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A combinatorial lemma and its applications

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Abstract

In this paper, we present a generalization of a combinatorial lemma we stated and proved in a recent work. Then we apply the generalized lemma to prove:
(1) a theorem on the existence of a zero for an excess demand mapping, (2) the existence of a continuum of zeros for a parameterized excess demand mapping, (3) Sperner's lemma on labelings of triangulations. Proofs of these results are constructive: they contain algorithms (based on the combinatorial lemma) for the computation of objects of interest or, at least, of their approximations.

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1 Introduction

In the paper [1], we stated and proved a combinatorial lemma with the help of which we then showed the existence of a zero for an excess demand functions and Brouwer's fixed point theorem. We also stated some open problems in the referred paper. The current work answers some of these questions.

First, we prove a generalization of the combinatorial lemma presented in [1]. Then we apply it to prove the existence of an equilibrium price vector for an excess demand mapping (Lemma 6 and Theorem 7). Next, we apply our combinatorial lemma to prove the existence of a continuum of zeros for a parameterized excess demand mapping (Theorem 8). Then we derive Sperner's lemma (Theorem 13) from our combinatorial result.

Let us emphasize the fact that the combinatorial Lemma 2 allows us to get algorithms for finding (approximations of) objects whose existence is claimed in Theorems 7 and 8 and a simplex enjoying properties stated in Sperner's lemma. Hence, our proofs are not only of existential character, but they enable the computation of objects of interest (or at least their approximations).[a]

In the next section, we set up notation and introduce preliminary notions from combinatorial topology. Then we prove the just mentioned combinatorial lemma (Lemma 2) and apply it to get the promised results. The last section comprises some comments.

2 Preliminaries

Let $\mathbb{N}$ denote the set of positive integers, and for any $n \in \mathbb{N}$, let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and let $[n] := \{1,\dots,n\}$, $[0] := \emptyset$, $[n]_0 := \{0,1,\dots,n\}$, and $[n]_i := \emptyset$ for $i > n$. We take on the convention $\sum_{i\in\emptyset} a_i = 0$. Moreover, e^i is the ith unit vector of the standard basis of $\mathbb{R}^n$, where $i \in [n]$. The (vector) inequality $x \geq y$ ($x > y$), $x, y \in \mathbb{R}^n$, means $x_i \geq y_i$ ($x_i > y_i$), $i \in [n]$. In what follows, for $n \in \mathbb{N}$, the set $\Delta^n := \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$, where $\mathbb{R}_+$ is the set of nonnegative real numbers, is the standard (closed) $(n-1)$-simplex, and $\operatorname{int}\Delta^n := \{x \in \Delta^n : x_i > 0, i \in [n]\}$ is its (relative) interior. For a set $X \subset \mathbb{R}^n$, $\partial(X)$, $\operatorname{int} X$, and $\operatorname{conv} X$ denote its boundary (or relative boundary of the closure of X if X is convex), interior (or relative interior if X is convex), and convex hull, respectively. For vectors $x, y \in \mathbb{R}^n$, their scalar product is $xy := \sum_{i=1}^n x_i y_i$. For sets $A, B \subset \mathbb{R}^n$, $AB := \{ab \in \mathbb{R} : a \in A, b \in B\}$ and $A + B := \{a + b \in \mathbb{R}^n : a \in A, b \in B\}$; for $a \in \mathbb{R}^n$, we briefly write aB and $a + B$ instead of $\{a\}B$ and $\{a\} + B$, respectively (similarly if the set B has one element only). If $A \subset \mathbb{R}^n$ and $b \in \mathbb{R}^n$, then $A \geq b$ ($A > b$) means that for each $a \in A$, $a \geq b$ ($a > b$). If $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$, then by $b \geq a$ we mean $b_i \geq a$, $i \in [n]$; similarly, for the strict inequality '$>$'. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted by $|x|$. For any set A, $\#A$ denotes its cardinality, and $\operatorname{diam} A := \sup\{|x-y| : x, y \in A\}$ is the diameter of the set A. For $r > 0$, $B_r := \{x \in \mathbb{R}^n : |x| < r\}$ is the open ball centered at $0 \in \mathbb{R}^n$ with radius r. For a multivalued mapping $F : A \multimap B$, where A, B are some sets, $F(C) := \bigcup_{c\in C} F(c)$ for any set $C \subset A$. For a sequence $k_q \in \mathbb{R}$, $q \in \mathbb{N}$, $k_q \nearrow +\infty$ means that k_q diverges to $+\infty$ strictly monotonically as q increases to $+\infty$.

We need some more or less standard definitions and facts from combinatorial topology, which can be found in [2] and [3]. Let us fix $n \in \mathbb{N}$.

- Let $v^j \in \mathbb{R}^n$, $j \in [k]$, $k \in [n+1]$, be affinely independent. The set σ defined by $\sigma := \{x \in \mathbb{R}^n : x = \sum_{j=1}^k \alpha_j v^j, \alpha \in \Delta^k\} = \operatorname{conv}\{v^1,\dots,v^k\}$ is called a $(k-1)$-simplex with vertices v^j, $j \in [k]$. We write it briefly as $\sigma = \langle v^j : j \in [k]\rangle$ or $\sigma = \langle v^1,\dots,v^k\rangle$ or $\sigma = \langle\{v^1,\dots,v^k\}\rangle$. Observe that the standard $(n-1)$-simplex Δ^n is an $(n-1)$-simplex since $\Delta^n = \langle e^1,\dots,e^n\rangle \subset \mathbb{R}^n$. If we know that σ is a $(k-1)$-simplex, then the set of its vertices is denoted by $V(\sigma)$. If $p = \sum_{j=1}^k \alpha_j^\sigma(p) v^j \in \sigma$, then the vector $\alpha^\sigma(p) := (\alpha_1^\sigma(p),\dots,\alpha_k^\sigma(p)) \in \Delta^k$ is called (the vector of) the barycentric coordinates of the point p in the simplex σ; in this case, we say that the barycentric coordinate $\alpha_j^\sigma(p)$ of p corresponds to the vertex v^j or, in short, that $\alpha_j^\sigma(p)$ is the ith barycentric coordinate of p in σ, $j \in [n]$. For each $p \in \sigma$, its vector $\alpha^\sigma(p)$ of the barycentric coordinates in the simplex σ is uniquely determined. If σ is a k-simplex and we do not order its vertices $V(\sigma)$, then it is sometimes convenient to think that the barycentric coordinates of a point $p \in \sigma$ in σ are determined according to the unique function $\alpha^\sigma(p) : V(\sigma) \to [0,1]$, $V(\sigma) \ni v \mapsto \alpha_v^\sigma(p) \in [0,1]$, with $\sum_{v\in V(\sigma)} \alpha_v^\sigma(p) = 1$ and $p = \sum_{v\in V(\sigma)} \alpha_v^\sigma(p) v$; it is said in this case that the barycentric coordinate $\alpha_v^\sigma(p)$ of the point $p \in \sigma$ corresponds to the vertex $v \in V(\sigma)$ in the simplex σ. Moreover, $\alpha^\sigma(p)$ is called the (mapping of) barycentric coordinates of p in S.
- If σ is a $(k-1)$-simplex, then $\langle A\rangle$, where $\emptyset \neq A \subset V(\sigma)$, is called a $(\#A-1)$-face of σ.
- If σ is a $(k-1)$-simplex and $A = V(\sigma)\backslash\{v\}$, where $v \in V(\sigma)$, then the $(k-2)$-simplex $\langle A\rangle$ is the $(k-2)$-face opposite to the vertex v. Obviously, to each vertex v, there corresponds a unique $(k-2)$-face opposite to v.
- A collection $T(S) = \{\sigma_j \subset S : j \in [J]\}$, $J \in \mathbb{N}$, of nonempty subsets of a $(k-1)$-simplex $S \subset \mathbb{R}^n$, $0 < k \leq n+1$, is called a triangulation of S if it meets the following conditions:

1. σ_j is a $(k-1)$-simplex, $j \in [J]$,
2. if $\sigma_j \cap \sigma_{j'} \neq \emptyset$ for $j, j' \in [J]$, then $\sigma_j \cap \sigma_{j'}$ is a common face of σ_j and $\sigma_{j'}$,
3. $S = \bigcup_{j\in[J]} \sigma_j$.

The collection of all vertices of simplices in $T(S)$ is denoted by $V(T(S))$. If there is no ambiguity, then we write T instead of $T(S)$.

- If T is a triangulation of an $(n-1)$-simplex S, then for $(n-2)$-face F of S, the set $\{\sigma \cap F : \sigma \cap F \text{ is an } (n-2)\text{-simplex}, \sigma \in T\}$ is a triangulation of F (see [4], p.27, Theorem 2.3(e)).
- Two different $(k-1)$-simplices σ_j, $\sigma_{j'}$, $j, j' \in [J]$, $j \neq j'$, in a triangulation T of a $(k-1)$-simplex S are said to be adjacent if $\langle V(\sigma) \cap V(\sigma')\rangle$ is a $(k-2)$-face for both of them. Each $(k-2)$-face of a simplex σ_j, $j \in [J]$, is a $(k-2)$-face for exactly two different simplices in the triangulation, provided that the $(k-2)$-face is not contained in $\partial(S)$.
- The K_m-triangulation of an $(n-1)$-simplex $S = \langle v^1, \dots, v^n\rangle \subset \mathbb{R}^n$ with grid size m^{-1}, where m is a positive integer,[b] is the collection of all $(n-1)$-simplices σ of the form $\sigma = \langle p^1, p^2, \dots, p^n\rangle$, where vertices $p^1, p^2, \dots, p^n \in S$ satisfy the following conditions:
  (1) the barycentric coordinate $\alpha^S_{v_i}(p^1)$ of p^1 corresponding to v_i in S, $i \in [n]$, is a nonnegative integer multiple of m^{-1},
  (2) $\alpha^S(p^{j+1}) = \alpha^S(p^j) + m^{-1}(e^{\pi_j} - e^{\pi_j+1})$, where $\pi = (\pi_1, \dots, \pi_{n-1})$ is a permutation of $[n-1]$, $l \in \{j, j+1\}$, $j \in [n-1]$.

  The K_m-triangulation of S with grid size m^{-1} is denoted by $K_m(S)$, and the set of all vertices of simplices in $K_m(S)$ is denoted by $V_m(S)$. Obviously, $V_m(S) = \bigcup_{\sigma\in K_m(S)} V(\sigma) = \{\alpha_1 v^1 + \cdots + \alpha_n v^n : \alpha \in \Delta^n, \alpha_i \in \{0, 1/m, \dots, 1-1/m, 1\}\}$. For any $\varepsilon > 0$ and for a sufficiently large m, each simplex in $K_m(S)$ has the diameter not greater than ε. Moreover, the vertex v^n belongs to exactly one simplex in $K_m(S)$.

We shall also need a variant of a result by Kuratowski [5], Theorem 5.47.6:

Lemma 1 *Let $G^q \subset X$, $q \in \mathbb{N}$, where $X \subset \mathbb{R}^n$ is a convex, compact, and connected set, be a sequence of nonempty compact and connected sets such that there exists a point $\overline{g} = \lim_{q\to+\infty} \overline{g}^q$, $\overline{g} \in X$, $\overline{g}^q \in G^q$, $q \in \mathbb{N}$. Then the (limit) set*

$$\Gamma := \left\{x \in X : x = \lim_{q\to+\infty} g^{k_q}, \text{where } k_q \nearrow +\infty \text{ as } q \to +\infty, g^{k_q} \in G^{k_q}, q \in \mathbb{N}\right\}$$

is compact and connected in X. Moreover, $\overline{g} \in \Gamma$.

Proof It is obvious that $\overline{g} \in \Gamma$. Let us assume that $\overline{g} \in \bigcap_{q\in\mathbb{N}} G^q$. Let $\gamma^k \in \Gamma \subset X$, $k \in \mathbb{N}$. By the Bolzano-Weierstrass theorem we may assume that $\gamma := \lim_{k\to+\infty} \gamma^k$ exists in X. Since $\gamma^k \in \Gamma$, by the definition of Γ we get that, for each k, there exist sequences $k_q \in \mathbb{N}$, $g^{k_q} \in G^{k_q}$, $q \in \mathbb{N}$, such that $\lim_{q\to+\infty} g^{k_q} = \gamma^k$. Hence, for each k, there exists $s(k) \in \mathbb{N}$ such that $|g^{k_{s(k)}} - \gamma^k| < 1/k$, and we may assume that $k_{s(k)} \nearrow +\infty$ as $k \to +\infty$. It is now obvious that $\gamma \in \Gamma$, which proves the compactness of Γ. Suppose that $A, A' \subset X$ are open sets such that $\overline{g} \in A$, $A \cap A' = \emptyset$, and $\Gamma \subset A \cup A'$. Notice that the connectedness of G^q and the fact that $\overline{g} \in G^q$, $q \in \mathbb{N}$, imply $G^q \subset A$, $q \in \mathbb{N}$. To show that Γ is connected, it suffices to demonstrate that $\Gamma \cap A' = \emptyset$. Let us now fix a point $x \in \Gamma$ and assume that $x \in A'$. Since A' is open, there exists $\varepsilon > 0$ with $x + B_\varepsilon \subset A'$. The last inclusion implies that for any sequence converging to x, almost all its terms belong to A'. From this, the fact that $G^q \subset A$, $q \in \mathbb{N}$, and the disjointness of A from A' it follows that $x \notin \Gamma$. Thus, $\Gamma \cap A' = \emptyset$.

