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Conditions for one direction convexity and starlikeness

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Abstract
We investigate several sufficient conditions on a function to be convex in one direction or starlike in one direction.

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1 Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and denote by $\mathcal{A}$ the class of analytic functions in $\mathcal{H}$ that are normalized by $f(0) = 0 = f'(0) - 1$. Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ composed of functions that are univalent in $\mathbb{D}$.

We say that a function f is starlike in one direction if f it maps $|z| = r$ for every r near 1 onto a contour C that is cut by a straight-line passing through the origin in two and no more than two points. Robertson [1] found the following sufficient condition for starlikeness in one direction.

Lemma 1 *Let $f(z)$ be analytic in $|z| \leq r$, and $f(z) \neq 0$ in $0 < |z| \leq r$. Further, let $f(0) = 0$. Suppose that*

$$\int_0^{2\pi} \left| \Re \frac{zf'(z)}{f(z)} \right| \mathrm{d}\theta < 4\pi, \quad z = \rho e^{i\theta}, \text{ for every } \rho \leq r.$$

Then, for every $\rho \leq r$, $f(z)$ maps $|z| = \rho$ onto a curve that is starlike in one direction.

A function is said to be convex in one direction in $|z| < r$ $(r > 0)$ if the function maps $|z| = \rho < r$ for every ρ near r into a contour that may be cut by every straight-line parallel to this direction in no more than two points. It is known (see [1]) that if $f \in \mathcal{A}$ and $zf'(z)$ is starlike in one direction, then $f(z)$ is convex in one direction and belongs to $\mathcal{S}$. Therefore, we can obtain the following lemma (see also [2–4]).

Lemma 2 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic for $|z| \leq 1$ and $f'(z) \neq 0$ on $|z| = r < 1$. Suppose that*

$$\int_0^{2\pi} \left| 1 + \Re\left\{ \frac{zf''(z)}{f'(z)} \right\} \right| \mathrm{d}\theta < 4\pi, \quad z = re^{i\theta}, \textit{ for every } r < 1.$$

Then $f(z)$ is convex in one direction, and hence $f(z)$ is univalent in $|z| \le 1$.

We may refer to [5–7] for more sufficient conditions on analytic functions to be convex in one direction.

In the present paper, we investigate several sufficient conditions on functions in $\mathcal{A}$ to be convex in one direction using various methods. Also, we find sufficient conditions for starlikeness in one direction.

2 Main results

Theorem 1 *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\left| 1 + \Re\left\{ \frac{zf''(z)}{f'(z)} \right\} \right| < 2\Re\left\{ \frac{1+z}{1-z} \right\} \quad (z \in \mathbb{D}). \tag{1}$$

Then $f(z)$ is convex in one direction, and hence $f(z)$ is univalent in $\mathbb{D}$.

Proof Let $0 \le r < 1$. From hypothesis (1) we have

$$\begin{aligned} &\int_0^{2\pi} \left| 1 + \Re\left\{ \frac{zf''(z)}{f'(z)} \right\} \right| \mathrm{d}\theta \\ &< 2\int_{|z|=r} \left\{ \Re\left\{ \frac{1+z}{1-z} \right\} \right\} \mathrm{d}\theta \\ &= 2\int_0^{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2} \mathrm{d}\theta \\ &= 4\pi. \end{aligned}$$

Therefore, by Lemma 2, $f(z)$ is convex in one direction in $\mathbb{D}$. □

Example 1 Consider the function $f_1 : \mathbb{D} \to \mathbb{C}$ defined by $f_1(z) = z/(1-z)$. Then we have

$$1 + \frac{zf_1''(z)}{f_1'(z)} = \frac{1+z}{1-z} \quad (z \in \mathbb{D}).$$

Moreocer, we can easily check that condition (1) holds for the function f_1. Therefore, by Theorem 1 the function f_1 is convex in one direction and univalent in $\mathbb{D}$.

