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An asymptotically sharp coefficients estimate for harmonic K-quasiconformal mappings

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Abstract

By using the improved Hübner inequalities, in this paper we obtain an asymptotically sharp lower bound estimate for the coefficients of harmonic K-quasiconformal self-mappings of the unit disk $\mathbb{D}$ which keep the origin fixed. The result partly improves the former results given by (Partyka and Sakan in Ann. Acad. Sci. Fenn., Math. 30:167-182, 2005) and (Zhu and Zeng in J. Comput. Anal. Appl. 13:1081-1087, 2011). Furthermore, using some estimate for the derivative of the boundary function of a harmonic K-quasiconformal self-mapping w of $\mathbb{D}$ which keeps the origin fixed, we obtain an upper bound estimate for the coefficients of w.

MSC: Primary 30C62; secondary 30C20; 30F15

Keywords: Heinz inequality; Hübner inequalities; coefficients estimate; harmonic quasiconformal mapping

1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disk, $w(z)$ be a harmonic mapping defined in $\mathbb{D}$. Then $w(z)$ can be presented as $w(z) = h(z) + \overline{g(z)}$, where

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

are both analytic in $\mathbb{D}$. By Lewy's theorem [3], we know that $w(z)$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its Jacobian satisfies the following inequality:

$$J_f(z) = \left|w_z(z)\right|^2 - \left|w_{\bar{z}}(z)\right|^2 = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 > 0$$

for all $z \in \mathbb{D}$. One of the basic properties for harmonic self-mappings of $\mathbb{D}$ is the Heinz inequality [4].

Lemma A *Let w map the unit disk harmonically onto itself with $w(0) = 0$. Then*

$$\left|w_z(0)\right|^2 + \left|w_{\bar{z}}(0)\right|^2 \geq c \tag{2}$$

for some absolute constant $c > 0$.

Subsequently, in 1982, Hall [5] obtained the sharp lower bound of c.

Theorem B *Let $w(z) = h(z) + \overline{g(z)} = \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ be a univalent harmonic mapping of the unit disk onto itself, then its coefficients satisfy the inequality*

$$|a_1|^2 + |b_1|^2 \geq \frac{27}{4\pi^2}. \tag{3}$$

The lower bound $\frac{27}{4\pi^2}$ is the best possible.

Let

$$p(r, x - \varphi) = \frac{1 - r^2}{2\pi(1 - 2r\cos(x - \varphi) + r^2)}$$

denote the Poisson kernel, then every bounded harmonic mapping w defined in $\mathbb{D}$ has the following representation:

$$w(z) = P[f](z) = \int_0^{2\pi} p(r, x - \varphi) f\left(e^{ix}\right) dx, \tag{4}$$

where $z = re^{i\varphi} \in \mathbb{D}$ and f is a bounded integrable function defined on the unit circle $\mathbf{T} := \partial\mathbb{D}$.

Suppose that $w(z)$ is a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto a domain $\Omega \subseteq \mathbb{C}$. Then $w(z)$ is a harmonic K-quasiconformal mapping if and only if

$$K(w) := \sup_{z\in\mathbb{D}} \frac{|w_z(z)| + |w_{\bar{z}}(z)|}{|w_z(z)| - |w_{\bar{z}}(z)|} \leq K.$$

Under the additional assumption that $w(z)$ is a K-quasiconformal mapping, in 2005 Partyka and Sakan [1] obtained an asymptotically sharp variant of Heinz's inequality as follows (see also [2]).

Theorem C *Let $w(z)$ be a harmonic K-quasiconformal mapping of $\mathbb{D}$ onto itself satisfying $w(0) = 0$. Then the inequality*

$$\left|\partial_z w(z)\right|^2 + \left|\partial_{\bar{z}} w(z)\right|^2 \geq \frac{1}{4}\left(1 + \frac{1}{K}\right)^2 \max\left\{\frac{4}{\pi^2}, L_K^2\right\} \tag{5}$$

holds for every $z \in \mathbb{D}$, where

$$L_K := \frac{2}{\pi}\int_0^{\frac{1}{\sqrt{2}}} \frac{d(\Phi_{1/K}(s)^2)}{s\sqrt{1 - s^2}} \tag{6}$$

is a strictly decreasing function of K. For $L > 0$, $\Phi_L(s)$ is the Hersch-Pfluger distortion function defined by the equalities $\Phi_L(s) := \mu^{-1}(\mu(s)/L)$, $0 < s < 1$; $\Phi_L(0) := 0$, $\Phi_L(1) := 1$, where $\mu(s)$ stands for the module of Grötzsch's extremal domain $\mathbb{D}\backslash[0, s]$.

In 2010, Qiu and Ren [6] improved the Hübner inequalities as follows.