Now, if $\bigcap_{q\in\mathbb{N}} G^q = \emptyset$, then let for each $q \in \mathbb{N}$, $h^q \in G^q$ be a point such that $|h^q - \overline{g}| = \inf\{|\overline{g} - h| : h \in G^q\}$; such a point exists by the compactness of G^q. Define the sets $H^q := G^q \cup \{t\overline{g} + (1-t)h^q : t \in [0,1]\}$, $q \in \mathbb{N}$. By the convexity of X, $H^q \subset X$, $q \in \mathbb{N}$. Moreover, the sets H^q, $q \in \mathbb{N}$, are compact, connected, and the point $\overline{g}$ belongs to each of them. Thus, the limit set Γ' (defined as Γ, but with H^q in place of G^q) is compact, nonempty, and connected. It suffices to prove that $\Gamma = \Gamma'$. Obviously, $\Gamma \subset \Gamma'$. If $x \in \Gamma'\backslash\Gamma$, then x is the limit of a sequence of points $g^{k_q} \in \{t\overline{g} + (1-t)h^q : t \in [0,1)\}\backslash G^{k_q}$ for a sequence $k_q \nearrow +\infty$, $q, k_q \in \mathbb{N}$. But $\lim_{q\to+\infty} h^q = \overline{g}$, and thus $x = \overline{g}$. Consequently, $x \in \Gamma$, which ends the proof. □

3 The combinatorial lemma and its applications

The result which is common for our proofs of the existence of zeros for excess demand mappings, continuum of zeros for parameterized excess demand mappings, and for a proof of Sperner's lemma is the following combinatorial Lemma 2, which generalizes the combinatorial lemma presented in [1].[c]

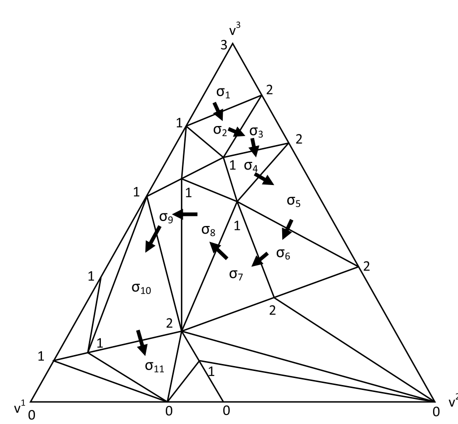
Lemma 2 *Let $S := \langle v^1, \dots, v^n\rangle \subset \mathbb{R}^n$ be an $(n-1)$-simplex, $T := T(S) = \{\gamma_j : j \in [Q]\}$, $Q \in \mathbb{N}$, be a triangulation of S, and let $V := V(T)$ denote the set of all vertices of simplices in T. Suppose that there exists exactly one simplex $\overline{\sigma} \in T$ such that $v^n \in \overline{\sigma}$. Assume also that $\overline{\sigma} \cap \langle v^1, \dots, v^{n-1}\rangle = \emptyset$. Let now $l : V \to [n]_0$ be a function satisfying for all $p \in V$ the following conditions*:

1. *for $i \in [n-1]$: $\alpha_i^S(p) = 0 \Leftrightarrow l(p) \neq i$,*
2. *$l(p) = 0 \Leftrightarrow \alpha_n^S(p) = 0$,*
3. *$l(p) = n \Leftrightarrow \alpha_n^S(p) = 1$,*
4. *$l(p) \in [n-1] \Leftrightarrow \alpha_n^S(p) \in (0,1)$,*

where $\alpha_i^S(p) = \alpha_{v^i}^S(p)$ is the ith barycentric coordinate of the point p in S, $i \in [n]$. Then there exists a unique finite sequence of simplices $\sigma_1, \dots, \sigma_J \in T$, $J \in \mathbb{N}$, such that σ_j and σ_{j+1} are adjacent for $j \in [J-1]$, $n \in l(\sigma_1)$, $0 \in l(\sigma_J)$, $[n-1] \subset l(\sigma_j)$, $j \in [J]$, and $0 \notin l(\sigma^j)$, $\sigma_{j+1} \notin \{\sigma_1, \dots, \sigma_j\}$, $j \in [J-1]$.[d] *See Figure 1.*

Proof Let $\sigma_1 := \overline{\sigma}$. Obviously, $n \in l(\sigma_1)$. Since σ_1 is the unique $(n-1)$-simplex in T containing v^n and $\sigma_1 \cap \langle v^1, \dots, v^{n-1}\rangle = \emptyset$, we have $\sigma_1 = \langle p^1, \dots, p^{n-1}, p^n\rangle$, where $p^j = t^j v^j + (1-t^j)v^n$ for some $t^j \in (0,1)$, $j \in [n-1]$. Hence, $\alpha^S(p^j)$, the vector of barycentric coordinates of the

Figure 1 Illustration of Lemma 2 for *n* = 3. Small triangles are members of a triangulation T of $\langle v^1, v^2, v^3\rangle$. The triangulation T satisfies the assumptions of the lemma. The number at a vertex of a simplex in T is the value of the function l assigned to the vertex, and it is clear that l satisfies the assumptions of Lemma 2. The sequence of simplices $\sigma_1, \dots, \sigma_{11}$ is constructed according to the rules presented in the proof of Lemma 2.



vertex p^j, is of the form

$$\alpha^S(p^j) = \big(0,\dots,0,\underbrace{t^j}_{j\text{th coordinate}},0,\dots,0,1-t^j\big),\quad j\in[n-1].$$

The equality $\alpha_i^S(p^j)=0$ implies $l(p^j)\neq i$, and therefore, due to the facts that $\alpha_j^S(p^j)=t^j>0$, $j\in[n-1]$, and $\alpha_n^S(p^n)=1$, we see that $l(p^j)=j$, $j\in[n]$, and $[n-1]\subset l(\sigma_1)$. Moreover, since for all $v\in V$, $\alpha_i^S(v)=0$ implies $l(v)\neq i$, $l(\sigma')=[n-1]$ entails that σ' is not contained in $\partial(S)$, where σ' is an $(n-2)$-face of some $\sigma\in T$. Thus, the (relative) boundary $\partial(S)$ of S contains no $(n-2)$-face σ' of a simplex $\sigma\in T$ such that $l(\sigma')=[n-1]$. From this we get that there exists exactly one $\sigma_2\in T\backslash\{\sigma_1\}$ that is adjacent to σ_1. Obviously, $[n-1]\subset l(\sigma_2)$, and if $0\in l(\sigma_2)$, then the proof is finished ($J=2$). Suppose that $[n-1]=l(\sigma_2)$ and let p^{n+1} be the only element of $V(\sigma_2)\backslash V(\sigma_1)$. Since $l(\{p^1,\dots,p^{n-1}\})=[n-1]$ and $l(p^{n+1})\in[n-1]$, there exists exactly one index $i_1\in[n]$ such that $l(p^{i_1})=l(p^{n+1})$ and $l(V(\sigma^2)\backslash\{p^{i_1}\})=[n-1]$. So we can find a simplex $\sigma_3\in T\backslash\{\sigma_1,\sigma_2\}$ adjacent to σ_2 with $[n-1]\subset l(\sigma_3)$, and if $0\in l(\sigma_3)$, then the process is complete, if not, then proceeding as before, we can find a simplex $\sigma_4\in T\backslash\{\sigma_1,\sigma_2,\sigma_3\}$ and so on.[e] Suppose that we have constructed the sequence $\sigma_1,\dots,\sigma_J$. If $0\in l(\sigma_J)$, then the sequence satisfies the claim. Suppose that $0\notin l(\sigma_J)$. Since each $(n-2)$-face that is not contained in $\partial(S)$ is shared by exactly two simplices of T, there exists precisely one simplex σ' in $T\backslash\{\sigma_1,\dots,\sigma_J\}$ such that σ_J and σ' share the $(n-2)$-face $\sigma'\cap\sigma_J$ with $l(\sigma'\cap\sigma_J)=[n-1]$; this ensures that $\sigma_{J+1}=\sigma'$ and that no simplex of T appears twice (or more) in the sequence $\sigma_1,\dots,\sigma_{J+1}$, and $l(\sigma_j)=[n-1]$, $j\in[J-1]\backslash\{1\}$. Thus, in view of the finiteness of T and the fact that no $(n-2)$-face σ' of some σ^j, $j\in[J+1]$, with $l(\sigma')=[n-1]$ is contained in $\partial(S)$, we conclude that there exists the first index $J\in\mathbb{N}$ such that $0\in l(\sigma_J)$. The uniqueness of the constructed sequence comes from the preceding sentences, properties of the simplex containing v^n, and the fact that each $(n-2)$-face in the (relative) interior of S is shared by exactly two simplices of the triangulation. □

3.1 The existence of equilibrium

Definition 3 Let us fix $n\in\mathbb{N}$. We say that a mapping $z:\operatorname{int}\Delta^n\multimap\mathbb{R}^n$ is an excess demand mapping if it satisfies the following conditions:

1. z is upper semicontinuous on $\operatorname{int}\Delta^n$ with nonempty convex and compact values $z(p)$, $p\in\operatorname{int}\Delta^n$,
2. Walras' law: $pz(p)=0$, $p\in\operatorname{int}\Delta^n$,
3. the boundary condition: if $p^q\in\operatorname{int}\Delta^n$, $y^q\in z(p^q)$, $q\in\mathbb{N}$, and $\lim_{q\to+\infty}p^q=p$, then

   $$p_i=0\quad\Rightarrow\quad\lim_{q\to+\infty}y_i^q=+\infty,\quad i\in[n],$$

4. z is bounded from below: there exists a negative number L such that

   $$\inf\{y_i\in\mathbb{R}:y\in z(p),p\in\operatorname{int}\Delta^n\}>L,\quad i\in[n].$$

Definition 4 Let $z:\operatorname{int}\Delta^n\multimap\mathbb{R}^n$ be an excess demand mapping, $n\in\mathbb{N}$. A point $p\in\operatorname{int}\Delta^n$ is called an equilibrium point for z if $0\in z(p)$.

Lemma 5 *Let z be an excess demand mapping. Then*

1. *there exists* $\varepsilon_1 \in (0,1/2]$ *such that for* $i \in [n]$ *and* $y \in z(p)$, $p \in \operatorname{int}\Delta^n$, *we have*

$$(p_i \leq \varepsilon_1 \Rightarrow y_i > 0) \quad \textit{and} \quad (p_i \geq 1 - \varepsilon_1 \Rightarrow y_i < 0),$$

2. *for any* $\varepsilon_2 \in (0,1/2]$, *there exists* $U > 0$ *such that* $z(p) \subset [L,U]^n$ *for each* $p \in \operatorname{int}\Delta^n$ *with* $p_i \geq \varepsilon_2$, $i \in [n]$, *where* L *is the constant appearing in Definition* 3, *condition* 4,
3. *for each* $\varepsilon_3 \in (0,1/2]$, *there exists* $\varepsilon_4 \in (0,\varepsilon_3/2]$ *such that for* $p \in \operatorname{int}\Delta^n$ *with* $p_n \leq 1 - \varepsilon_3$, *we have that, for* $i \in [n-1]$ *and* $y \in z(p)$,

$$\bigl(p_i \leq \varepsilon_4 \Rightarrow (1-p_n)y_i + p_n y_n > 0\bigr) \quad \textit{and} \quad \bigl(p_i \geq 1 - \varepsilon_4 \Rightarrow (1-p_n)y_i + p_n y_n < 0\bigr).$$

4. *for* ε_3, ε_4 *for which claim* 3 *and its premises hold, there exists* $\Lambda \in (0,+\infty)$ *such that, for* $i \in [n-1]$,

$$(1-p_n)y_i + p_n y_n > \Lambda,$$

whenever $y \in z(p)$, $p \in \operatorname{int}\Delta^n$, $p_i \leq \varepsilon_4$, $p_n \leq 1-\varepsilon_3$, *and* $p_j \in [\varepsilon_4/2n, 1-\varepsilon_4/2n]$, $j \in [n]$.

Proof Suppose that the left-hand side implication in claim 1 is not true. Then there exist $i \in [n]$ and sequences $p^q \in \operatorname{int}\Delta^n$, $y^q \in z(p^j)$, $j \in \mathbb{N}$, such that $\lim_{q\to+\infty} p^q = p$, $p_i = 0$, and $\limsup_{q\to+\infty} y_i^q \leq 0$, which is impossible due to the boundary condition. Hence, there exists $\varepsilon_1 \in (0,1/2]$ such that the considered implication is satisfied. To prove the right-hand side implication in claim 1, observe that $p_i \geq 1-\varepsilon_1$ implies $p_{i'} \leq \varepsilon_1$, $i \neq i'$, and $y_{i'} > 0$, $i' \neq i$, for $y \in z(p)$. Finally, by Walras' law, $0 < \sum_{i'\neq i} p_{i'}y_{i'} = -p_i y_i$, and consequently $y_i < 0$.

