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New criteria for $\mathcal{H}$-tensors and an application

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Abstract

Some new criteria for $\mathcal{H}$-tensors are obtained. As an application, some sufficient conditions of the positive definiteness for an even-order real symmetric tensor are given. The advantages of the results obtained are illustrated by numerical examples.

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Keywords: $\mathcal{H}$-tensor; symmetric tensor; positive definiteness; irreducible; nonzero elements chain

1 Introduction

Let $\mathbb{C}(\mathbb{R})$ be the complex (real) field and $N = \{1, 2, \dots, n\}$. We call $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ a complex (real) order m dimension n tensor, if

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{C}(\mathbb{R}),$$

where $i_j = 1, \dots, n$ for $j = 1, \dots, m$ [1, 2]. A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is called symmetric [3], if

$$a_{i_1 i_2 \cdots i_m} = a_{\pi(i_1 i_2 \cdots i_m)}, \quad \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. Furthermore, an order m dimension n tensor $\mathcal{I} = (\delta_{i_1 i_2 \cdots i_m})$ is called the unit tensor [4], if its entries

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an order m dimension n complex tensor. If there exist a complex number λ and a nonzero complex vector $x = (x_1, x_2, \dots, x_n)^T$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then we call λ an eigenvalue of $\mathcal{A}$ and x the eigenvector of $\mathcal{A}$ associated with λ [5–7], $\mathcal{A}x^{m-1}$, and $\lambda x^{[m-1]}$ are vectors, whose ith components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \dots, i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$\left(x^{[m-1]}\right)_i = x_i^{m-1},$$

respectively. If the eigenvalue λ and the eigenvector x are real, then λ is called an H-eigenvalue of $\mathcal{A}$ and x is its corresponding H-eigenvector [1].

Throughout this paper, we will use the following definitions.

Definition 1 [8] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a tensor of order m dimension n. $\mathcal{A}$ is called a diagonally dominant tensor if

$$|a_{ii\cdots i}| \geq \sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|, \quad \forall i \in N. \tag{1}$$

If all inequalities in (1) hold, then we call $\mathcal{A}$ a strictly diagonally dominant tensor.

Definition 2 [9] Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an order m dimension n complex tensor. $\mathcal{A}$ is called an $\mathcal{H}$-tensor if there is a positive vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ such that

$$|a_{ii\cdots i}| x_i^{m-1} > \sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| x_{i_2} \cdots x_{i_m}, \quad i = 1, 2, \ldots, n.$$

Definition 3 [10] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a tensor of order m dimension n, $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$. Denote

$$\mathcal{B} = (b_{i_1 \cdots i_m}) = \mathcal{A} X^{m-1}, \qquad b_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}, \quad i_j \in N, j \in N,$$

we call $\mathcal{B}$ the product of the tensor $\mathcal{A}$ and the matrix X.

Definition 4 [11] A complex tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ of order m dimension n is called reducible, if there exists a nonempty proper index subset $I \subset N$ such that

$$a_{i_1 i_2 \cdots i_m} = 0, \quad \forall i_1 \in I, \forall i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

Definition 5 Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an order m dimension n complex tensor, for $i, j \in N$ $(i \neq j)$, if there exist indices $k_1, k_2, \ldots, k_r$ with

$$\sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{k_s i_2\cdots i_m}=0 \\ k_{s+1} \in \{i_2,\ldots,i_m\}}} |a_{k_s i_2 \cdots i_m}| \neq 0, \quad s = 0, 1, \ldots, r,$$

where $k_0 = i$, $k_{r+1} = j$, we call there is a nonzero elements chain from i to j.

For an mth-degree homogeneous polynomial of n variables $f(x)$ can be denoted

$$f(x) = \sum_{i_1, i_2, \ldots, i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \tag{2}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The homogeneous polynomial $f(x)$ in (2) is equivalent to the tensor product of an order m dimensional n symmetric tensor $\mathcal{A}$ and x^m defined by

$$f(x) \equiv \mathcal{A}x^m = \sum_{i_1, i_2, \ldots, i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \tag{3}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ [1].

