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On the Keller limit and generalization

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Abstract

Let c be any real number and let

$$u_n(c) = (n+1)\left(1+\frac{1}{n+c}\right)^{n+c} - n\left(1+\frac{1}{n+c-1}\right)^{n+c-1} - e.$$

In this note, we establish an integral expression of $u_n(c)$, which provides a direct proof of Theorem 1 in (Mortici and Jang in Filomat 7:1535-1539, 2015).

MSC: 35B05; 35B10

Keywords: Keller's limit; constant e; integral expression

1 Introduction motivation

The limit

$$\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}}\right) = e$$

is well known in the literature as the Keller's limit, see [2]. Such a limit is very useful in many mathematical contexts and contributes as a tool for establishing some interesting inequalities [3–6].

In the recent paper [1], Mortici *et al.* have constructed a new proof of the limit and have discovered the following new results which generalize the Keller limit.

Theorem 1 *Let c be any real number and let*

$$u_n(c) = (n+1)\left(1+\frac{1}{n+c}\right)^{n+c} - n\left(1+\frac{1}{n+c-1}\right)^{n+c-1} - e.$$

Then

$$\lim_{n\to\infty} u_n(c) = 0, \tag{1.1}$$

$$\lim_{n\to\infty} n^2 u_n(c) = \frac{e}{24}(1-12c), \tag{1.2}$$

$$\lim_{n\to\infty} n^3 u_n\left(\frac{1}{12}\right) = \frac{5e}{144}. \tag{1.3}$$

The proof of Theorem 1 given in [1] is based on the following double inequality for every x in $0 < x \le 1$:

$$a(x) < (1+x)^{1/x} < b(x),$$

where

$$a(x) = e - \frac{e}{2x} + \frac{11ex^2}{24} - \frac{21ex^3}{48} + \frac{2{,}447ex^4}{5{,}760} - \frac{959ex^5}{2{,}304}$$

and

$$b(x) = a(x) + \frac{959ex^5}{2{,}304}.$$

But, this proof has a major objection, namely, for the reader it is very difficult to observe the behavior of $u_n(c)$ as $n \to \infty$.

In this note, we will establish an integral expression of $u_n(c)$, which tells us that Theorem 1 is a natural result.

2 Main results

To establish an integral expression of $u_n(c)$, we first recall the following result we obtained in [7].

Theorem 2 *Let* $h(s) = \frac{\sin(\pi s)}{\pi} s^s (1-s)^{1-s}$, $0 \le s \le 1$. *Then for every* $x > 0$, *we have*

$$\left(1 + \frac{1}{x}\right)^x = e\left(1 - \sum_{j=1}^{\infty} \frac{b_j}{(1+x)^j}\right), \tag{2.1}$$

where

$$b_1 = \frac{1}{2}, \tag{2.2}$$

$$b_j = \frac{1}{e}\int_0^1 s^{j-2} h(s)\, ds \quad (j = 2, 3, \ldots). \tag{2.3}$$

In [8] (see also [9, 10]) Yang has proved that $b_2 = \frac{1}{24}$, $b_3 = \frac{1}{48}$.

Hence

$$\int_0^1 h(s)\, ds = \frac{e}{24}, \tag{2.4}$$

$$\int_0^1 s h(s)\, ds = \frac{e}{48}. \tag{2.5}$$

Now, we establish an integral expression of $u_n(c)$. Equation (2.1) implies the following results:

$$\left(1 + \frac{1}{n+c}\right)^{n+c} = e\left(1 - \sum_{j=1}^{\infty} \frac{b_j}{(1+n+c)^j}\right), \tag{2.6}$$

$$\left(1+\frac{1}{n+c-1}\right)^{n+c-1}=e\left(1-\sum_{j=1}^{\infty}\frac{b_j}{(n+c)^j}\right). \tag{2.7}$$

Hence by (2.2), (2.3), (2.6), and (2.7), we have

$$\begin{aligned} u_n(c) &= \frac{e}{2}\left(\frac{n}{n+c}-\frac{n+1}{1+n+c}\right)+\int_0^1 h(s)\sum_{j=2}^{\infty}\frac{ns^{j-2}}{(n+c)^j}\,ds \\ &\quad -\int_0^1 h(s)\sum_{j=2}^{\infty}\frac{(n+1)s^{j-2}}{(1+n+c)^j}\,ds. \end{aligned} \tag{2.8}$$

Note that

$$\frac{e}{2}=\int_0^1 12h(s)\,ds, \tag{2.9}$$

$$\sum_{j=2}^{\infty}\frac{ns^{j-2}}{(n+c)^j}=\frac{1}{(n+c)(n+c-s)}, \tag{2.10}$$

$$\sum_{j=2}^{\infty}\frac{(n+1)s^{j-2}}{(1+n+c)^j}=\frac{1}{(1+n+c)(1+n+c-s)}. \tag{2.11}$$

Therefore, from (2.8)-(2.11), we obtain the desired result:

$$u_n(c)=\int_0^1 h(s)\frac{(1-12c)n^2+(1-12c+24cs-24c^2)n+K}{(n+c)(1+n+c)(n+c-s)(1+n+c-s)}\,ds, \tag{2.12}$$

where

$$K=s^2-(1+2c)s+c+c^2.$$

From (2.12), we get immediately

$$\begin{aligned} \lim_{n\to\infty} u_n(c) &= 0, \\ \lim_{n\to\infty} n^2u_n(c) &= \int_0^1 (1-12c)h(s)\,ds \\ &= \frac{e}{24}(1-12c), \\ \lim_{n\to\infty} n^3u_n\left(\frac{1}{12}\right) &= \int_0^1\left(2s-\frac{1}{6}\right)h(s)\,ds \\ &= \frac{e}{24}-\frac{1}{6}\times\frac{e}{24} \\ &= \frac{5e}{144}. \end{aligned}$$

3 Conclusions

We have established an integral expression of $u_n(c)$, which provides a direct proof of Theorem 1 in [1] and tell us that Theorem 1 is a natural result. We believe that the expression will lead to a significant contribution toward the study of Keller's limit.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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