Statement 2 is true since the restriction of the mapping z to the (compact) set $\{p \in \operatorname{int}\Delta^n : p_i \geq \varepsilon_2, i \in [n]\}$ is an upper semicontinuous mapping with compact values, and such mappings transform compact sets into compact sets [6], p.560.[f]

To prove assertion 3, suppose that there exists $\varepsilon_3 \in (0,1/2]$ such that for any $q \in \mathbb{N}$, $k \geq 2$, there exist $p^q \in \operatorname{int}\Delta^n$: $p_n^q \leq 1-\varepsilon_3$ and $i_q \in [n-1]$: $p_{i_q}^q \leq \varepsilon_3/q$ with $(1-p_n^q)y_{i_q}^q + p_n^q y_n^q \leq 0$ for some $y^q \in z(p^q)$. Without loss of generality, we assume that $i_q = 1$, $q \in \mathbb{N}$. The boundary condition now implies that $\lim_{q\to+\infty} y_1^q = +\infty$. Since $1 - p_n^q \geq \varepsilon_3$, $q \in \mathbb{N}$, and z is bounded from below by the constant L, we obtain that $(1-p_n^q)y_1^k + p_n^q y_n^q \geq \varepsilon_3 y_1^q + L > 0$ for large q. This contradicts our assumption that $(1-p_n^q)y_{i_q}^q + p_n^q y_n^q \leq 0$ for $q \in \mathbb{N}$. Hence, for any $\varepsilon_3 \in (0,1/2]$, there exists $\varepsilon_4 \in (0,\varepsilon_3/2]$ such that the first implication in claim 3 is satisfied for any $p \in \Delta^n$: $p_n \leq 1-\varepsilon_3$. Observe that for fixed ε_3 and ε_4 for which the first implication in claim 3 holds, it follows that if $p \in \operatorname{int}\Delta^n$: $p_n \leq 1-\varepsilon_3$ and $p_i \geq 1-\varepsilon_4$ for some $i \in [n-1]$, then $p_j \leq \varepsilon_4$, $j \in [n-1]$: $j \neq i$, and by the first implication of assertion 3 we get $(1-p_n)y_i + p_n y_n = -\sum_{j\in[n-1]\setminus\{i\}}[(1-p_n)y_j + p_n y_n] < 0$.

Let now ε_3, ε_4 be as in claim 3 and suppose that claim 4 is false. Thus, there exist sequences $p^q \in \operatorname{int}\Delta^n$: $p_n^q \leq 1-\varepsilon_3$, $p_j^q \in [\varepsilon_4/2n, 1-\varepsilon_4/2n]$, $j \in [n]$, $y^q \in z(p^q)$, $q \in \mathbb{N}$, and $\bar{i} \in [n-1]$ such that $p_{\bar{i}}^q \leq \varepsilon_4$ and $(1-p_n^q)y_{\bar{i}}^q + p_n^q y_n^q \leq 1/q$, $q \in \mathbb{N}$. By the boundary and lower boundedness conditions on z, the boundedness of the standard simplex, and by the upper semicontinuity of z and compactness of its values, from the sequences $p^q, y^q, q \in \mathbb{N}$, we can extract subsequences converging to $p \in \operatorname{int}\Delta^n$ and $y \in z(p)$, respectively. But then $p_i \in [\varepsilon_4/2n, 1-\varepsilon_4/2n]$, $i \in [n]$, $p_n \leq 1-\varepsilon_3$, and $p_{\bar{i}} \leq \varepsilon_4$. By the contradictory assumption we get $(1-p_n)y_{\bar{i}} + p_n y_n \leq 0$, which is impossible due to the choice of ε_3 and ε_4. □

We are in position to prove the first consequence of the combinatorial Lemma 2.

Lemma 6 *Let z be an excess demand mapping. For each $\varepsilon > 0$, there exist $p \in \operatorname{int}\Delta^n$ and $y \in z(p)$ such that $y_i \leq \varepsilon$, $i \in [n]$.*

Proof Fix $\varepsilon > 0$. The claim is trivial for $n = 1$, so assume that $n \geq 2$. To ease the reading, we divide the proof into four parts.

Part 1: *A restriction of the mapping z to a simplex $S \subset \operatorname{int}\Delta^n$*. Let us fix ε_1 for which the assertion of Lemma 5, statement 1 is satisfied. Let ε_4 correspond to $\varepsilon_3 := \varepsilon_1/2$ according to Lemma 5, statement 3. Finally, suppose that U fulfills Lemma 5, statement 2 for $\varepsilon_2 = \varepsilon_4$ and Λ corresponds to ε_3 ($= \varepsilon_1/2$) and ε_4 as in Lemma 5, statement 4. Notice that $1 - \varepsilon_1 < 1 - \frac{1}{2}\varepsilon_1 = 1 - \varepsilon_3 < 1 - \varepsilon_4 < 1 - \frac{n-1}{2n}\varepsilon_4 \leq 1 - \frac{1}{2n}\varepsilon_4$.

Without loss of generality, we can assume that ε_4 is so small that the vectors

$$\overline{v}^i := \Big(\frac{\varepsilon_4}{2n}, \dots, \frac{\varepsilon_4}{2n}, \underbrace{1 - \frac{n-1}{2n}\varepsilon_4}_{i\text{th coordinate}}, \frac{\varepsilon_4}{2n}, \dots, \frac{\varepsilon_4}{2n}\Big) \in \operatorname{int}\Delta^n, \quad i \in [n], \tag{1}$$

are linearly independent. Let now $S := \langle \overline{v}^i : i \in [n]\rangle$, and let $\alpha_i^S(p) := \alpha_{\overline{v}^i}^S(p)$ denote the ith barycentric coordinate of $p \in S$ in the simplex S, $i \in [n]$. Hence, $p = \sum_{i\in[n]} \alpha_i^S(p)\overline{v}^i$. It is obvious that $S \subset \operatorname{int}\Delta^n$, the set S is an $(n-1)$-simplex with the vertices $\overline{v}^i$, $i \in [n]$, and $p_i \in [\varepsilon_4/2n, 1 - \frac{n-1}{2n}\varepsilon_4]$ for $p \in S$. Moreover, for each $i \in [n]$, any value in $[\varepsilon_4/2n, 1 - \frac{n-1}{2n}\varepsilon_4]$ is taken on by p_i for a point $p \in S$. Observe that if for a point $p \in S$, $\alpha_i^S(p) = 0$ for some $i \in [n]$ ($\Leftrightarrow p_i = \varepsilon_4/2n$), then $p_i < \varepsilon_1$ and $y_i > 0$ for $y \in z(p)$; if $\alpha_i^S(p) = 1$ ($\Leftrightarrow p_i = 1 - \frac{n-1}{2n}\varepsilon_4$) for some $i \in [n]$, then $p_i > 1 - \varepsilon_1$ and $y_i > 0$ for $y \in z(p)$. Further, if $p \in S$, $p_n \leq 1 - \varepsilon_3 = 1 - \varepsilon_1/2$, and $\alpha_i^S(p) = 0$ for some $i \in [n-1]$, then $p_i \leq \varepsilon_4$ and $(1-p_n)y_i + p_n y_n > \Lambda$ for $y \in z(p)$; in the case where $\alpha_i^S(p) = 1$ for some $i \in [n-1]$, we have $p_i \geq 1 - \varepsilon_4$ and $(1-p_n)y_i + p_n y_n < 0$ for $y \in z(p)$.

Part 2: *Two triangulations of S*. From the upper semicontinuity of z we obtain that, for any $p \in S$, there exists $\delta_p \in (0, \varepsilon/2) > 0$ such that $z((p + B_{\delta_p}) \cap S) \subset z(p) + B_{\varepsilon/2}$. By the compactness of S there exist points $p^1, \dots, p^Q \in S$, $Q \in \mathbb{N}$, such that the sets $p^i + B^i$, $i \in [Q]$, where $B^i := B_{\delta_{p^i}}$, are an open cover of S. Now, let $D := 1 + \sup\{|y| : y \in z(p), p \in S\}$ ($D \in [1, +\infty)$ by Lemma 5, statement 2). Define $\delta := \min_{i\in[Q]} \delta_{p^i}$ and let $\lambda \in (0, \min\{\Lambda, \delta/2\})$ be a Lebesgue number of the covering $\{p^i + B^i : i \in [Q]\}$ of S satisfying the inequality

$$\lambda \leq \frac{(n-1)\delta\varepsilon_4}{8n}. \tag{2}$$

Hence, any set $A \subset S$ whose diameter is less than λ satisfies the condition $A \subset p^i + B^i$ for some $i \in [Q]$. Let us now fix $m_1 \in \mathbb{N}$ so large that $\operatorname{diam}\sigma < \min\{\lambda, \varepsilon_1/6\}$ for $\sigma \in K_{m_1}(S)$, and to each vertex $v \in V_{m_1}(S)$, arbitrarily assign a point $y(v) \in z(v)$; notice that $|y(v)| \leq D$, $v \in V_{m_1}(S)$, and if a simplex $\sigma \in K_{m_1}(S)$ contains a point p such that $p_n \geq 1 - \varepsilon_1/2$, then $p'_n \geq 1 - 2\varepsilon_1/3$, $p' \in \sigma$. Let us also fix $m_2 \in \mathbb{N}$ such that for each simplex σ in the triangulation $T^S := K_{m_1\times m_2}(S)$, we have $\operatorname{diam}\sigma < \min\{\lambda/D, \varepsilon_1/3\}$. Observe that $V_{m_1}(S) \subset V^S := V_{m_1\times m_2}(S)$. For any $\sigma \in T^S$, by $\hat{\sigma}$ we denote the unique simplex $\hat{\sigma} \in K_{m_1}(S)$ such that $\sigma \subset \hat{\sigma}$. So, it makes sense to speak on the barycentric coordinates of points $p \in \sigma \in T^S$ in $\hat{\sigma} \in K_{m_1}(S)$. We denote the barycentric coordinates of $p \in \sigma \in T^S$ in the simplex

$\hat{\sigma} \in K_{m_1}(S)$ by $\alpha^{\hat{\sigma}}(p)$ and the barycentric coordinate of p corresponding to $v \in V(\hat{\sigma})$ in $\hat{\sigma}$ by $\alpha^{\hat{\sigma}}_v(p)$, $v \in V(\hat{\sigma})$. Hence, $p = \sum_{v\in V(\hat{\sigma})} \alpha^{\hat{\sigma}}_v(p)$ and $\sum_{v\in V(\hat{\sigma})} \alpha^{\hat{\sigma}}_v(p) = 1$, $\alpha^{\hat{\sigma}}_v(p) \in [0,1]$, $v \in V(\hat{\sigma})$. Since the set $K_{m_1}(S)$ is finite and λ and D are fixed independently of the triangulation of S and the barycentric coordinates of a point p in a simplex are continuous functions of p, without loss of generality (increasing m_2 if necessary), we may assume that $\sum_{v\in V(\hat{\sigma})} |\alpha^{\hat{\sigma}}_v(p) - \alpha^{\hat{\sigma}}_v(p')| < \lambda/D$ for any $\sigma \in T^S$ and $p, p' \in \sigma$, that is, the barycentric coordinates of points $p, p' \in \sigma \in T$ in the simplex $\hat{\sigma} \in K_{m_1}(S)$ differ by less than λ/D.