Theorem D *For all $s \in (0,1)$ and $K \in (1,\infty)$, we have*

$$4^{1-K}s^K \le \Phi_{1/K}(s) < 4^{D(s)(1-K)}s^K \tag{7}$$

and

$$s^{1/K} \le \Phi_K(s) < 4^{(1-s^2)^{\frac{3}{4}}(1-1/K)}s^{1/K}, \tag{8}$$

where $D(s) = (1-s)(1+s)^{1/\ln 4}$.

A sense-preserving harmonic mapping of $\mathbb{D}$ onto itself can be represented as the Poisson extension of the boundary function $f(e^{it}) = e^{i\gamma(t)}$, where $\gamma(t)$ is a continuous non-decreasing function with $\gamma(2\pi) - \gamma(0) = 2\pi$ and $\gamma(t+2\pi) = \gamma(t) + 2\pi$ (*cf.* [7, 8]). The coefficients a_n and b_n have an alternative interpretation as Fourier coefficients of the periodic function $e^{i\gamma(t)}$, and so Heinz's lemma can be viewed as a statement about Fourier series.

In this paper, assuming that $w(z)$ is a harmonic K-quasiconformal mapping of $\mathbb{D}$ onto itself satisfying $w(0) = 0$, by using Theorem D we obtain a sharp lower bound for its coefficients as follows:

$$|a_1|^2 + |b_1|^2 \ge B_1(K) := 2 - 2^{2(1-1/K)(2+2^{5/4})}\frac{2K^2\Gamma(\frac{2}{K})}{(K+1)\Gamma^2(\frac{1}{K})} \tag{9}$$

which satisfies $\lim_{K\to 1^+} B_1(K) = 1$, where Γ is the gamma function.

For $n \ge 2$ we have

$$|a_n|^2 + |b_n|^2 \ge B_n(K), \tag{10}$$

where

$$B_n(K) := \chi(K) + \frac{2^{2(1-1/K)(2+2^{5/4})}\Gamma(1+\frac{2}{K})(n-1-\frac{1}{K})!}{\Gamma^2(\frac{1}{K})(n+\frac{1}{K})\frac{1}{K}(n-1+\frac{1}{K})!}, \tag{11}$$

$$\left(n-1-\frac{1}{K}\right)! := \left(n-1-\frac{1}{K}\right)\left(n-2-\frac{1}{K}\right)\cdots\left(1-\frac{1}{K}\right),$$

$$\left(n-1+\frac{1}{K}\right)! := \left(n-1+\frac{1}{K}\right)\left(n-2+\frac{1}{K}\right)\cdots\left(1+\frac{1}{K}\right),$$

and

$$\chi(K) := 2 - \frac{2^{2(1-1/K)(2+2^{5/4})}\Gamma(1+\frac{2}{K})}{\Gamma^2(1+\frac{1}{K})} \tag{12}$$

is a decreasing function of K with $\chi(1) = 0$.

Assume that $w(z) = P[f](z)$ is a harmonic K-quasiconformal mapping of $\mathbb{D}$ onto itself with the boundary function $f(e^{it}) = e^{i\gamma(t)}$, satisfying $w(0) = 0$. In Theorem 3.2 of [9], Partyka and Sakan proved that the following inequalities:

$$\frac{2^{5(1-K^2)/2}}{(K^2+K-1)^K} \le \left|f'(z)\right| \le K^{3K}2^{5(K-\frac{1}{K})/2} \tag{13}$$

hold for a.e. $z = e^{it} \in \mathbf{T}$. Applying the above inequalities we obtain an upper bound for the coefficients of a harmonic K-quasiconformal self-mapping $w(z)$ of $\mathbb{D}$ satisfying $w(0) = 0$ as follows:

$$|a_n|^2 + |b_n|^2 \leq A_n(K) := \frac{16}{n^2\pi^2} K^{6K} 2^{5(K-1/K)}. \tag{14}$$

Furthermore we show that (9) and (10) are sharp as $K \to 1$.

2 Auxiliary results

Lemma 1 *Let $K > 1$ be a constant. Then the equality*

$$\int_0^\pi \sin^{\frac{2}{K}}(t)\cos(2nt)\,dt = \frac{\pi}{4^{\frac{1}{K}}} \frac{(-1)^n \Gamma(1+\frac{2}{K})}{\Gamma(1+\frac{1}{K}-n)\Gamma(1+\frac{1}{K}+n)} \tag{15}$$

holds for all nonnegative integer numbers $n = 0, 1, 2, \ldots$.