The positive definiteness of homogeneous polynomials have applications in automatic control [12, 13], polynomial problems [14], magnetic resonance imaging [15, 16], and spectral hypergraph theory [17, 18]. However, for $n > 3$ and $m > 4$, it is a hard problem to identify the positive definiteness of such a multivariate form. For solving this problem, Qi [1] pointed out that $f(x)$ defined by (3) is positive definite if and only if the real symmetric tensor $\mathcal{A}$ is positive definite, and Qi provided an eigenvalue method to verify the positive definiteness of $\mathcal{A}$ when m is even (see Theorem 1).

Theorem 1 [1] *Let $\mathcal{A}$ be an even-order real symmetric tensor, then $\mathcal{A}$ is positive definite if and only if all of its H-eigenvalues are positive.*

Although from Theorem 1 we can verify the positive definiteness of an even-order symmetric tensor $\mathcal{A}$ (the positive definiteness of the mth-degree homogeneous polynomial $f(x)$) by computing the H-eigenvalues of $\mathcal{A}$, it is difficult to compute all these H-eigenvalues when m and n are large. Recently, by introducing the definition of $\mathcal{H}$-tensor, Li *et al.* [9] provided a practical sufficient condition for identifying the positive definiteness of an even-order symmetric tensor (see Theorem 2).

Theorem 2 [9] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an even-order real symmetric tensor of order m dimension n with $a_{k \cdots k} > 0$ for all $k \in N$. If $\mathcal{A}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is positive definite.*

Theorem 2 provides a method for identifying the positive definiteness of an even-order symmetric tensor by determining $\mathcal{H}$-tensors. Thus the identification of $\mathcal{H}$-tensors is useful in checking the positive definiteness of homogeneous polynomials. In this paper, some new criteria for identifying $\mathcal{H}$-tensors are presented, which is easy to calculate since it only depends on the entries of tensors. As an application of these criteria, some sufficient conditions of the positive definiteness for an even-order real symmetric tensor are obtained. Numerical examples are also given to verify the corresponding results.

2 Main results

In this section, we give some new criteria for $\mathcal{H}$-tensors. First of all, we give some notation and lemmas.

Let S be a nonempty subset of N and let $N \setminus S$ be the complement of S in N. Given an order m dimension n complex tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, we denote

$$R_i(\mathcal{A}) = \sum_{\substack{i_2, \ldots, i_m \in N \\ \delta_{i i_2 \ldots i_m} = 0}} |a_{i i_2 \cdots i_m}| = \sum_{i_2, \ldots, i_m \in N} |a_{i i_2 \cdots i_m}| - |a_{i i \cdots i}|,$$

$$N_1 = \bigl\{ i \in N : 0 < |a_{i i \cdots i}| \leq R_i(\mathcal{A}) \bigr\}, \qquad N_2 = \bigl\{ i \in N : |a_{i i \cdots i}| > R_i(\mathcal{A}) \bigr\},$$

$$s_i = \frac{|a_{i i \cdots i}|}{R_i(\mathcal{A})}, \qquad t_i = \frac{R_i(\mathcal{A})}{|a_{i i \cdots i}|}, \qquad r = \max \Bigl\{ \max_{i \in N_1} s_i, \max_{i \in N_2} t_i \Bigr\},$$

$$S^{m-1} = \{i_2 i_3 \cdots i_m : i_j \in S, j = 2, 3, \ldots, m\},$$
$$N^{m-1} \setminus S^{m-1} = \{i_2 i_3 \cdots i_m : i_2 i_3 \cdots i_m \in N^{m-1} \text{ and } i_2 i_3 \cdots i_m \notin S^{m-1}\}.$$

It is obvious that if $N_1 = \emptyset$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor. It is known that, for an $\mathcal{H}$-tensor $\mathcal{A}$, $N_2 \neq \emptyset$ [9]. So we always assume that both N_1 and N_2 are not empty. Otherwise, we assume that $\mathcal{A}$ satisfies: $a_{ii\cdots i} \neq 0$, $R_i(\mathcal{A}) \neq 0$, $\forall i \in N$.