Part 3: *A labeling of* V^S. Recall that to each vertex $v \in K_{m_1}(S) \subset T^S$, we have arbitrarily assigned $y(v) \in z(v)$ in Part 2 of the proof. Let

$$h(p) = \bigl(h_1(p), \dots, h_n(p)\bigr) := \sum_{v\in V(\hat{\sigma})} \alpha^{\hat{\sigma}}_v(p) y(v) \in \mathbb{R}^n, \tag{3}$$

where $p \in S$, and σ is any simplex in T^S with $p \in \sigma$; it is clear that $h(p)$ is independent of the choice of σ as long as $p \in \sigma$. For $p, p' \in \sigma \in T^S$, we have

$$\begin{aligned} \bigl|h(p) - h(p')\bigr| &= \biggl|\sum_{v\in V(\hat{\sigma})} \bigl(\alpha^{\hat{\sigma}}_v(p) - \alpha^{\hat{\sigma}}_v(p')\bigr) y(v)\biggr| \le \sum_{v\in V(\hat{\sigma})} \bigl|\alpha^{\hat{\sigma}}_v(p) - \alpha^{\hat{\sigma}}_v(p)\bigr|\bigl|y(v)\bigr| \\ &\le \sum_{v\in V(\hat{\sigma})} \bigl|\alpha^{\hat{\sigma}}_v(p) - \alpha^{\hat{\sigma}}_v(p')\bigr| D \le \frac{\lambda}{D} D = \lambda. \end{aligned} \tag{4}$$

Further, for any $p \in \sigma \in T^S$, we have

$$ph(p) = \sum_{v\in V(\hat{\sigma})} \alpha^{\hat{\sigma}}_v(p) p y(v) = \sum_{v\in V(\hat{\sigma})} \alpha^{\hat{\sigma}}_v(p) \bigl(v + \gamma(p,v)\bigr) y(v) = (\star),$$

where $\gamma(p,v) := v - p$, $v \in V(\hat{\sigma})$, and by the choice of m_2 we have $|\gamma(p,v)| \le \min\{\lambda/D, \varepsilon_1/3\}$, $v \in V(\hat{\sigma})$, so by Schwarz's inequality

$$(\star) = \sum_{v\in V(\hat{\sigma})} \underbrace{\alpha^{\hat{\sigma}}_v(p) v y(v)}_{=0} + \sum_{v\in V(\hat{\sigma})} \alpha^{\hat{\sigma}}_v(p) \gamma(p,v) y(v) \le \sum_{v\in V(\hat{\sigma})} \alpha^{\hat{\sigma}}_v(p) \frac{\lambda}{D} D \le \lambda,$$

and hence, for each $p \in S$,

$$ph(p) \le \lambda. \tag{5}$$

It is an elementary task to check that, for $p \in S$,

$$(1 - p_n) ph(p) = \sum_{i\in[n-1]} p_i \bigl((1 - p_n) h_i(p) + p_n h_n(p)\bigr). \tag{6}$$

Conditions (5) and (6) imply that, for $p \in S$, we have

$$\sum_{i\in[n-1]} p_i \bigl((1 - p_n) h_i(p) + p_n h_n(p)\bigr) \le \lambda.$$

By Lemma 5, statement 1 and the choice of m_2, $h_n(p) < 0$ for $p \in S$ with $p_n \ge 1 - \varepsilon_1/2$ and $h_n(p) > 0$ for $p \in S$ with $p_n \le \varepsilon_1/2$. Further, by Lemma 5, statement 4, $(1 - p_n) h_i(p) +$

$p_n h_n(p) > \Lambda > \lambda$ for $p \in S$, $p_n \le 1 - \varepsilon_1/2$, and $p_i = \varepsilon_4/2n$ ($\Leftrightarrow \alpha_i^S(p) = 0$) for some $i \in [n-1]$. These considerations show that the function $l : V^S \to [n]_0$ defined by the formula[g]

$$\forall p \in V^S:$$
$$l(p) := \begin{cases} n & \text{if } p_n = 1 - \frac{n-1}{2n}\varepsilon_4, \\ 0 & \text{if } p_n = \frac{\varepsilon_4}{2n}, \\ \min\{i \in [n-1] : \alpha_i^S(p) > 0\} & \text{if } p_n \in [1 - \frac{\varepsilon_1}{2}, 1 - \frac{n-1}{2n}\varepsilon_4), \\ \min\{i \in [n-1] : (1-p_n)h_i(p) + p_n h_n(p) \le \lambda\} & \text{if } p_n \in (\frac{\varepsilon_4}{2n}, 1 - \frac{\varepsilon_1}{2}), \end{cases} \tag{7}$$

the simplex S, the triangulation T^S of S, and its vertices V^S satisfy the assumptions of Lemma 2.

Part 4: *The existence of the asserted point.* By Lemma 2 and Part 3 we obtain that there exists a sequence of different simplices $\sigma_1, \dots, \sigma_J \in T$, $J \in \mathbb{N}$, such that $\overline{v}^n \in \sigma_1$, $0 \in \sigma_J$, σ_j and σ_{j+1} are adjacent, $j \in [J-1]$, and $[n-1] \subset l(\sigma_j)$, $j \in [J]$, $0 \notin l(\sigma_j)$, $j \in [J-1]$. It is obvious that the set $\bigcup_{j \in J} \sigma_j$ is an arcwise connected subset of S joining the 'upper' vertex $\overline{v}^n$ of S with the 'bottom' $\langle \overline{v}^i : i \in [n-1] \rangle$ of S. Since $0 \in l(\sigma_J)$, we have that $h_n(v) > 0$ for the unique vertex of σ_J for which $l(v) = 0$. It is also clear that there is the first simplex σ_{j_1}, $j_1 \in [J]$, such that $\sigma_j \subset \{p \in S : p_n \le 1 - \varepsilon_1/2\}$, $j \in j_1, \dots, J$. The choice of ε_1 and the fact that the diameters of simplices in T are sufficiently small guarantee $\sigma_{j_1} \subset \{p \in S : p_n \in [1-\varepsilon_1, 1-\varepsilon_1/2]\}$. Hence, $h_n(v) < 0$ for $v \in V(\sigma_{j_1})$. The arcwise connectedness of the set $\bigcup_{j=j_1}^J \sigma_j$ implies now that there is σ_{j_2}, $j_2 \in \{j_1, \dots, J\}$ such that $h_n(v) > 0$ for some $v \in V(\sigma_{j_2})$ and $h_n(v') \le 0$ for some $v' \in V(\sigma_{j_2}) > 0$, which in view of the construction of vectors $h(p)$, $p \in S$, and the convexity of σ_{j_2}, implies that there exists $p \in \sigma_{j_2}$: $h_n(p) = 0$. From (4) we obtain that $|h_n(v)| \le \lambda$, $v \in V(\sigma_{j_2})$. Notice that for each $v \in V(\sigma_{j_2})$ with $l(v) = i$ for some $i \in [n-1]$, we have $(1-v_n)h_i(v) + v_n h_n(v) \le \lambda$. In view of the inequality $v_n \le 1 - \frac{n-1}{n}\varepsilon_4$ for all $i \in [n-1]$, we have

$$h_i(v) \le \frac{1}{1-v_n}\lambda - \frac{v_n}{1-v_n}h_n(v) \le \frac{\lambda}{1-v_n} + \frac{v_n\lambda}{1-v_n} \le \frac{\lambda}{1-v_n}(1+v_n) \le \frac{2\lambda}{1-v_n} \le \frac{4n\lambda}{(n-1)\varepsilon_4},$$

from which, according to (2), we get

$$h_i(v) \le \delta/2 \quad \text{if } l(v) = i, i \in [n-1], v \in V(\sigma_{j_2}).$$

Since for each $i \in [n-1]$, there is $v' \in V(\sigma_{j_2})$ such that $l(v') = i$, by (4) we see that $h_i(v) \le \delta/2 + \lambda < \delta$, $v \in \sigma_{j_2}$, $i \in [n-1]$, and $h_n(v) \le \lambda < \delta$. Hence, $h_i(v) < \delta$ for any $v \in \sigma_{j_2}$, $i \in [n]$. To end the proof, notice that $v \in \sigma_{j_2} \subset \hat{\sigma}_{j_2} \in K_{m_1}(S)$, and thus $h(v) \in \operatorname{conv}(z(p) + B_{\varepsilon/2})$ for some $p \in \{p^i : i \in [Q]\}$ (see Part 2 of the proof). But $z(p) + B_{\varepsilon/2}$ is a convex set, so that $h(v) \in z(p) + B_{\varepsilon/2}$, and there are $y \in z(p)$ and $x \in B_{\varepsilon/2}$ such that $h(v) = y + x$. We have

$$y_i = h_i(v) - x_i \le \delta + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad i \in [n].$$

The points $p \in S \subset \operatorname{int}\Delta^n$ and $y \in z(p)$ satisfy the assertion. See Figure 2. □

From Lemma 6 and its proof we obtain the following.

Theorem 7 *Let z be as in Lemma* 6. *There exists an equilibrium point for z.*

Proof Observe that the simplex S constructed in Part 1 of the proof is independent of $\varepsilon > 0$. Hence, there are points $p^q \in S$ and $y^q \in z(p^q)$, satisfying $y_i^q \leq 1/q$, $i \in [n]$, $q \in \mathbb{N}$. The compactness of S allows us to assume that the sequence p^q converges to a point $p \in S$. From the upper semicontinuity of the mapping z and from the compactness of its values we may also assume that the corresponding sequence of points $y^q \in z(p^q)$ converges to a point $y \in z(p)$ with $y_i \leq 0$, $i \in [n]$. Since $p \in S \subset \operatorname{int} \Delta^n$, $y = 0$ (by Walras' law). □

3.2 A continuum of equilibria for parameterized excess demand mappings

The main result of this section is a version of Browder fixed point theorem for excess demand mappings.

Theorem 8 *Let $z : \operatorname{int} \Delta^{n-1} \times [0,1] \multimap \mathbb{R}^{n-1}$, $n \in \mathbb{N}$, be a nonempty, convex, and compact-valued upper semicontinuous mapping such that*

1. $pz(p,t) = 0$, $(p,t) \in \operatorname{int} \Delta^{n-1} \times [0,1]$,
2. *if* $(p^q,t^q) \in \operatorname{int} \Delta^{n-1} \times [0,1]$, $y^q \in z(p^q,t^q)$, $q \in \mathbb{N}$, $\lim_{q\to+\infty}(p^q,t^q) = (p,t) \in \Delta^{n-1} \times [0,1]$, *then*

$$p_i = 0 \quad \Rightarrow \quad \lim_{q+\infty} y_i^q = +\infty, \quad i \in [n-1],$$

3. *there exists a negative number L such that*

$$\inf\{y_i \in \mathbb{R} : y \in z(p,t), (p,t) \in \operatorname{int} \Delta^{n-1} \times [0,1]\} > L, \quad i \in [n-1].$$

Then there exists a compact and connected set $E \subset \operatorname{int} \Delta^{n-1} \times [0,1]$ such that $E \cap (\operatorname{int} \Delta^{n-1} \times \{0\}) \neq \emptyset$, $E \cap (\operatorname{int} \Delta^{n-1} \times \{1\}) \neq \emptyset$, and $0 \in z(p,t)$, $(p,t) \in E$.

Before we present a proof of Theorem 8, let us remark that for any $t \in [0,1]$, the mapping $z(\cdot,t)$ is an excess demand mapping in the sense of Definition 3. Theorem 8, assumptions 2 and 3 (and the assumption of upper semicontinuity) impose some uniformity conditions on the family of mappings $\{z(\cdot,t) : t \in [0,1]\}$, and we suppose that the claim of Theorem 8 may not be valid for a nonempty, convex, and compact-valued upper semicontinuous mapping $z : \operatorname{int} \Delta^{n-1} \times [0,1]$ satisfying Theorem 8, assumption 1 and such that each mapping $z(\cdot,t)$ is an excess demand mapping, $t \in [0,1]$, but either Theorem 8, assumption 2 or Theorem 8, assumption 3 is not satisfied. However, we were not able to construct an example of such mapping nor to deliver a proof of Theorem 8 without introducing the just mentioned conditions.

Proof of Theorem 8 The claim is certainly true for $n = 1$, so assume that $n \geq 2$. Let us define the mapping $\xi : \operatorname{int} \Delta^n \multimap \mathbb{R}^n$ by

$$\xi(p) := \begin{cases} z(\frac{p_1}{\sum_{i\in[n-1]} p_i}, \dots, \frac{p_{n-1}}{\sum_{i\in[n-1]} p_i}, 0) \times \{0\} & \text{if } p_n \leq 1/3, \\ z(\frac{p_1}{\sum_{i\in[n-1]} p_i}, \dots, \frac{p_{n-1}}{\sum_{i\in[n-1]} p_i}, 3p_n - 1) \times \{0\} & \text{if } p_n \in [1/3, 2/3], \\ z(\frac{p_1}{\sum_{i\in[n-1]} p_i}, \dots, \frac{p_{n-1}}{\sum_{i\in[n-1]} p_i}, 1) \times \{0\} & \text{if } p_n \geq 2/3, \end{cases}$$

for $p \in \operatorname{int} \Delta^n$. By the assumptions on the mapping z and the construction of ξ, reasoning similarly as in the proof of Lemma 5 (with ξ in place of z), we deduce that:

(i) for each $\varepsilon_2 \in (0, 1/12)$, there exists $U > 0$ such that $\xi(p) \subset [L, U]^n$ for each $p \in \operatorname{int} \Delta^n$ with $p_i \geq \varepsilon_2$, $i \in [n]$,

(ii) for each $\varepsilon_3 \in (0, 1/12)$, there exists $\varepsilon_4 \in (0, \varepsilon_3/2]$ such that for $p \in \operatorname{int} \Delta^n$ with $p_n \leq 1 - \varepsilon_3$, we have that, for $i \in [n-1]$ and $y \in \xi(p)$,

$$\bigl(p_i \leq \varepsilon_4 \Rightarrow (1-p_n)y_i + p_n y_n > 0\bigr) \quad \textit{and} \quad \bigl(p_i \geq 1 - \varepsilon_4 \Rightarrow (1-p_n)y_i + p_n y_n < 0\bigr),$$

(iii) for ε_3, ε_4 for which claim (ii) and its premises hold, there exists $\Lambda \in (0, +\infty)$ such that, for $i \in [n-1]$,

$$(1-p_n)y_i + p_n y_n > \Lambda,$$

whenever $y \in \xi(p)$, $p \in \operatorname{int} \Delta^n$, $p_i \leq \varepsilon_4$, $p_n \leq 1 - \varepsilon_3$, and $p_j \in [\varepsilon_4/2n, 1 - \varepsilon_4/2n]$, $j \in [n]$.