Theorem 2 *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| \le \left| \frac{1+z}{1-z} \right| \quad (z \in \mathbb{D}). \tag{2}$$

Then $f(z)$ is convex in one direction in $|z| < r_0 = 0.251652\cdots$, where r_0 is the root of the equation

$$2\pi\left(\frac{1+r}{1-r} \right) + 4\log\left(\frac{1+r}{1-r} \right) = 4\pi. \tag{3}$$

Proof Let $0 \le r < 1$. From inequality (2) we have

$$\begin{aligned}
&\int_0^{2\pi} \left|1 + \Re\left\{\frac{zf''(z)}{f'(z)}\right\}\right| \mathrm{d}\theta \\
&\quad \le \int_0^{2\pi} \left|1 + \frac{zf''(z)}{f'(z)}\right| \mathrm{d}\theta \\
&\quad \le \int_0^{2\pi} \left|\frac{1+z}{1-z}\right| \mathrm{d}\theta \\
&\quad \le \int_0^{2\pi} \left(\left|\frac{1-r^2}{1-2r\cos\theta + r^2}\right| + \left|\frac{2r\sin\theta}{1-2r\cos\theta + r^2}\right|\right) \mathrm{d}\theta \\
&\quad \le 2\pi\left(\frac{1+r}{1-r}\right) + 4\log\left(\frac{1+r}{1-r}\right).
\end{aligned}$$

Define the function $g : [0,1) \to \mathbb{R}$ by

$$g(r) = 2\pi\left(\frac{1+r}{1-r}\right) + 4\log\left(\frac{1+r}{1-r}\right).$$

Then $g(0) = 2\pi$ and $g(r) \to \infty$ as $r \to 1^-$. Also, we have that the function g is increasing on $[0,1)$ since

$$g'(r) = \frac{4\pi}{(1-r)^2} + \frac{8}{1-r^2} > 0$$

for all $r \in [0,1)$. Therefore, there exists a unique root r_0 in $[0,1)$ such that $g(r) = 4\pi$. Hence, we have

$$\int_0^{2\pi} \left|1 + \Re\left\{\frac{zf''(z)}{f'(z)}\right\}\right| \mathrm{d}\theta < 4\pi$$

for $|z| < r_0$. It follows from Lemma 2 that $f(z)$ is convex in one direction in $|z| < r_0$. $\square$

Theorem 3 *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\left|\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} - \frac{1}{2}\right| < \Re\left\{\frac{1+z}{1-z}\right\} + \frac{1}{2} \quad (z \in \mathbb{D}). \tag{4}$$

Then $f(z)$ is convex in one direction, and hence $f(z)$ is univalent in $\mathbb{D}$.

Proof Let $0 \le r < 1$. From hypothesis (4) we have

$$\begin{aligned}
&\int_{|z|=r} \left\{\left|\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| - \frac{1}{2}\right\} \mathrm{d}\theta \\
&\quad \le \int_{|z|=r} \left|\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} - \frac{1}{2}\right| \mathrm{d}\theta \\
&\quad < \int_{|z|=r} \left\{\Re\left\{\frac{1+z}{1-z}\right\} + \frac{1}{2}\right\} \mathrm{d}\theta \\
&\quad = 3\pi.
\end{aligned}$$

Therefore, we have

$$\int_{|z|=r}\left|\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right|\mathrm{d}\theta<4\pi$$

for $|z|=r<1$. This shows that $f(z)$ is convex in one direction in $\mathbb{D}$. □

Corollary 1 *Let $f(z)\in\mathcal{A}$ and suppose that*

$$\left|\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right|<\Re\left\{\frac{1+z}{1-z}\right\}+1\quad(z\in\mathbb{D}).$$

Then $f(z)$ is convex in one direction, and hence $f(z)$ is univalent in $\mathbb{D}$.