Lemma 1 [8] *If $\mathcal{A}$ is a strictly diagonally dominant tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.*

Lemma 2 [10] *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be a complex tensor of order m dimension n. If there exists a positive diagonal matrix X such that $\mathcal{A}X^{m-1}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.*

Lemma 3 [9] *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be a complex tensor of order m dimension n. If $\mathcal{A}$ is irreducible,*

$$|a_{i\cdots i}| \geq R_i(\mathcal{A}), \quad \forall i \in N,$$

and strictly inequality holds for at least one i, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Lemma 4 *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be an order m dimension n complex tensor. If*
(i) $|a_{ii\cdots i}| \geq R_i(\mathcal{A})$, $\forall i \in N$,
(ii) $J(\mathcal{A}) = \{i \in N : |a_{ii\cdots i}| > R_i(\mathcal{A})\} \neq \emptyset$,
(iii) *for any $i \notin J(\mathcal{A})$, there exists a nonzero elements chain from i to j, such that $j \in J(\mathcal{A})$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.*

Proof It is evident that the result holds with $J(\mathcal{A}) = N$. Next, we assume that $J(\mathcal{A}) \neq N$. Suppose $J(\mathcal{A}) = \{k+1, \ldots, n\}$, $N \setminus J(\mathcal{A}) = \{1, \ldots, k\}$, $1 \leq k < n$. By hypothesis,

$$|a_{kk\cdots k}| = R_k(\mathcal{A}).$$

By the condition (iii), there exist indices $k_1, k_2, \ldots, k_r$ such that

$$\sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{k_s i_2\cdots i_m} = 0 \\ k_{s+1} \in \{i_2,\ldots,i_m\}}} |a_{k_s i_2\cdots i_m}| \neq 0, \quad s = 0, 1, \ldots, r,$$

where $k_0 = k$, $k_{r+1} = j$, $j \in J(\mathcal{A})$. Then

$$\sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{k_r i_2\cdots i_m} = 0 \\ j \in \{i_2,\ldots,i_m\}}} |a_{k_r i_2\cdots i_m}| \neq 0.$$

Further, without loss of generality, we assume that $k_1, \ldots, k_r \notin J(\mathcal{A})$, that is, $1 \leq k_1, \ldots, k_r < k$. From $j \in J(\mathcal{A})$, we have $|a_{jj\cdots j}| > R_j(\mathcal{A})$, so there exists $0 < \varepsilon < 1$ such that $\varepsilon |a_{jj\cdots j}| > R_j(\mathcal{A})$.

Construct a positive diagonal matrix $X_{k_r} = \operatorname{diag}(x_1, \ldots, x_n)$, where

$$x_i = \begin{cases} \varepsilon^{\frac{1}{m-1}}, & i = j, \\ 1, & i \neq j. \end{cases}$$

Let $\mathcal{A}_{k_r} = [a^{(k_r)}_{i_1 i_2 \cdots i_m}] = \mathcal{A} X^{m-1}_{k_r}$. Then

$$\begin{aligned}
&\left|a^{(k_r)}_{ii\cdots i}\right| = |a_{ii\cdots i}| = R_i(\mathcal{A}) \geq R_i(\mathcal{A}_{k_r}), \quad 1 \leq i \leq k, i \neq k_r, \\
&\left|a^{(k_r)}_{k_r k_r \cdots k_r}\right| = |a_{k_r k_r \cdots k_r}| = R_{k_r}(\mathcal{A}) > R_{k_r}(\mathcal{A}_{k_r}), \\
&\left|a^{(k_r)}_{ii\cdots i}\right| = |a_{ii\cdots i}| > R_i(\mathcal{A}) \geq R_{k_r}(\mathcal{A}_{k_r}), \quad i \in J(\mathcal{A}), i \neq j, \\
&\left|a^{(k_r)}_{jj\cdots j}\right| = \varepsilon |a_{jj\cdots j}| > R_j(\mathcal{A}) \geq R_{k_r}(\mathcal{A}_{k_r}).
\end{aligned}$$

Obviously, $\mathcal{A}_{k_r}$ is also a diagonally dominant tensor, and $J(\mathcal{A}_{k_r}) = J(\mathcal{A}) \cup \{k_r\}$.