Remark that for $p \in \operatorname{int} \Delta^n$ and $y \in \xi(p)$, we have $y_n = 0$, and thus the inequality $(1-p_n)y_i + p_n y_n > 0$ is equivalent to $y_i > 0$ for $i \in [n-1]$; similarly, if we replace '>' with '<'. Let ε_4 correspond to some fixed $\varepsilon_3 \in (0, 1/12)$ according to (ii) and $\varepsilon_1 := 2\varepsilon_3$, $\varepsilon_2 := \varepsilon_4$.[h] Moreover, let U, Λ correspond to ε_2, ε_3, and ε_4 as in (i), (iii), respectively. Finally, fix $\varepsilon > 0$. Observe, that ε_i are chosen independently of ε. Literally repeating the passage of the proof of Lemma 6 starting from '*Without loss of generality, we can assume that* ε_4 *is so small that…*' in Part 1 and ending with '*It is obvious that the set* $\bigcup_{j \in J} \sigma_j$ *is an arcwise connected subset of S joining the 'upper' vertex* $\overline{v}^n$ *of S with the 'bottom'* $\langle \overline{v}^i : i \in [n-1] \rangle$ *of S*' in Part 4, with the exceptions that

- we replace z with ξ, and
- we keep in mind that $y_n = 0$ whenever y belongs to the image of ξ, so that, in consequence, $h_n(p) = 0$, $p \in S$,

we obtain the simplex S, its triangulation T^S, and a sequence of consecutively pairwise adjacent $(n-1)$-simplices $\sigma_1, \ldots, \sigma_J$ (with no simplex appearing twice in the sequence) in T^S joining $\overline{v}^n$ and $\langle \overline{v}^1, \ldots, \overline{v}^{n-1} \rangle$ and satisfying the following equivalences: $0 \in l(\sigma^j) \Leftrightarrow j = J$, $n \in l(\sigma_j) \Leftrightarrow j = 1$ for $j \in [J]$, $l(\sigma^j) \subset [n-1]$. We also have that there exists the first $j_1 \in [J]$ such that $p \in \sigma_j$, $j > j_1$, implies $p_n \leq 2/3$, and there exists the greatest $j \in \{j_1 + 1, \ldots, J\}$ such that $p \in \sigma_j$, $j \in \{j_1, j_1 + 1, \ldots, j_2 - 1\}$, implies $p_n \geq 1/3$. Let

$$A := \{j_1, j_1 + 1, \ldots, j_2\}, \qquad E_A := \Bigl(\bigcup_{j \in A} \sigma_j\Bigr) \cap \bigl\{p \in S : p_n \in [1/3, 2/3]\bigr\}.$$

Observe that since the diameter of each simplex in T^S is small, $p \in \sigma^{j_1} \cup \sigma^{j_2}$ implies $p_n \in (\varepsilon_4/2n, 1 - \varepsilon_1/2)$. It is also clear that there are $p \in \sigma_{j_1}$: $p_n = 2/3$ and $p' \in \sigma_{j_2}$: $p'_n = 1/3$. Moreover, for any $j', j'' \in [J]$, $j' \leq j''$, the set $\bigcup_{j \in [J]: j' \leq j \leq j''} \sigma_j$ is arcwise connected.

The definition of the function l and the fact that $l(\sigma_j) = [n-1]$, $j \in A$, entail that for $i \in [n-1]$ and $j \in A$, there is $v \in V(\sigma_j)$ with $(1 - v_n)h_i(v) + v_n h_n(v) \leq \lambda$; hence,

$$h_i(v) \leq \delta/2,$$

and thus (see Part 4 of the proof of Lemma 6)

$$h_i(v) < \delta \leq \varepsilon/2, \quad i \in [n-1], v \in \sigma_j, j \in A.$$

For each $j \in A$, there are $\hat{\sigma}_j \in K_{m_1}(S)$ with $\operatorname{conv}\{h(v) : v \in \sigma_j\} \subset \operatorname{conv}\{y(w) : w \in \hat{\sigma}_j\}$ and $i_j \in [Q]$ with $\sigma_j \subset \hat{\sigma}_j \subset p^{i_j} + B^{i_j}$ (see formula (3) and Part 2 of the proof of Lemma 6). Hence, $\operatorname{conv}\{h(v) : v \in \sigma_j\} \subset \xi(p^{i_j}) + B_{\varepsilon/2}$ and $y_i \leq \delta \leq \varepsilon/2$ for $y \in \operatorname{conv}\{h(v) : v \in \sigma_j\}$, $i \in [n-1]$, $j \in A$. By the arcwise connectedness of the set E_A there exists a continuous function $g : [0,1] \to S$ such that

(iv) $g([0,1]) \subset \{p \in S : p_n \in [1/3, 2/3]\}$,
(v) $g_n(0) = 2/3$, $g_n(1) = 1/3$,
(vi) for each $t \in [0,1]$, $h_i(g(t)) \leq \varepsilon/2$, $i \in [n-1]$, $h_n(g(t)) = 0$,
(vii) for each $t \in [0,1]$, there are $p \in S$ and $y \in \xi(p)$ with $|p - g(t)| < \delta$, $|y - h_i(g(t))| < \varepsilon/2$, $i \in [n-1]$, and thus $y_i \leq \varepsilon$, $i \in [n-1]$.

From up-to-now considerations we conclude that for each $\varepsilon > 0$, there exist $\delta \in (0, \varepsilon/2)$ (see Part 2 of the proof of Lemma 6) and continuous functions $g^\varepsilon : [0,1] \to S$ and $h^\varepsilon : S \to \mathbb{R}^n$ satisfying properties (iv)-(vii). Let $\varepsilon^q = 1/q$, $q \in \mathbb{N}$, and let g^q, h^q be functions meeting properties (iv)-(vii) (with g^ε, h^ε instead of g, h, respectively). Without loss of generality, let us assume that $\lim_{q\to+\infty} g^q(0) = \overline{g}$. By (v), $\overline{g}_n = 2/3$. Notice that the continuity of g^q implies that the set $\Gamma^q := g^q([0,1])$ is a compact and connected subset of the convex compact set S, $q \in \mathbb{N}$. Define

$$\Gamma := \left\{x \in X : x = \lim_{q\to+\infty} x^{k_q}, \text{where } k_q \nearrow +\infty \text{ as } q \to +\infty, x^{k_q} \in G^{k_q}, q \in \mathbb{N}\right\}.$$

By Lemma 1 the set Γ is a connected and compact subset of S. By property (v) there is a point $\overline{g}' \in \Gamma : \overline{g}'_n = 1/3$. Observe also that $\overline{g} \in \Gamma$ and recall that $\overline{g}_n = 2/3$. From (vii) we see that $0 \in \xi(g)$, $g \in \Gamma$. Indeed, if $x \in \Gamma$, then there exist sequences $k_q \in \mathbb{N}$, $x^q \in g^{k_q}([0,1])$, $q \in \mathbb{N}$, $k_q \nearrow +\infty$, such that $\lim_{q\to+\infty} x^q = x$. By property (vii), for each $q \in \mathbb{N}$, there exist $p^q \in S$ and $y^q \in \xi(p^q)$ such that $|p^q - x^q| < 1/q$, $y_i^q \leq 1/q$, $i \in [n-1]$. Obviously, $\lim_{q\to+\infty} p^q = x$, and since $\xi(p^q) \subset [L,U]^n$, $q \in \mathbb{N}$, we may assume (extracting a subsequence if necessary) that $\lim_{q\to+\infty} y^q = y \in [L,U]^n$, where $y_i \leq 0$, $i \in [n-1]$. By the upper semicontinuity of the compact-valued mapping ξ on the set S, $y \in \xi(p)$. Now, since $p \in S \subset \operatorname{int} \Delta^n$, by assumption 1 on the mapping z and the definition of ξ we get

$$\sum_{i\in[n]} p_i y_i = \sum_{i\in[n-1]} p_i y_i = \left(\sum_{j\in[n-1]} p_j\right) \sum_{i\in[n-1]} \frac{p_i}{\sum_{j\in[n-1]} p_j} y_i = 0,$$

which, due to the inequalities $y \leq 0$ and $p > 0$, implies $y = 0$, and hence $0 \in \xi(p)$.

Let now the function $w : \{p \in \operatorname{int} \Delta^n : p_n \in [1/3, 2/3]\} \to \operatorname{int} \Delta^{n-1} \times [0,1]$ be defined by

$$w(p) := \left(\frac{p_1}{\sum_{i\in[n-1]} p_i}, \dots, \frac{p_{n-1}}{\sum_{i\in[n-1]} p_i}, 3p_n - 1\right),$$

$p \in \operatorname{int} \Delta^n$, $p_n \in [1/3, 2/3]$. The function w is a homeomorphism between $\{p \in \operatorname{int} \Delta^n : p_n \in [1/3, 2/3]\}$ and $\operatorname{int} \Delta^{n-1} \times [0,1]$, $w_n(p) = 0 \Leftrightarrow p_n = 1/3$, and $w_n(p) = 1 \Leftrightarrow p_n = 2/3$, $p \in \operatorname{int} \Delta^n$: $p_n \in [1/3, 2/3]$. It follows that the set $E := w^{-1}(\Gamma)$ satisfies the assertion. See Figure 2. □

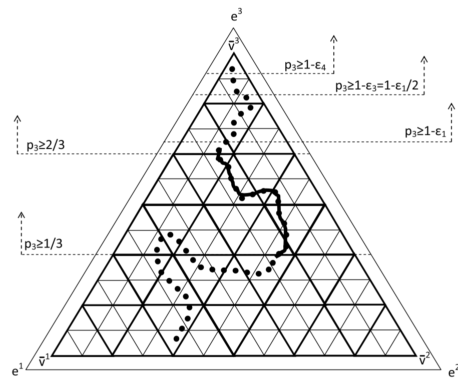


Figure 2 This figure explains the idea of the proofs of Lemma 6 and Theorem 8 for $n = 3$. The vertices $\overline{v}^i$, $i \in \{1, 2, 3\}$, are defined by (1). The simplex $S := \langle v^1, v^2, v^3\rangle$ is triangulated with thick-lined triangles ($K_6(S)$ triangulation, $m_1 = 6$) and then with thin-lined triangles ($K_{12}(S)$ triangulation, $m_2 = 2$). The path P of simplices marked with thick-dots or thick-line is determined by adequate labeling (dependent on the theorem being considered) and combinatorial Lemma 2. For Lemma 6: there is a simplex $\sigma \in P$ below the line $p_3 \geq 1 - \varepsilon_1/2$ and points $v, v' \in \sigma$ such that $h_3(v) < 0$ and $h_3(v') \geq 0$ (see Part 4 of the proof). For Theorem 8: the simplices in P marked with thick-line represent the compact connected set E_A corresponding to the given accuracy level ε and joining level-lines $p_3 = 1/3$ and $p_3 = 2/3$.

3.3 Sperner's lemma

Let us fix $n \in \mathbb{N}\backslash\{1\}$ and define, for $i \in [n-1]$,

$$\begin{aligned} w^i &:= \frac{1}{3}e^i + \frac{2}{3}e^n, \\ v^i &:= \frac{2}{3}e^i + \frac{1}{3}e^n, \end{aligned} \tag{8}$$

where $e^i \in \mathbb{R}^n$ denotes the ith unit vector of $\mathbb{R}^n$, $i \in [n]$. We can easily see that $w^i, v^i \in \Delta^n$, $i \in [n-1]$, and that the vectors $w^1, \dots, w^{n-1}$ and $v^1, \dots, v^{n-1}$, are linearly independent, so $\langle w^i : i \in [n-1]\rangle$ and $\langle v^i : i \in [n-1]\rangle$ are $(n-2)$-simplices contained in Δ^n. Moreover, directly from formula (8) we obtain that each of the sets $\{v^1, \dots, v^{n-1}, w^1\}$, $\{v^2, \dots, v^{n-1}, w^1, w^2\}$, $\dots$, $\{v^{n-1}, w^1, \dots, w^{n-1}\}$ is a set of affinely independent vectors, and thus these sets generate $(n-1)$-simplices

$$\begin{aligned} &S^1 := \langle v^1, \dots, v^{n-1}, w^1\rangle, \\ &S^2 := \langle v^2, \dots, v^{n-1}, w^1, w^2\rangle, \\ &\dots, \\ &S^{n-1} := \langle v^{n-1}, w^1, \dots, w^{n-1}\rangle. \end{aligned} \tag{9}$$

In what follows, by $v^i, \dots, w^{i'}$ we mean $v^i, \dots, v^{n-1}, w^1, \dots, w^{i'}$ for any $i, i' \in [n-1]$. We first present some lemmas. Their proofs are postponed to the Appendix.