Lemma 2 *Let $\varphi(t) := |\cos\frac{t}{2}|^{\frac{3}{2}} + |\sin\frac{t}{2}|^{\frac{3}{2}}$, for any $t \in [0, 2\pi]$. Then*

$$\max_{0 \leq t \leq 2\pi} \varphi(t) = \varphi\left(\frac{\pi}{2}\right) = \sqrt[4]{2}. \tag{16}$$

Lemma 3 *Let $w = P[f](z)$ be a harmonic K-quasiconformal self-mapping of $\mathbb{D}$ with the boundary function $f(e^{it}) = e^{i\gamma(t)}$. For every $z_1 = e^{i(s+t)}, z_2 = e^{i(s-t)} \in \mathbf{T}$, let $\theta = \gamma(s+t) - \gamma(s-t)$. Then $f(z_1) = e^{i\theta} f(z_2)$ and the inequalities*

$$2^{10-10K} \sin^{2K}(t) \leq \sin^2\left(\frac{\theta}{2}\right) \leq 2^{2(1-1/K)(1+2^{5/4})} \sin^{2/K}(t) \tag{17}$$

hold for every $0 \leq s < 2\pi$, $0 \leq t \leq \pi$.

Proof According to the quasi-invariance of the harmonic measure (see (1.9) in [1]), we have

$$\Phi_{1/K}\left(\cos\frac{t}{2}\right) \leq \cos\frac{\theta}{4} \leq \Phi_K\left(\cos\frac{t}{2}\right) \tag{18}$$

for every $0 \leq s < 2\pi$, $0 \leq t \leq \pi$, and $\theta = \gamma(s+t) - \gamma(s-t)$. Since $\Phi_K^2(x) + \Phi_{1/K}^2(\sqrt{1-x^2}) = 1$ holds for every $0 \leq x \leq 1$, this shows that

$$\Phi_{1/K}\left(\sin\frac{t}{2}\right) \leq \sin\frac{\theta}{4} \leq \Phi_K\left(\sin\frac{t}{2}\right). \tag{19}$$

Using the Hübner inequalities, (7) and (8), we see that $4^{1-K}s^K \leq \Phi_{1/K}(s) < 4^{D(s)(1-K)}s^K$ and $s^{1/K} \leq \Phi_K(s) < 4^{(1-s^2)^{\frac{3}{4}}(1-1/K)}s^{1/K}$. Applying (18), (19), and the above two inequalities, we have

$$\sin^2\left(\frac{\theta}{2}\right) \geq 4\Phi_{1/K}^2\left(\sin\frac{t}{2}\right)\Phi_{1/K}^2\left(\cos\frac{t}{2}\right) \geq 2^{10(1-K)} \sin^{2K} t$$

and

$$\sin^2\left(\frac{\theta}{2}\right) \leq 4\Phi_K^2\left(\sin\frac{t}{2}\right)\Phi_K^2\left(\cos\frac{t}{2}\right) \leq 2^{2(1-1/K)\{2[|\cos(\frac{t}{2})|^{\frac{3}{2}}+|\sin(\frac{t}{2})|^{\frac{3}{2}}]+1\}}\sin^{\frac{2}{K}}(t).$$

By using Lemma 2 we see that

$$\left|\cos\frac{t}{2}\right|^{\frac{3}{2}} + \left|\sin\frac{t}{2}\right|^{\frac{3}{2}} \leq \sqrt[4]{2}.$$

This implies that

$$2^{10-10K}\sin^{2K}(t) \leq \sin^2\left(\frac{\theta}{2}\right) \leq 2^{2(1-1/K)(1+2^{5/4})}\sin^{2/K}(t)$$

hold for every $0 \leq s < 2\pi$, $0 \leq t \leq \pi$, and $\theta = \gamma(s+t) - \gamma(s-t)$.

This completes the proof. □

3 Main results

Theorem 1 *Given $K > 1$, let $w(z) = P[f](z) = h(z) + \overline{g(z)}$ be a harmonic K-quasiconformal self-mapping of $\mathbb{D}$ satisfying $w(0) = 0$ with the boundary function $f(e^{it}) = e^{i\gamma(t)}$, where*

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{20}$$

are both analytic in $\mathbb{D}$. Then

$$|a_1|^2 + |b_1|^2 \geq B_1(K),$$

where $B_1(K)$ is given by (9) *and satisfies* $\lim_{K\to 1^+} B_1(K) = 1$. *For $n \geq 2$,*

$$|a_n|^2 + |b_n|^2 \geq B_n(K),$$

where $B_n(K)$ is given by (11) *and satisfies* $\lim_{n\to\infty}\lim_{K\to 1^+} B_n(K) = 0$.