If $J(\mathcal{A}_{k_r}) = N$, then $\mathcal{A}_{k_r}$ is strictly diagonally dominant. By Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

If $N \setminus J(\mathcal{A}_{k_r}) \neq \emptyset$, then $\mathcal{A}_{k_r}$ also satisfies the conditions of the lemma, that is, for any $i \in N \setminus J(\mathcal{A}_{k_r})$, there exist indices $l_1, l_2, \ldots, l_s$, such that

$$\sum_{\substack{i_2,\ldots,i_m \in N, \\ \delta_{l_t i_2 \cdots i_m} = 0, \\ l_{t+1} \in \{i_2,\ldots,i_m\}}} |a_{l_t i_2 \cdots i_m}| \neq 0, \quad t = 0, 1, \ldots, s,$$

where $l_0 = i$, $l_{s+1} = j$, $j \in J(\mathcal{A}_{k_r})$. Then

$$\sum_{\substack{i_2,\ldots,i_m \in N, \\ \delta_{l_s i_2 \cdots i_m} = 0, \\ j \in \{i_2,\ldots,i_m\}}} |a_{l_s i_2 \cdots i_m}| \neq 0.$$

Similar to the above argument, for $\mathcal{A}_{k_r}$, there exists a positive diagonal matrix X_{l_s} such that $\mathcal{A}_{l_s} = \mathcal{A}_{k_r} X^{m-1}_{l_s}$ is diagonally dominant, and $J(\mathcal{A}_{l_s}) = J(\mathcal{A}_{k_r}) \cup \{l_s\}$.

If $J(\mathcal{A}_{l_s}) = N$, then $\mathcal{A}_{l_s}$ is strictly diagonally dominant. By Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

If $N \setminus J(\mathcal{A}_{l_s}) \neq \emptyset$, then $\mathcal{A}_{l_s}$ also satisfies the conditions of the lemma. Similarly as the above argument, for $\mathcal{A}_{l_s}$, there exist at most k positive diagonal matrices $X_{k_r}, X_{l_s}, \ldots, X_{p_q}$ such that $\mathcal{B}$ is strictly diagonally dominant, where $\mathcal{B} = \mathcal{A}(X_{k_r} X_{l_s} \cdots X_{p_q})^{m-1}$. Hence, $\mathcal{B}$ is an $\mathcal{H}$-tensor, and by Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor. The proof is completed. □

Theorem 3 *Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an order m dimension n complex tensor. If*

$$|a_{ii\cdots i}| s_i > r \sum_{\substack{i_2 \cdots i_m \in N^{m-1} \setminus N_2^{m-1} \\ \delta_{ii_2 \ldots i_m} = 0}} |a_{ii_2 \cdots i_m}| + \sum_{i_2 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,\ldots,i_m\}} \{t_j\} |a_{ii_2 \cdots i_m}|, \quad \forall i \in N_1, \tag{4}$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Let

$$M_i = \frac{|a_{ii\cdots i}| s_i - r \sum_{\substack{i_2 \cdots i_m \in N^{m-1} \setminus N_2^{m-1} \\ \delta_{ii_2 \ldots i_m} = 0}} |a_{ii_2 \cdots i_m}| - \sum_{i_2 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,\ldots,i_m\}} \{t_j\} |a_{ii_2 \cdots i_m}|}{\sum_{i_2 \cdots i_m \in N_2^{m-1}} |a_{ii_2 \cdots i_m}|},$$

$$\forall i \in N_1. \tag{5}$$

If $\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|=0$, we denote $M_i=+\infty$. From inequality (4), we obtain $M_i>0$ ($i\in N_1$). Hence, there exists a positive number $\varepsilon>0$ such that

$$0<\varepsilon<\min\Bigl\{\min_{i\in N_1}M_i, 1-\max_{i\in N_2}t_i\Bigr\}. \tag{6}$$

Let the matrix $X=\operatorname{diag}(x_1,x_2,\ldots,x_n)$, where

$$x_i=\begin{cases}(s_i)^{\frac{1}{m-1}}, & i\in N_1,\\ (\varepsilon+t_i)^{\frac{1}{m-1}}, & i\in N_2.\end{cases}$$

By inequality (6), we have $(\varepsilon+t_i)^{\frac{1}{m-1}}<1$ ($i\in N_2$). As $\varepsilon\neq+\infty$, so $x_i\neq+\infty$, which implies that X is a diagonal matrix with positive entries. Let $\mathcal{B}=(b_{i_1i_2\cdots i_m})=\mathcal{A}X^{m-1}$. Next, we will prove that $\mathcal{B}$ is strictly diagonally dominant.