The next lemma is a bit technical, but the geometry behind it is rather intuitive. Namely, the lemma reveals that it is possible to decompose the polytope $\mathrm{conv}\{v^1, \dots, v^n, w^1, \dots, w^n\}$ into the simplices $S^1, \dots, S^{n-1}$, and the intersection of any pair of those simplices is their common face.[i]

Lemma 9 *Let* w^i, v^i, $i \in [n-1]$, *be as in* (8), *and* S^i, $i \in [n-1]$, *as in* (9). *Then*

$$S^i \cap S^{i'} = \langle v^{i'}, \dots, w^i\rangle, \quad i, i' \in [n-1], i < i' \tag{10}$$

and

$$\operatorname{conv}\{v^1,\dots,w^{n-1}\} = S^1 \cup S^2 \cup \cdots \cup S^{n-1}. \tag{11}$$

Remark 10 Similar decomposition applies for the set $\operatorname{conv}\{v^1,\dots,v^{n-1},e^1,\dots,e^{n-1}\}$. The idea for a proof is identical to that presented in the proof above (only some minor changes are necessary).

The situation presented further in Lemma 11 can be imagined easily.

Lemma 11 *Fix $n \geq 3$, $n \in \mathbb{N}$. Let T be a triangulation of an $(n-2)$-simplex $S = \langle p^1,\dots,p^{n-1}\rangle \subset \mathbb{R}^n$, and let $S' = \langle p^1,\dots,p^{n-1},r\rangle$ be an $(n-1)$-simplex. Then the collection $T' := \{\langle V(\sigma)\cup\{r\}\rangle : \sigma \in T\}$ is a triangulation of S'. Moreover, each simplex $\sigma \in T$ and each simplex σ' of the form $\langle V(F)\cup\{r\}\rangle$, where F is an $(i-1)$-face of a simplex in T, $i \in [n-2]$, belong to the set of all i-faces of simplices in T', $i \in [n-2]$.*

We shall now show that there is a special triangulation of the standard closed simplex Δ^n. The special triangulation allows us to embed a triangulated $(n-2)$-simplex S in the standard simplex Δ^n keeping simplices from the triangulation of S as faces of simplices in the triangulation of Δ^n. This lemma allows us to apply our combinatorial Lemma 2 to detect a simplex satisfying the assertion of Sperner's lemma.

Lemma 12 *Fix $n \geq 3$, $n \in \mathbb{N}$. Let v^j, w^j, $j \in [n-1]$, be defined by* (8). *Suppose that T is a triangulation of the $(n-2)$-simplex $\langle v^1,\dots,v^{n-1}\rangle$. Then there exists a triangulation T' of Δ^n such that $\langle v^n,w^1,\dots,w^{n-1}\rangle \in T'$ and, for each $\sigma \in T$, there is a simplex $\sigma' \in T'$ for which σ is its an $(n-2)$-face. We also have*

$$V(T') = V(T) \cup \{e^1,\dots,e^n,w^1,\dots,w^{n-1}\}. \tag{12}$$

Moreover, if for a simplex $\sigma' \in T'$, there is exactly one $i \in [n-1]$ such that $e^i \in V(\sigma')$, then $\sigma' = \langle V(\sigma)\cup\{e^i\}\rangle$ for some $\sigma \in T$.

Theorem 13 (Sperner's lemma) *Fix $n \geq 3$, $n \in \mathbb{N}$. Suppose that T is a triangulation of an $(n-2)$-simplex $S = \langle p^1,\dots,p^{n-1}\rangle$. Let $l : V(T) \to [n]$ be a function such that $l(v) \neq i$ whenever $\alpha_i(v) = 0$, where $\alpha_i(v) := \alpha^S_{p^i}(v)$ is the ith barycentric coordinate of the vector v in S. There exists $\sigma \in T$ with $l(\sigma) = [n-1]$. Moreover, the number of such simplices is odd.*

Proof Let v^i, w^i, $i \in [n-1]$, be defined by (8). Without loss of generality, assume that $p^i = v^i$, $i \in [n-1]$. Since T triangulates $\langle v^1,\dots,v^{n-1}\rangle$, we may assume that T' is a triangulation of Δ^n whose existence is asserted in Lemma 12. Let now $l' : V(T') \to [n]_0$ be defined by

$$l'(p) := \begin{cases} n & \text{if } p = e^n, \\ 0 & \text{if } p \in \{e^1,\dots,e^{n-1}\}, \\ i & \text{if } p = w^i, \text{for some } i \in [n-1], \\ l(p) & \text{if } p \in V(T), \end{cases}$$

where $p \in V(T')$. The function l' and triangulation T' of Δ^n evidently meet the assumptions of Lemma 2. Hence, there exists a sequence $\sigma_1,\dots,\sigma_J$ of adjacent simplices in T' such

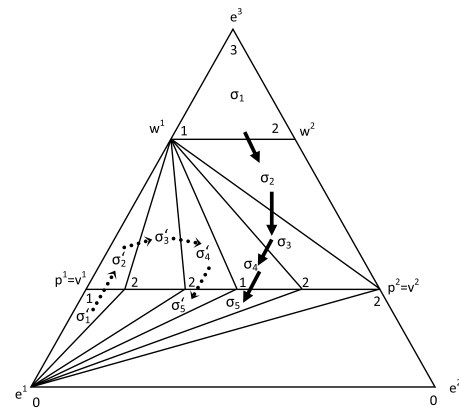


Figure 3 The idea of the proof of Sperner's lemma (Theorem 13) for $n = 3$. Then simplex $\langle p^1, p^2\rangle$ is embedded in $\langle e^1, e^2, e^3\rangle$ as $\langle v^1, v^2\rangle$. The triangulation T of that simplex consists of small sectors in $\langle v^1, v^2\rangle$ whose vertices are labeled by the function l. The labels at the other vertices are produced by l' (see the proof). The simplices $\sigma_1, \dots, \sigma_5$ come from combinatorial Lemma 1. The simplices $\sigma'_1, \dots, \sigma'_5$ come from the pairing of simplices by the procedure described in the proof of Sperner's lemma (see also endnote j). Finally, observe that the simplex $\langle e^1, e^2, e^3\rangle$ is triangulated as in the claim (and proof) of Lemma 12.

that σ_J is the first simplex with a vertex contained in $\{e^1, \dots, e^{n-1}\}$. Obviously, by Lemma 2, $l'(\sigma_J) = [n-1]_0$, and from Lemma 12 we see that $\sigma_J = \langle V(\overline{\sigma}) \cup \{e^i\}\rangle$ for some $\overline{\sigma} \in T$ and $i \in [n-1]$. Hence, $l'(\overline{\sigma}) = l(\overline{\sigma}) = [n-1]$, which proves the existence of a simplex in T satisfying the desired property. Now, if $\sigma \in T$, $\sigma \neq \overline{\sigma}$, then using a similar procedure to that presented in the proof of Lemma 2 to generate the sequence $\sigma_1, \dots, \sigma_J$, we obtain a unique sequence $\sigma'_1, \dots, \sigma'_{J'}$ of simplices in T' such that $\sigma'_{J'}$ is the first simplex in that sequence different from σ'_1 that possesses a face $\overline{\sigma}' \in T\backslash\{\overline{\sigma}\}$ with $l(\overline{\sigma}') = [n-1]$.[j] This way we can pair the simplex σ with a different simplex $\overline{\sigma}' \in T$, where $l(\sigma) = l(\overline{\sigma}')$. So, to each simplex σ in $T\backslash\{\overline{\sigma}\}$ with $l(\sigma) = [n-1]$, we can uniquely assign a different simplex $\sigma' \in T\backslash\{\overline{\sigma}\}$ with $l(\sigma') = [n-1]$. It follows that the number of simplices satisfying the claim is odd since $\overline{\sigma}$ has no paired simplex (obtainable using the presented procedure of generating sequences of simplices in T'). See Figure 3. □

4 Final remarks

4.1 An algorithm for finding a zero of an excess demand mapping

From the proof of Lemma 2 we obtain the following algorithm for finding approximate solutions to the equation $y = 0$ for $y \in z(p)$, $p \in \operatorname{int} \Delta^n$, where z is an excess demand mapping.

Step 0: Fix accuracy level $\varepsilon > 0$. Determine $\varepsilon_1, \varepsilon_3, \varepsilon_4, \Lambda$ and vertices $\overline{v}^i$, $i \in [n]$, and simplex $S := \langle \overline{v}^i : i \in [n]\rangle$ as in Part 1 of the proof of Lemma 6. Fix numbers δ, λ, m_1, m_2 as in Part 2 of the proof. Let also $T := K_{m_1 \times m_2}(S)$. To each vertex $v \in V(K_{m_1}(S))$, arbitrarily assign $y(w) \in z(v)$. Further, to each $v \in V(T)$, assign the point $h(v)$ according to formula (3) and label each $v \in V(T)$ as $l(v) \in [n]_0$, where labeling l is defined by (7). Let also σ_1 be the only simplex in T with $\overline{v}^n \in \sigma_1$. Put $F := V(\sigma_1)\backslash\{\overline{v}^n\}$, $\overline{v} := \overline{v}^n$, and go to Step 1.

Step 1: Determine the only vertex $v \in T$ such that $v \neq \overline{v}$ and $\langle F \cup \{v\}\rangle \in T$. Go to Step 2.

Step 2: If $h_n(v) \geq 0$, then STOP: there is a point $p \in (v + B_\delta) \cap \operatorname{int} \Delta^n$ and $y \in z(p)$ with $y_i \leq \varepsilon$, $i \in [n]$. Otherwise, assign the only element of $l^{-1}(l(v)) \cap F$ as the value of $\overline{v}$. Set $F := (F\backslash\{\overline{v}\}) \cup \{v\}$ and go to Step 1.

Let us emphasize once again that the proofs of Theorems 8 and 13 also contain descriptions of algorithms for finding the asserted objects, although we leave their detailed formulations to the reader.

4.2 Kakutani fixed point theorem and Gale-Debreu-Nikaido lemma

Suppose that $F : \Delta^n \multimap \Delta^n$ is an upper semicontinuous nonempty convex and compact-valued mapping. Kakutani's fixed point theorem ensures that such a mapping possesses a fixed point [3], p.27. It is possible to derive Kakutani's theorem from our Theorem 7. To this end, for any fixed $\varepsilon > 0$, define the mapping $z^\varepsilon : \operatorname{int}\Delta^n \multimap \mathbb{R}^n$ by

$$z(p) := g^\varepsilon(p) + \left\{ y - \frac{py}{pp} p : y \in F(p) \right\}, \quad p \in \operatorname{int}\Delta^n,$$

where $g^\varepsilon(p) = \varepsilon(\frac{1}{np_1} - 1, \dots, \frac{1}{np_n} - 1)$. We can prove that z^ε is an excess demand mapping in the sense of Definition 3. By Theorem 7, for each $\varepsilon > 0$, there exists $p^\varepsilon \in \operatorname{int}\Delta^n$ such that $0 \in z^\varepsilon(p^\varepsilon)$, and thus $0 = g^\varepsilon(p^\varepsilon) + y^\varepsilon - \frac{p^\varepsilon y^\varepsilon}{p^\varepsilon p^\varepsilon} p^\varepsilon$ for some $y^\varepsilon \in F(p^\varepsilon)$. Hence, for a sequence ε^q, $q \in \mathbb{N}$, converging to 0^+, there exist sequences $p^q, y^q \in F(p^q)$, $q \in \mathbb{N}$, with

$$0 = \varepsilon^q \left(\frac{1}{np_1^q} - 1, \dots, \frac{1}{np_n^q} - 1 \right) + y^q - \frac{p^q y^q}{p^q p^q} p^q$$

that converge to some $p \in \Delta^n$ and $y \in F(p)$, respectively. Then, for each $i \in [n]$, we have

$$1 - \frac{\varepsilon^q}{np_i^q} = y_i^q - \frac{p^q y^q}{p^q p^q} p_i^q,$$

and since the right-hand side sequence is bounded, we can assume that the left-hand side terms converge to some $a \in \mathbb{R}$. Multiplying the last equation by p_i (for each $i \in [n]$), summing over $i \in [n]$, and taking the limit as $q \to +\infty$, we see that $a = 0$ (by Walras' law). Hence,

$$y = \frac{py}{pp} p,$$

and $y, p \in \Delta^n$ implies that $\frac{py}{pp} = 1$, so $p = y \in F(p)$.

Now, if $F : \Delta^n \multimap \mathbb{R}^n$ is an upper semicontinuous nonempty convex and compact-valued mapping satisfying a weak version of Walras' law $pF(p) \leq 0$, $p \in \Delta^n$, then the Gale-Debreu-Nikaido lemma asserts that there are $p \in \Delta^n$ and $y \in F(p)$ with $y \leq 0$ [7], p.81. Going along the lines above for Kakutani's fixed point theorem, we obtain points $p \in \Delta^n$, $y \in F(p)$ with $y = \frac{py}{pp} p$. By the weak Walras law we get $py \leq 0$, hence, due to the inequality $p \geq 0$, we have $y \leq 0$.