Theorem 4 *Let $f(z)\in\mathcal{A}$ and suppose that*

$$\left|1+\frac{zf''(z)}{f'(z)}\right|<\sqrt{7}\quad(z\in\mathbb{D}).\tag{5}$$

Then $f(z)$ is convex in one direction, and hence $f(z)$ is univalent in $\mathbb{D}$.

Proof Let $0\leq r<1$. From (5) we have

$$\int_{|z|=r}\left|1+\frac{zf''(z)}{f'(z)}\right|^2\mathrm{d}\theta<14\pi.\tag{6}$$

Note that

$$\int_{|z|=r}\left(1+\frac{zf''(z)}{f'(z)}\right)^2\mathrm{d}\theta=\int_{|z|=r}\left\{1+2\frac{zf''(z)}{f'(z)}+\left(\frac{zf''(z)}{f'(z)}\right)^2\right\}\mathrm{d}\theta=2\pi\tag{7}$$

and

$$\int_{|z|=r}\overline{\left(1+\frac{zf''(z)}{f'(z)}\right)^2}\mathrm{d}\theta=2\pi.\tag{8}$$

Therefore, from (6), (7), and (8) we have

$$\begin{aligned}&\int_{|z|=r}\left(1+\Re\frac{zf''(z)}{f'(z)}\right)^2\mathrm{d}\theta\\&=\frac{1}{4}\int_{|z|=r}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^2+2\left|1+\frac{zf''(z)}{f'(z)}\right|^2+\overline{\left(1+\frac{zf''(z)}{f'(z)}\right)^2}\right]\mathrm{d}\theta\\&<8\pi.\end{aligned}\tag{9}$$

Hence, applying the Cauchy-Schwarz inequality and (9), we get

$$\int_{|z|=r}\left|1+\Re\frac{zf''(z)}{f'(z)}\right|\mathrm{d}\theta\leq\sqrt{2\pi\int_{|z|=r}\left|1+\Re\frac{zf''(z)}{f'(z)}\right|^2\mathrm{d}\theta}<4\pi.$$

This completes the proof of Theorem 4. □

Example 2 Consider the function $f_2 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_2(z) = \frac{\sqrt{7}}{7}\left(\left(\frac{z+\sqrt{7}}{\sqrt{7}}\right)^7 - 1\right) \quad (z \in \mathbb{D}).$$

Then we have

$$\left|1 + \frac{zf_2''(z)}{f_2'(z)}\right| = \left|\sqrt{7}\frac{\sqrt{7}z+1}{\sqrt{7}+z}\right| < \sqrt{7} \quad (z \in \mathbb{D}).$$

Hence, it follows from Theorem 4 that the function f_2 is convex in one direction. In fact, the function f_2 is convex in the direction of the positive real axis.

Applying the same method as that used in the proof of the aforementioned theorems and Lemma 1, we have the following sufficient conditions on analytic functions to be starlike in one direction.

Theorem 5 *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\left|\Re\left\{\frac{zf'(z)}{f(z)}\right\}\right| < 2\Re\left\{\frac{1+z}{1-z}\right\} \quad (z \in \mathbb{D}).$$

Then $f(z)$ is starlike in one direction in $\mathbb{D}$.

Theorem 6 *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\left|\frac{zf'(z)}{f(z)}\right| \le \left|\frac{1+z}{1-z}\right| \quad (z \in \mathbb{D}).$$

Then $f(z)$ is starlike in one direction in $|z| < r_0 = 0.251652\cdots$, where r_0 is the root of equation (3).

Theorem 7 *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\left|\Re\left\{\frac{zf'(z)}{f(z)}\right\} - \frac{1}{2}\right| < \Re\left\{\frac{1+z}{1-z}\right\} + \frac{1}{2} \quad (z \in \mathbb{D}).$$

Then $f(z)$ is starlike in one direction in $\mathbb{D}$.

Theorem 8 *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\left|\frac{zf'(z)}{f(z)}\right| < \sqrt{7} \quad (z \in \mathbb{D}).$$

Then $f(z)$ is starlike in one direction in $\mathbb{D}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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