For all $i\in N_1$, if $\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|=0$, then by inequality (4), we have

$$\begin{aligned}R_i(\mathcal{B}) &= \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|b_{ii_2\cdots i_m}|+\sum_{i_2\cdots i_m\in N_2^{m-1}}|b_{ii_2\cdots i_m}|\\ &= \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}+\sum_{i_2\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}\\ &\leq r\sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|<|a_{ii\cdots i}|s_i=|b_{ii\cdots i}|.\end{aligned} \tag{7}$$

If $\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|\neq 0$, then by inequalities (5) and (6), we obtain

$$\begin{aligned}R_i(\mathcal{B}) &= \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}+\sum_{i_2\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}\\ &= \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}\\ &\quad+\sum_{i_2\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|(\varepsilon+t_{i_2})^{\frac{1}{m-1}}\cdots(\varepsilon+t_{i_m})^{\frac{1}{m-1}}\\ &\leq r\sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|+\sum_{i_2\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|\Bigl(\varepsilon+\max_{j\in\{i_2,\ldots,i_m\}}\{t_j\}\Bigr)\\ &= r\sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|+\sum_{i_2\cdots i_m\in N_2^{m-1}}\max_{j\in\{i_2,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}|\\ &\quad+\varepsilon\sum_{i_2\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|\\ &< |a_{ii\cdots i}|s_i=|b_{ii\cdots i}|.\end{aligned} \tag{8}$$

Now, we consider $i \in N_2$. Since $|a_{ii\cdots i}| > R_i(\mathcal{A})$, we have

$$|a_{ii\cdots i}| - \sum_{\substack{i_2 i_3\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| > 0 \tag{9}$$

and

$$r \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\ldots,i_m\}} \{t_j\} |a_{ii_2\cdots i_m}| - R_i(\mathcal{A}) \le 0. \tag{10}$$

By inequalities (9), (10), and $\varepsilon > 0$, we get

$$\varepsilon > \frac{r \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}| - R_i(\mathcal{A})}{|a_{ii\cdots i}| - \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|}. \tag{11}$$

From inequality (11), for any $i \in N_2$, we obtain

$$\begin{aligned} |b_{ii\cdots i}| - R_i(\mathcal{B}) &= |a_{ii\cdots i}|(\varepsilon + t_i) - \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} \\ &\quad - \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|(\varepsilon + t_{i_2})^{\frac{1}{m-1}} \cdots (\varepsilon + t_{i_m})^{\frac{1}{m-1}} \\ &\ge |a_{ii\cdots i}|(\varepsilon + t_i) - r \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}} |a_{ii_2\cdots i_m}| \\ &\quad - \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \Big(\varepsilon + \max_{j\in\{i_2,\ldots,i_m\}}\{t_j\}\Big) \\ &= \varepsilon \Bigg(|a_{ii\cdots i}| - \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \Bigg) + R_i(\mathcal{A}) \\ &\quad - r \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}} |a_{ii_2\cdots i_m}| - \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}| \\ &> 0. \end{aligned} \tag{12}$$

Therefore, from inequalities (7), (8), and (12), we obtain $|b_{ii\cdots i}| > R_i(\mathcal{B})$ for all $i \in N$, that is, $\mathcal{B}$ is strictly diagonally dominant. By Lemma 1 and Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor. The proof is completed. □

Theorem 4 *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be an order m dimension n complex tensor. If $\mathcal{A}$ is irreducible and*

$$|a_{ii\cdots i}| s_i \ge r \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}|, \quad \forall i \in N_1, \tag{13}$$

and a strict inequality holds for at least one $i \in N_1$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Let the matrix $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$, where

$$x_i = \begin{cases} (s_i)^{\frac{1}{m-1}}, & i \in N_1, \\ (t_i)^{\frac{1}{m-1}}, & i \in N_2. \end{cases}$$

By the irreducibility of $\mathcal{A}$, we have $x_i \neq +\infty$, then X is a diagonal matrix with positive diagonal entries. Let $\mathcal{B} = [b_{i_1\cdots i_m}] = \mathcal{A}X^{m-1}$.

Adopting the same procedure as in the proof of Theorem 3, we can obtain $|b_{ii\cdots i}| \geq R_i(\mathcal{B})$ ($\forall i \in N$), and there exists at least an $i_0 \in N_1$ such that $|b_{i_0 i_0\cdots i_0}| > R_{i_0}(\mathcal{B})$.