4.3 Browder theorem

Browder's fixed point theorem for mappings states that if $F : \Delta^n \times [0,1] \multimap \Delta^n$ is an upper semicontinuous mapping with nonempty, convex, and compact values, then there is a compact connected subset $E \subset \{(x,t) \in \Delta^n \times [0,1] : x \in F(x,t)\}$ with $E \cap (\Delta^n \times \{0\}) \neq \emptyset$ and $E \cap (\Delta^n \times \{1\}) \neq \emptyset$ [3], p.29. It is possible to derive this theorem from our Theorem 8 and Lemma 1. Indeed, let for $\varepsilon > 0$, $z^\varepsilon : \operatorname{int}\Delta^n \times [0,1] \multimap \mathbb{R}^n$ be defined by

$$z(p,t) := g^\varepsilon(p) + \left\{ y - \frac{py}{pp} p : y \in F(p,t) \right\}, \quad (p,t) \in \operatorname{int}\Delta^n \times [0,1],$$

where g^ε is defined in Section 4.2. The mapping z^ε satisfies the assumptions of Theorem 8. So, there exists a compact and connected set $E^\varepsilon \subset \operatorname{int}\Delta^{n-1} \times [0,1]$ such that $E^\varepsilon \cap (\operatorname{int}\Delta^{n-1} \times \{0\}) \neq \emptyset$, $E^\varepsilon \cap (\operatorname{int}\Delta^{n-1} \times \{1\}) \neq \emptyset$, and $0 \in z^\varepsilon(p,t)$, $(p,t) \in E^\varepsilon$. By Lemma 1 we obtain that for a sequence $\varepsilon^q > 0$, $q \in \mathbb{N}$, converging to 0^+, and nonempty compact and connected sets $E^q \subset \Delta^n \times [0,1]$ ($E^q := E^{\varepsilon^q}$), the limit set $E \subset \Delta^n \times [0,1]$ is compact, connected, and $E \cap (\Delta^{n-1} \times \{0\}) \neq \emptyset$, $E \cap (\Delta^{n-1} \times \{1\}) \neq \emptyset$. Arguing as in Section 4.2, we conclude that each point $(p,t) \in E$ is a fixed point of $F(\cdot,t)$: $p \in F(p,t)$.

Let us notice that in a recent work [8], there was also proved a generalization of Browder's theorem. However, it seems that our less general approach is simpler than that presented in [8].

4.4 A bit on economics

Our findings in [1] were well motivated by economics. However, it appears that our results have natural origins in economics. Indeed, Theorem 7 allows us to state that there exists an equilibrium for a pure exchange economy (where agents' excess demands are multivalued mappings), whereas Theorem 8 ensures the existence of equilibrium in an exchange economy with price rigidities ([3], Chapter 2 or [8]). Even Sperner's lemma has an interesting economic/social implication; it enables us to deduce that there exists a fair division of a good (see, *e.g.*, a nice introduction in [9]).

Appendix

Recall that $v^i, \ldots, w^{i'}$ means $v^i, \ldots, v^{n-1}, w^1, \ldots, w^{i'}$ for any $i, i' \in [n-1]$.

Proof of Lemma 9 Let $i, i' \in [n-1]$, $i < i'$. It is obvious that $\langle v^{i'}, \ldots, w^i \rangle \subset S^i \cap S^{i'}$. To prove (10), it suffices to show that $\langle v^{i'}, \ldots, w^i \rangle \subset S^i \cap S^{i'}$. On the contrary, suppose that there is $x \in (S^i \cap S^{i'}) \setminus \langle v^{i'}, \ldots, w^i \rangle$. Define

$$\lambda_j := \alpha^{S^i}_{v^j}(x), \quad j \in \{i, i+1, \ldots, n-1\}, \qquad \gamma_j := \alpha^{S^i}_{w^j}(x), \quad j \in \{1, 2, \ldots, i\},$$
$$\lambda'_j := \alpha^{S^{i'}}_{v^j}(x), \quad j \in \{i', i+1, \ldots, n-1\}, \qquad \gamma'_j := \alpha^{S^{i'}}_{w^j}(x), \quad j \in \{1, 2, \ldots, i'\},$$

where $\alpha^S_v(x)$ denotes the barycentric coordinate of a point $x \in S$ corresponding to the vertex $v \in V(S)$. So, $x = \sum_{j=i}^{n-1} \lambda_j v^j + \sum_{j=1}^{i} \gamma_j w^j$, $x = \sum_{j=i'}^{n-1} \lambda'_j v^j + \sum_{j=1}^{i'} \gamma'_j w^j$. Observe that by the assumption on x there exists $j \in \{i, i+1, \ldots, i'-1\}$: $\lambda_j > 0$. We have

$$\begin{aligned}
&\sum_{j=i}^{n-1} \lambda_j v^j + \sum_{j=1}^{i} \gamma_j w^j = \sum_{j=i'}^{n-1} \lambda'_j v^j + \sum_{j=1}^{i'} \gamma'_j w^j \\
&\quad\Leftrightarrow\quad \sum_{j=i}^{i'-1} \lambda_j v^j = \sum_{j=i'+1}^{n-1} (\lambda'_j - \lambda_j) v^j + \sum_{j=1}^{i-1} (\gamma'_j - \gamma_j) w^j + (\lambda'_{i'} - \lambda_{i'}) v^{i'} \\
&\qquad\qquad + (\gamma'_i - \gamma_i) w^i + \gamma'_{i'} w^{i'} + \sum_{j=i+1}^{i'-1} \gamma'_j w^j,
\end{aligned}$$

which, due to (8), imply $\lambda_j' - \lambda_j = 0, j \in \{i'+1, i'+2, \dots, n-1\}$ and $\gamma_j' - \gamma_j = 0, j \in \{1, 2, \dots, i-1\}$, so

$$\sum_{j=i}^{i'-1} \lambda_j v^j = (\lambda_{i'}' - \lambda_{i'}) v^{i'} + (\gamma_i' - \gamma_i) w^i + \gamma_{i'}' w^{i'} + \sum_{j=i+1}^{i'-1} \gamma_j' w^j.$$

The last equation and (8) imply

$$\frac{2}{3}\lambda_i = \frac{1}{3}(\gamma_i' - \gamma_i) \geq 0, \qquad \frac{2}{3}\lambda_{i+1} = \frac{1}{3}\gamma_{i+1}', \dots, \frac{2}{3}\lambda_{i'-1} = \frac{1}{3}\gamma_{i'-1}', \qquad \frac{2}{3}(\lambda_{i'}' - \lambda_{i'}) = -\frac{1}{3}\gamma_{i'}'$$

and

$$\frac{1}{3}\sum_{j=i}^{i'-1} \lambda_j = \frac{1}{3}(\lambda_{i'}' - \lambda_{i'}) + \frac{2}{3}(\gamma_i' - \gamma_i) + \frac{2}{3}\gamma_{i'}' + \frac{2}{3}\sum_{j=i+1}^{i'-1} \gamma_j',$$

from which we obtain by multiplying both sides by 3 and carefully substituting γ's for λ's

$$\frac{1}{2}(\gamma_i' - \gamma_i) + \frac{1}{2}\sum_{j=i+1}^{i'-1} \gamma_j' = -\frac{1}{2}\gamma_{i'}' + 2(\gamma_i' - \gamma_i) + 2\gamma_{i'}' + 2\sum_{j=i+1}^{i'-1} \gamma_j',$$

which is equivalent to

$$0 = 3(\gamma_i' - \gamma_i) + 3\gamma_{i'}' + 3\sum_{j=i+1}^{i'-1} \gamma_j'.$$

But $\gamma_i' - \gamma_i \geq 0$ and γ's are nonnegative, so $\gamma_{i'}' - \gamma_{i'} = 0$, $\gamma_j' = 0, j \in \{i+1, i+2, \dots, i'\}$, which entails $\lambda_j = 0$ for $j \in \{i, i+1, \dots, i'-1\}$, which is impossible due to the assumptions on the point x. Hence, equation (10) holds.

To prove (11), we shall first show that

$$\operatorname{conv}\{v^1, \dots, w^{n-1}\} = S^1 \cup \operatorname{conv}\{v^2, \dots, w^{n-1}\}.$$

Suppose that $x \in \operatorname{conv}\{v^1, \dots, v^{n-1}, w^1, \dots, w^{n-1}\}$, $x \neq v^1$, so there exist nonnegative numbers $\lambda_j, \gamma_j, j \in [n-1]$, $\sum_{j\in[n-1]}(\lambda_j + \gamma_j) = 1$, $\lambda_1 < 1$, for which

$$x = \sum_{j\in[n-1]} \lambda_j v^j + \sum_{j\in[n-1]} \gamma_j w^j. \tag{13}$$

By (8), $e^n = 2w^1 - v^1$, $w^j = \frac{1}{2}v^j + \frac{1}{2}e^n$, $j \in \{2, 3, \dots, n-1\}$, and, due to (13), we get

$$\begin{aligned} x &= \sum_{j\in[n-1]} \lambda_j v^j + \gamma_1 w^1 + \sum_{j=2}^{n-1} \gamma_j \left(\frac{1}{2}v^j + \frac{1}{2}(2w^1 - v^1)\right) \\ &= \lambda_1 v^1 - \left(\sum_{j=2}^{n-1} \frac{\gamma_j}{2}\right) v^1 + \sum_{j=2}^{n-1}\left(\lambda_j + \frac{\gamma_j}{2}\right) v^j + \left(\sum_{j=1}^{n-1} \gamma_j\right) w^1 \end{aligned}$$

$$= \left(1 - \sum_{j=2}^{n-1} \lambda_j - \sum_{j=1}^{n-1} \gamma_j\right) v^1 - \left(\sum_{j=2}^{n-1} \frac{\gamma_j}{2}\right) v^1 + \sum_{j=2}^{n-1} \left(\lambda_j + \frac{\gamma_j}{2}\right) v^j + \left(\sum_{j=1}^{n-1} \gamma_j\right) w^1$$
$$= \left(1 - \left(\sum_{j=1}^{n-1} \gamma_j + \sum_{j=2}^{n-1} \left(\lambda_j + \frac{1}{2}\gamma_j\right)\right)\right) v^1 + \left(\sum_{j=1}^{n-1} \gamma_j\right) w^1 + \sum_{j=2}^{n-1} \left(\lambda_j + \frac{\gamma_j}{2}\right) v^j.$$

Let $t := \sum_{j=1}^{n-1} \gamma_j + \sum_{j=2}^{n-1} (\lambda_j + \frac{1}{2}\gamma_j)$. Since $\lambda_1 < 1$, $t > 0$. Define now $t_1 := \frac{\sum_{j=1}^{n-1} \gamma_j}{t}$, $t_j := \frac{\lambda_j + \frac{\gamma_j}{2}}{t}$, $j \in \{2, \dots, n-1\}$. Then $t_j \geq 0$, $j \in [n-1]$, $\sum_{j \in [n-1]} t_j = 1$, and

$$x = (1-t)v^1 + t\left(t_1 w^1 + \sum_{j=2}^{n-1} t_j v^j\right).$$

Notice that if $t \in [0,1]$, then $x \in S^1$, and if $t = 1$, then $x \in \langle v^2, \dots, w^1 \rangle$, and x belongs to the $(n-2)$-face of S^1 opposite to the vertex v^1.

If $t > 1$, then

$$\frac{1}{t}x + \frac{t-1}{t}v^1 = t_1 w^1 + \sum_{j=2}^{n-1} t_j v^j \in \langle v^2, \dots, w^1 \rangle.$$

Let now $v := \frac{1}{t}x + \frac{t-1}{t}v^1$. Then $v \in \langle v^2, \dots, w^1 \rangle \subset S^1$ and $v^1 = \frac{t}{t-1}v - \frac{1}{t-1}x$. By (13),

$$x = \lambda_1 \left(\frac{t}{t-1}v - \frac{1}{t-1}x\right) + \sum_{j=2}^{n-1} \lambda_j v^j + \sum_{j=1}^{n-1} \gamma_j w^j,$$

and, subsequently,

$$\left(1 + \frac{\lambda_1}{t-1}\right)x = \frac{\lambda_1 t}{t-1}v + \sum_{j=2}^{n-1} \lambda_j v^j + \sum_{j=1}^{n-1} \gamma_j w^j,$$

which results in

$$x = \frac{\frac{\lambda_1 t}{t-1}}{1 + \frac{\lambda_1}{t-1}} v + \sum_{j=2}^{n-1} \frac{\lambda_j}{1 + \frac{\lambda_1}{t-1}} v^j + \sum_{j=1}^{n-1} \frac{\gamma_j}{1 + \frac{\lambda_1}{t-1}} w^j.$$

Remark that all coefficients at the vectors v, v^j, w^j on the right-hand side are nonnegative and

$$\frac{\frac{\lambda_1 t}{t-1}}{1 + \frac{\lambda_1}{t-1}} + \sum_{j=2}^{n-1} \frac{\lambda_j}{1 + \frac{\lambda_1}{t-1}} + \sum_{j=1}^{n-1} \frac{\gamma_j}{1 + \frac{\lambda_1}{t-1}} = \frac{\frac{\lambda_1 t}{t-1} + \sum_{j=2}^{n-1} \lambda_j + \sum_{j=1}^{n-1} \gamma_j}{1 + \frac{\lambda_1}{t-1}} = \frac{\frac{\lambda_1 t}{t-1} + 1 - \lambda_1}{1 + \frac{\lambda_1}{t-1}} = 1,$$

and hence $x \in \operatorname{conv}\{v^2, \dots, w^{n-1}\}$ (if $t > 1$).

