)q}}|dx|\right]^{\frac{1}{q}} \\
&\le\left[\int_{E_{11}}\frac{1}{\|\omega\|_{\partial\Omega}^{\frac{1}{n}(n+\alpha-1)q}}|dx|+\int_{E_{12}}\frac{1}{|x-y|^{(n+\alpha-1)q}}|dx|\right]^{\frac{1}{q}}.
\end{aligned}
\tag{4.3}
$$

In addition, from Lemma 2.5, we have

$$
\begin{aligned}
&\int_{E_{11}}\frac{1}{\|\omega\|_{\partial\Omega}^{\frac{1}{n}(n+\alpha-1)q}}|dx| \\
&=\|\omega\|_{\partial\Omega}^{\frac{1}{n}(n+\alpha-1)q}A_{E_{11}}\le\|\omega\|_{\partial\Omega}^{-\frac{1}{n}(n+\alpha-1)q}A_{E_{11}\cup E_{12}} \\
&\le C\|\omega\|_{\partial\Omega}^{[1-\frac{1}{n}(n+\alpha-1)q]}=C\|\omega\|_{\partial\Omega}^{\frac{1}{n}[n-nq+(1-\alpha)q]}.
\end{aligned}
\tag{4.4}
$$

Again when $p > n/(1-\alpha)$, $1 < q < n/(n+\alpha-1)$. Thus $(n+\alpha-1)q-(n-1) < 1$. So from the local generalized spherical coordinate, we obtain

$$\begin{aligned}
&\int_{E_{12}} \frac{1}{|x-y|^{(n+\alpha-1)q}} |dx| \\
&\quad \leq J_{26} \int_0^{\|\omega\|_{\partial\Omega}^{\frac{1}{n}}} \frac{1}{\rho_0^{[(n+\alpha-1)q-(n-1)]}} \, d\rho_0 = J_{26} \|\omega\|_{\partial\Omega}^{\frac{1}{n}[n-nq+(1-\alpha)q]}.
\end{aligned} \tag{4.5}$$

Thus from (4.2)-(4.5) and $1/p + 1/q = 1$, we get

$$\begin{aligned}
I_1 &\leq J_{25} L_p[f,\Omega] \bigl[C \|\omega\|_{\partial\Omega}^{\frac{1}{n}[n-nq+(1-\alpha)q]} + J_{26} \|\omega\|_{\partial\Omega}^{\frac{1}{n}[n-nq+(1-\alpha)q]}\bigr]^{\frac{1}{q}} \\
&\leq J_{27} L_p[f,\Omega] \|\omega\|_{\partial\Omega}^{\frac{n-nq+(1-\alpha)q}{nq}} \\
&= J_{27} L_p[f,\Omega] \|\omega\|_{\partial\Omega}^{\frac{(1-\alpha)p-n}{np}} \\
&= J_{27} L_p[f,\Omega] \|\omega\|_{\partial\Omega}^{\gamma},
\end{aligned} \tag{4.6}$$

where $\gamma = (1-\alpha)/n - 1/p > 0$.

Similarly, we have

$$I_2 \leq J_{28} L_p[f,\Omega] \|\omega\|_{\partial\Omega}^{\gamma}. \tag{4.7}$$

Therefore from (4.1), (4.6), and (4.7), we obtain

$$\bigl|\bigl(T_{\Omega_\omega}[f]\bigr)(y) - \bigl(T_\Omega[f]\bigr)(y)\bigr| \leq J_{29} L_p[f,\Omega] \|\omega\|_{\partial\Omega}^{\gamma}.$$

We take $M_7 = J_{29}$, that is,

$$\bigl|\bigl(T_{\Omega_\omega}[f]\bigr)(y) - \bigl(T_\Omega[f]\bigr)(y)\bigr| \leq M_7 L_p[f,\Omega] \|\omega\|_{\partial\Omega}^{\gamma}.$$

From the above proof process, we know that $M_7 = J_{29}$ is only dependent on n, p, α, Ω. □

Competing interests
The author declares that they have no competing interests.

Author's contributions
LW has done all contributions to the article.

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