On the other hand, since $\mathcal{A}$ is irreducible and so is $\mathcal{B}$. Then by Lemma 3, we see that $\mathcal{B}$ is an $\mathcal{H}$-tensor. By Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor. The proof is completed. □

Let

$$J_1 = \left\{ i \in N_1 : |a_{ii\cdots i}| s_i > r \sum_{\substack{i_2\cdots i_m \in N^{m-1} \setminus N_2^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,\ldots,i_m\}} \{t_j\} |a_{ii_2\cdots i_m}| \right\},$$

$$J_2 = \left\{ i \in N_2 : |a_{ii\cdots i}| t_i > r \sum_{i_2\cdots i_m \in N^{m-1} \setminus N_2^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} \max_{j \in \{i_2,\ldots,i_m\}} \{t_j\} |a_{ii_2\cdots i_m}| \right\}.$$

Theorem 5 *Let $\mathcal{A} = (a_{i_1\cdots i_m})$ be an order m dimension n complex tensor. If*

$$|a_{ii\cdots i}| s_i \geq r \sum_{\substack{i_2\cdots i_m \in N^{m-1} \setminus N_2^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,\ldots,i_m\}} \{t_j\} |a_{ii_2\cdots i_m}|, \tag{14}$$

$J_1 \cup J_2 \neq \emptyset$, and for $\forall i \in (N_1 \setminus J_1) \cup (N_2 \setminus J_2)$, there exists a nonzero elements chain from i to j such that $j \in J_1 \cup J_2$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Let the matrix $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$, where

$$x_i = \begin{cases} (s_i)^{\frac{1}{m-1}}, & i \in N_1, \\ (t_i)^{\frac{1}{m-1}}, & i \in N_2. \end{cases}$$

Obviously $x_i \neq +\infty$, then X is a diagonal matrix with positive diagonal entries. Let $\mathcal{B} = [b_{i_1\cdots i_m}] = \mathcal{A}X^{m-1}$. Similarly as in the proof of Theorem 3, we can obtain $|b_{ii\cdots i}| \geq R_i(\mathcal{B})$ ($\forall i \in N$). From $J_1 \cup J_2 \neq \emptyset$, there exists at least an $i_0 \in N$ such that $|b_{i_0 i_0\cdots i_0}| > R_{i_0}(\mathcal{B})$.

On the other hand, if $|b_{ii\cdots i}| = R_i(\mathcal{B})$, then $i \in (N_1 \setminus J_1) \cup (N_2 \setminus J_2)$, by the assumption, we know that there exists a nonzero elements chain from i to j of $\mathcal{A}$ such that $j \in J_1 \cup J_2$. Then there exists a nonzero elements chain from i to j of $\mathcal{B}$ with j satisfying $|b_{jj\cdots j}| > R_j(\mathcal{B})$.

Based on above analysis, we conclude that the tensor $\mathcal{B}$ satisfies the conditions of Lemma 4, so $\mathcal{B}$ is an $\mathcal{H}$-tensor. By Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor. The proof is completed. □

Theorem 6 *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ be a complex tensor of order m dimension n. If*

$$|a_{ii\cdots i}| s_i > r \sum_{\substack{i_2\cdots i_m \in N^{m-1} \setminus N_2^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|, \quad \forall i \in N_1 \tag{15}$$

and

$$\sum_{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}} |a_{ii_2\cdots i_m}| = 0, \quad \forall i \in N_2, \tag{16}$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof By inequality (15), for each $i \in N_1$, there exists a positive number $K_i > 1$, such that

$$|a_{ii\cdots i}|s_i > r \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| + \frac{1}{K_i}\left(\sum_{i_2\cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}|\right). \tag{17}$$

Let $K \equiv \max_{i\in N_1}\{K_i\}$. By inequality (17), we obtain

$$|a_{ii\cdots i}|s_i > r \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| + \frac{1}{K}\left(\sum_{i_2\cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}|\right), \quad \forall i \in N_1. \tag{18}$$

Since $|a_{ii\cdots i}| \le R_i(\mathcal{A})$ $(i \in N_1)$ and inequality (15), so

$$\sum_{i_2\cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}| > 0, \quad \forall i \in N_1. \tag{19}$$

For any $i \in N_1$, denote

$$T_i = \frac{|a_{ii\cdots i}|s_i - r\sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| - \frac{1}{K}(\sum_{i_2\cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}|)}{\sum_{i_2 i_3\cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}|}.$$