We have just proved that

$$\operatorname{conv}\{v^1, \dots, w^{n-1}\} = S^1 \cup \operatorname{conv}\{v^2, \dots, w^{n-1}\}.$$

Proceeding similarly, we get $\mathrm{conv}\{v^2,\dots,w^{n-1}\} = S^2 \cup \mathrm{conv}\{v^3,\dots,w^{n-1}\}$, then $\mathrm{conv}\{v^3,\dots,w^{n-1}\} = S^3 \cup \mathrm{conv}\{v^4,\dots,w^{n-1}\}$, $\mathrm{conv}\{v^4,\dots,w^{n-1}\} = S^4 \cup \mathrm{conv}\{v^5,\dots,w^{n-1}\},\dots$ until we meet the set $\mathrm{conv}\{v^{n-1},\dots,w^{n-1}\} = S^{n-1}$. Decomposition (11) follows. □

Proof of Lemma 11 Since S' is an $(n-1)$-simplex, $\langle V(\sigma)\cup\{r\}\rangle$ is also an $(n-1)$-simplex for any $\sigma \in T$. Indeed, the inclusion $\sigma \in T$ implies that vertices $V(\sigma)$ are affinely independent, and from this and due to the assumption that S' is an $(n-1)$-simplex, we can conclude that the vectors $V(\sigma)\cup\{r\}$ are affinely independent, which implies that T' is a collection of $(n-1)$-simplices.

Each point $x \in S'$ is uniquely represented as $x = \sum_{i\in[n-1]} \lambda_i p^i + \gamma r$ with nonnegative λs and γ that sum up to 1. Therefore, $x = \gamma r + (1-\gamma)\sum_{i\in[n-1]} \frac{\lambda_i}{\sum_{j\in[n-1]}\lambda_j} p^i$. Let $p := \sum_{i\in[n-1]} \frac{\lambda_i}{\sum_{j\in[n-1]}\lambda_j} p^i \in \langle p^1,\dots,p^n\rangle$. Hence, $p \in \sigma$ for some $\sigma \in T$, and $\gamma \in [0,1]$, and according to the fact that $\gamma \in [0,1]$, $x = \gamma r + (1-\gamma)p \in \sigma' \in T'$ for some $\sigma' \in T'$. Observe that for any $\sigma_1', \sigma_2' \in T'$, there exist $\sigma_1, \sigma_2 \in T$: $\sigma_1' = \langle V(\sigma_1)\cup\{r\}\rangle$, $\sigma_2' = \langle V(\sigma_2)\cup\{r\}\rangle$. Thus, $\sigma_1' \cap \sigma_2' = \langle (V(\sigma_1)\cap V(\sigma_2))\cup\{r\}\rangle$, and since $\sigma_1\cap\sigma_2$ (if nonempty) is a common face of simplices σ_1, σ_2 and due to the fact that r and vertices of $V(\sigma_1\cap\sigma_2)$ are affinely independent, the set $\langle (V(\sigma_1)\cap V(\sigma_2))\cup\{r\}\rangle$ is a face of both σ_1' and σ_2'. If $\sigma_1\cap\sigma_2 = \emptyset$, then $\sigma_1'\cap\sigma_2' = \langle V(\sigma_1)\cup\{r\}\rangle \cap \langle V(\sigma_2)\cup\{r\}\rangle = \langle r\rangle$. We conclude that T' is a triangulation of S'. Now, because each $\sigma' \in T'$ is of the form $\sigma' = \langle h_1,\dots,h_{n-1},r\rangle$ for some $\langle h_1,\dots,h_{n-1}\rangle \in T$, we get that its nonzero faces are of the form $\langle \{h_i : i\in A\}\cup\{r\}\rangle$, where $\emptyset \neq A \subset [n-1]$, or $\langle \{h_i : i \in A\}\rangle$, $A \subset [n-1]$, $\#A \geq 2$. The assertion follows. □

Proof of Lemma 12 Let S^i, $i\in[n-1]$, be determined by (9). Put $T^1 := T$ and define

$$\overline{T}^1 := \{\langle V(\sigma)\cup\{w^1\}\rangle : \sigma \in T^1\}.$$

By Lemma 11, $\overline{T}^1$ is a triangulation of S^1. Now, recursively define for $i \in [n-1]$, $i \geq 2$,

$$\begin{aligned} T^i &:= \{\sigma \cap \langle v^i,\dots,w^{i-1}\rangle : \sigma \in \overline{T}^{i-1}, \sigma\cap\langle v^i,\dots,w^{i-1}\rangle \text{ is an } (n-2)\text{-face of } \sigma\},\\ \overline{T}^i &:= \{\langle V(\sigma)\cup\{w^i\}\rangle : \sigma\in T^i\}. \end{aligned}$$

For each $i\in[n-1]$, the collection T^i is a triangulation of $\langle v^i,\dots,w^{i-1}\rangle$ (see Section 2), and by Lemma 11, $\overline{T}^i$ is a triangulation of S^i.

If $F \subset \langle v^i,\dots,v^{n-1}\rangle$ is an $(n-i-1)$-face of some $\sigma\in\overline{T}^1$, $i\in[n-1]$, then by Lemma 11, $\langle v^i,\dots,v^{n-1},w^1\rangle$ is an $(n-i)$-face of some simplex in $\overline{T}^1$. Again by Lemma 11, $\langle v^i,\dots,v^{n-1}, w^1,w^2\rangle$ is an $(n-i+1)$-face of some simplex in $\overline{T}^2$. Repeating the reasoning, we obtain that $\langle v^i,\dots,w^{i-1}\rangle$ is an $(n-2)$-face of a simplex in $\overline{T}^{i-1}$, and hence $\langle v^i,\dots,w^i\rangle \in \overline{T}^i$.

Observe that $(n-2)$-faces of a simplex $\sigma_1 \in \overline{T}^1$ contained in $\langle v^2,\dots,w^1\rangle$ are of the form $\langle V(F_1)\cup\{w^1\}\rangle$, where $F_1 \subset \langle v^2,\dots,v^{n-1}\rangle$ is an $(n-3)$-face of σ_1. Thus, simplices in $\overline{T}^2$ are of the form $\langle V(F_1)\cup\{w^1,w^2\}\rangle$, where $F_1\subset\langle v^2,\dots,v^{n-1}\rangle$ is an $(n-3)$-face of some $\sigma_1\in\overline{T}^1$. Further, $(n-2)$-faces contained in $\langle v^3,\dots,w^2\rangle$ of a simplex $\sigma_2 \in \overline{T}^2$ are of the form $\langle V(F_2)\cup\{w^2\}\rangle$, where $F_2\subset\langle v^3,\dots,w^1\rangle$ is an $(n-3)$-face of σ_2. But $w^1 \in V(\sigma_2)$, so $w^1\in F_2$. Now, since $F_2\subset\langle v^3,\dots,w^1\rangle$ is an $(n-3)$-face of σ_2 and $\sigma_2 = \langle V(F_1)\cup\{w^1,w^2\}\rangle$, where $F_1 \subset \langle v^2,\dots,v^{n-1}\rangle$ is an $(n-3)$-face of some $\sigma_1\in\overline{T}^1$, we conclude that simplices in $\overline{T}^3$ are of the form $\langle V(F)\cup\{w_1,w_2,w_3\}\rangle$, where $F\subset\langle v^3,\dots,v^{n-1}\rangle$ is an $(n-4)$-face of a simplex $\sigma \in \overline{T}^1$. Continuing that way, we obtain that, for $i \in [n-2]$,

$$\overline{T}^i = \left\{\left\langle V(F) \cup \left\{w^1, \dots, w^i\right\}\right\rangle : F \subset \left\langle v^i, \dots, v^{n-1}\right\rangle \text{ is an } (n-i-1)\text{-face of } \sigma, \sigma \in \overline{T}^1\right\}$$

and

$$\overline{T}^{n-1} = \left\{\left\langle v^{n-1}, \dots, w^{n-1}\right\rangle\right\} = \left\{S^{n-1}\right\}.$$

Since $\overline{T}^1$ is a triangulation of S^1, it is clear that if $\sigma \in \overline{T}^i$ and $\sigma' \in \overline{T}^{i'}$, $i, i' \in [n-1]$, then $\sigma \cap \sigma'$ is a common face of σ, σ'.

Now, taking e^i, $i \in [n-1]$, in place of w^i and observing that we can treat $\operatorname{conv}\{v^1, \dots, v^{n-1}, e^1, \dots, e^{n-1}\}$ symmetrically to $\operatorname{conv}\{v^1, \dots, v^{n-1}, w^1, \dots, w^{n-1}\}$, we obtain a collection C of $(n-1)$-simplices contained in $\operatorname{conv}\{v^1, \dots, v^{n-1}, e^1, \dots, e^{n-1}\}$ with properties 1 and 2 mentioned in the definition of triangulation of a simplex and whose union is $\operatorname{conv}\{v^1, \dots, v^{n-1}, e^1, \dots, e^{n-1}\}$. By the construction, each $\sigma \in T$ is a common face of a simplex in the family $\overline{T} := \bigcup_{i \in [n-1]} \overline{T}^i$ and a simplex in C. Moreover, by symmetry, there is exactly one simplex in C whose face is $\langle e^1, \dots, e^{n-1}\rangle$. Observe that the family of simplices $T' := C \cup \overline{T} \cup \{\langle w^1, \dots, w^{n-1}, e^n\rangle\}$ is the desired triangulation of Δ^n.

The above construction ensures the correctness of (12) and the claim following it. □

Competing interests
The author declares that he has no competing interests.

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Endnotes

[a] In the last section of the paper, we present an algorithm for finding an (approximate) zero of an excess demand mapping; the other algorithms may be derived from proofs of Theorems 8 and 13, but we leave the details for the reader. A comprehensive review of existing algorithms is presented in [3]. We would like to stress that we just show some novel ways leading to the existing/known results. These ways have a common factor, Lemma 2.

[b] The K_m-triangulation is also called the $K_2(m)$-triangulation [3], p.64.

[c] The generalization consists in allowing for a wider class of triangulations of the simplex; the only triangulation considered in [1] was our K_m-triangulation.

[d] To simplify the notation, $l(\sigma) := l(V(\sigma))$, $\sigma \in T$, or σ is a face of a simplex in T. The proof below is an adaptation of the proof of Lemma 1 in [1].

[e] The method of construction of the sequence is similar to that used in the proof of the correctness of the Scarf algorithm; see [3], p.68.

[f] In [6], the 'hemicontinuity' is used in the sense of our 'semicontinuity'.

[g] Let us recall that $\boldsymbol{\alpha}^S(p)$ denotes the vector of the barycentric coordinates of $p \in S$ in the simplex S.

[h] We introduce ε_1 only to enable literal usage of passages of the proof of Lemma 6.

[i] So, from the formal point of view, the family of simplices $S^1, \dots, S^{n-1}$ is a triangulation the polytope $\operatorname{conv}\{v^1, \dots, v^n, w^1, \dots, w^n\}$. For a definition of triangulation of a polytope, see, *e.g.*, Definition 1.4.3 in [3].

[j] We describe this procedure in short: we 'start' from a simplex in T' whose face is $\boldsymbol{\sigma}$ and which has a vertex contained in $\{e^1, \dots, e^{n-1}\}$; call this simplex $\boldsymbol{\sigma}_1'$, then choose $\boldsymbol{\sigma}_2'$ to be the simplex adjacent to $\boldsymbol{\sigma}_1'$ that shares the face $\boldsymbol{\sigma}$ with $\boldsymbol{\sigma}_1'$. Next, we use the same rule for rejection of a vertex to get the next adjacent simplex of the sequence as in the proof of Lemma 2. At each step, $[n-1] \subset l(\boldsymbol{\sigma}_i')$. We can continue the procedure until we meet the first simplex $\boldsymbol{\sigma}_{J'}' \in T'$ that possesses a vertex in $\{e^1, \dots, e^{n-1}\}$. The constructed sequence $\boldsymbol{\sigma}_j'$, $j \in [J']$, is unique and has no simplex common with the sequence $\boldsymbol{\sigma}_j$, $j \in [J]$. Moreover, if we start from $\boldsymbol{\sigma}_{J'}'$, then the procedure leads us back to $\boldsymbol{\sigma}_1'$. See the proof of Lemma 2 for details.

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