From inequalities (18) and (19), we have $T_i > 0$. Therefore there exists a positive number $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\left\{\min_{i\in N_1} T_i, 1 - \max_{i\in N_2}\frac{t_i}{K}\right\}.$$

Let the matrix $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$, where

$$x_i = \begin{cases} (s_i)^{\frac{1}{m-1}}, & i \in N_1, \\ (\varepsilon + \frac{t_i}{K})^{\frac{1}{m-1}}, & i \in N_2. \end{cases}$$

Mark $\mathcal{B} = \mathcal{A}X^{m-1}$. Similarly as in the proof of Theorem 3, we can prove that $\mathcal{B}$ is strictly diagonally dominant. By Lemma 1 and Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor. The proof is completed. □

There is no inclusion relation between the conditions of Theorem 3 and the conditions of Theorem 6. This can be seen from the following examples.

Example 1 Consider a tensor $\mathcal{A} = (a_{ijk})$ of order 3 dimension 3 defined as follows:

$$\begin{aligned}
&\mathcal{A} = \bigl[A(1,:,:), A(2,:,:), A(3,:,:)\bigr],\\
&A(1,:,:) = \begin{pmatrix} 15 & 1 & 0\\ 1 & 10 & 0\\ 1 & 1 & 10\end{pmatrix}, \qquad A(2,:,:) = \begin{pmatrix} 1 & 1 & 0\\ 0 & 12 & 0\\ 1 & 0 & 1\end{pmatrix},\\
&A(3,:,:) = \begin{pmatrix} 2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 15\end{pmatrix}.
\end{aligned}$$

Obviously,

$$\begin{aligned}
&|a_{111}| = 15, \qquad R_1(\mathcal{A}) = 24, \qquad |a_{222}| = 12,\\
&R_2(\mathcal{A}) = 4, \qquad |a_{333}| = 15, \qquad R_3(\mathcal{A}) = 5,
\end{aligned}$$

so $N_1 = \{1\}$, $N_2 = \{2,3\}$. By calculation, we have

$$s_1 = \frac{|a_{111}|}{R_1(\mathcal{A})} = \frac{15}{24}, \qquad t_2 = \frac{R_2(\mathcal{A})}{|a_{222}|} = \frac{1}{3}, \qquad t_3 = \frac{R_3(\mathcal{A})}{|a_{333}|} = \frac{1}{3}, \qquad r = \frac{15}{24}.$$

Since

$$\begin{aligned}
r\sum_{\substack{jk\in N^2\setminus N_2^2\\ \delta_{1jk}=0}} |a_{1jk}| + \sum_{jk\in N_2^2} \max_{l\in\{j,k\}}\{t_l\}|a_{1jk}| &= \frac{15}{24}(1+0+1+1) + \frac{1}{3}(0+1+10+10)\\
&= \frac{213}{24} < \frac{225}{24} = |a_{111}|s_1,
\end{aligned}$$

we know that $\mathcal{A}$ satisfies the conditions of Theorem 3, then $\mathcal{A}$ is an $\mathcal{H}$-tensor. But

$$\sum_{jk\in N^2\setminus N_2^2} |a_{2jk}| = 3 \neq 0, \qquad \sum_{jk\in N^2\setminus N_2^2} |a_{3jk}| = 2 \neq 0.$$

so $\mathcal{A}$ does not satisfy the conditions of Theorem 6.

Example 2 Consider a tensor $\mathcal{A} = (a_{ijk})$ of order 3 dimension 3 defined as follows:

$$\begin{aligned}
&\mathcal{A} = \bigl[A(1,:,:), A(2,:,:), A(3,:,:)\bigr],\\
&A(1,:,:) = \begin{pmatrix} 8 & 1 & 0\\ 1 & 10 & 0\\ 1 & 1 & 10\end{pmatrix}, \qquad A(2,:,:) = \begin{pmatrix} 0 & 0 & 0\\ 0 & 8 & 2\\ 0 & 1 & 1\end{pmatrix},\\
&A(3,:,:) = \begin{pmatrix} 0 & 0 & 0\\ 0 & 2.5 & 1.5\\ 0 & 1 & 10\end{pmatrix}.
\end{aligned}$$

Obviously,

$$|a_{111}| = 8, \qquad R_1(\mathcal{A}) = 24, \qquad |a_{222}| = 8,$$
$$R_2(\mathcal{A}) = 4, \qquad |a_{333}| = 10, \qquad R_3(\mathcal{A}) = 5,$$

so $N_1 = \{1\}$, $N_2 = \{2,3\}$. By calculation, we have

$$s_1 = \frac{|a_{111}|}{R_1(\mathcal{A})} = \frac{1}{3}, \qquad t_2 = \frac{R_2(\mathcal{A})}{|a_{222}|} = \frac{1}{2}, \qquad t_3 = \frac{R_3(\mathcal{A})}{|a_{333}|} = \frac{1}{2}, \qquad r = \frac{1}{2}.$$

Since

$$r \sum_{\substack{jk \in N^2 \setminus N_2^2 \\ \delta_{1jk}=0}} |a_{1jk}| = \frac{1}{2}(1+0+1+1) = \frac{3}{2} < \frac{8}{3} = |a_{111}| s_1$$

and

$$\sum_{jk \in N^2 \setminus N_2^2} |a_{2jk}| = 0, \qquad \sum_{jk \in N^2 \setminus N_2^2} |a_{3jk}| = 0,$$

we see that $\mathcal{A}$ satisfies the conditions of Theorem 6, then $\mathcal{A}$ is an $\mathcal{H}$-tensor. But

$$\begin{aligned} r \sum_{\substack{jk \in N^2 \setminus N_2^2 \\ \delta_{1jk}=0}} |a_{1jk}| + \sum_{jk \in N_2^2} \max_{l \in \{j,k\}} \{t_l\} |a_{1jk}| &= \frac{1}{2}(1+0+1+1) + \frac{1}{2}(0+1+10+10) \\ &= 12 > \frac{8}{3} = |a_{111}| s_1, \end{aligned}$$

so $\mathcal{A}$ does not satisfy the conditions of Theorem 3.

3 An application

In this section, based on the criteria for $\mathcal{H}$-tensors in Section 2, we present some criteria for identifying the positive definiteness of an even-order real symmetric tensor (the positive definiteness of a multivariate form).

From Theorems 2-6, we obtain easily the following result.

Theorem 7 *Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an even-order real symmetric tensor of order m dimension n with $a_{kk\cdots k} > 0$ for all $k \in N$. If $\mathcal{A}$ satisfies one of the following conditions, then $\mathcal{A}$ is positive definite*:

(i) *all the conditions of Theorem* 3;
(ii) *all the conditions of Theorem* 4;
(iii) *all the conditions of Theorem* 5;
(iv) *all the conditions of Theorem* 6.

Example 3 Let $f(x) = \mathcal{A}x^4 = 11x_1^4 + 18x_2^4 + 18x_3^4 + 12x_4^4 + 12x_1^2x_2x_3 - 24x_1x_2x_3x_4$ be a 4th-degree homogeneous polynomial. We can get an order 4 dimension 4 real symmetric tensor $\mathcal{A} = (a_{i_1i_2i_3i_4})$, where

$$a_{1111} = 11, \qquad a_{2222} = 18, \qquad a_{3333} = 18, \qquad a_{4444} = 12,$$

$$a_{1123} = a_{1132} = a_{1213} = a_{1312} = a_{1231} = a_{1321} = 1,$$

$$a_{2113} = a_{2131} = a_{2311} = a_{3112} = a_{3121} = a_{3211} = 1,$$

$$a_{1234} = a_{1243} = a_{1324} = a_{1342} = a_{1423} = a_{1432} = -1,$$

$$a_{2134} = a_{2143} = a_{2314} = a_{2341} = a_{2413} = a_{2431} = -1,$$

$$a_{3124} = a_{3142} = a_{3214} = a_{3241} = a_{3412} = a_{3421} = -1,$$

$$a_{4123} = a_{4132} = a_{4213} = a_{4231} = a_{4312} = a_{4321} = -1,$$

and other $a_{i_1i_2i_3i_4} = 0$. It can be verified that $\mathcal{A}$ satisfies all the conditions of Theorem 3. Thus, from Theorem 7, we see that $\mathcal{A}$ is positive definite, that is, $f(x)$ is positive definite.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors contributed equally to this work. Both authors read and approved the final manuscript.

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