Proof Since $w(z) = P[f](z) = \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$, using Parseval's relation (*cf.* [7]) we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i[\gamma(s+t)-\gamma(s-t)]}\,ds = \sum_{n=1}^{\infty}\left(|a_n|^2 e^{2int} + |b_n|^2 e^{-2int}\right)$$

for arbitrary $t \in R$. Taking real parts, we arrive at the formula

$$1 - 2J(t) = \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right)\cos(2nt), \tag{21}$$

where

$$J(t) = \frac{1}{2\pi}\int_0^{2\pi} \sin^2\left(\frac{\gamma(s+t) - \gamma(s-t)}{2}\right) ds.$$

Since $w(z)$ is a harmonic K-quasiconformal mapping, by Lemma 3 we have

$$2^{10-10K}\sin^{2K}(t)\leq J(t)\leq 2^{2(1-1/K)(1+2^{5/4})}\sin^{2/K}(t). \tag{22}$$

Hence $(|a_n|^2+|b_n|^2)\int_0^\pi\cos(2nt)(1+\cos(2nt))\,dt=\int_0^\pi(1-2J(t))(1+\cos(2nt))\,dt$. Using (15) we also obtain

$$\begin{aligned}
|a_n|^2+|b_n|^2 &= \frac{2}{\pi}\left(\pi-2\int_0^\pi J(t)\bigl(1+\cos(2nt)\bigr)\,dt\right)\\
&\geq \frac{2}{\pi}\left(\pi-2\cdot 2^{2(1-1/K)(1+2^{5/4})}\int_0^\pi \sin^{\frac{2}{K}}(t)\bigl(1+\cos(2nt)\bigr)\,dt\right)\\
&= 2-2^{2(1-1/K)(2+2^{5/4})}\frac{\Gamma(1+\frac{2}{K})}{\Gamma^2(1+\frac{1}{K})}-\frac{2^{2(1-1/K)(2+2^{5/4})}(-1)^n\Gamma(1+\frac{2}{K})}{\Gamma(1+\frac{1}{K}-n)\Gamma(1+\frac{1}{K}+n)}\\
&:= \chi(K)+\frac{2^{2(1-1/K)(2+2^{5/4})}(-1)^{n+1}\Gamma(1+\frac{2}{K})}{\Gamma(1+\frac{1}{K}-n)\Gamma(1+\frac{1}{K}+n)}.
\end{aligned}$$

For $n=1$, using the formula $\Gamma(z+1)=z\Gamma(z)$ and simplifying the above result we obtain the following inequality:

$$|a_1|^2+|b_1|^2\geq B_1(K):=2-2^{2(1-1/K)(2+2^{5/4})}\frac{2K^2\Gamma(\frac{2}{K})}{(K+1)\Gamma^2(\frac{1}{K})}.$$

By computation we know that $B_1(K)$ is a decreasing function of K and satisfies

$$\lim_{K\to 1^+}B_1(K)=1.$$

The above estimate is sharp. Consider the conformal mapping $w(z)=e^{ix}z$, where $x\in\mathbb{R}$ is a real number. Then we have $|a_1|+|b_1|=1$.

For $n\geq 2$, we have

$$\begin{aligned}
\Gamma\left(1+\frac{1}{K}-n\right)&=\frac{\Gamma(\frac{1}{K})}{(1+\frac{1}{K}-n)(2+\frac{1}{K}-n)\cdots(\frac{1}{K}-1)}=\frac{(-1)^{n-1}\Gamma(\frac{1}{K})}{(n-1-\frac{1}{K})!},\\
\Gamma\left(1+\frac{1}{K}+n\right)&=\left(n+\frac{1}{K}\right)\left(n+\frac{1}{K}-1\right)\cdots\left(\frac{1}{K}\right)\Gamma\left(\frac{1}{K}\right)=\frac{1}{K}\Gamma\left(\frac{1}{K}\right)\left(n+\frac{1}{K}\right)!,
\end{aligned}$$

then

$$|a_n|^2+|b_n|^2\geq\chi(K)+\frac{2^{2(1-1/K)(2+2^{5/4})}\Gamma(1+\frac{2}{K})(n-1-\frac{1}{K})!}{\Gamma^2(\frac{1}{K})(n+\frac{1}{K})\frac{1}{K}(n-1+\frac{1}{K})!}:=B_n(K).$$

By calculating we see that $\chi(K)$ is a decreasing function of K with $\chi(1)=0$. The function $B_n(K)$ is a continuous function of K with $\lim_{K\to 1^+}B_n(K)=\frac{2}{(n+1)n(n-1)}$. This implies that $B_n(K)>0$ holds for all $n\geq 2$ and some $K>1$.

The proof is completed. □

Remark 1 By computation we obtain

$$B_1(K)>\frac{27}{4\pi^2}$$

for all $1 \le K \le 1.05174$. This shows that under the additional assumption that w is a K-quasiconformal mapping, the lower bound of the inequality (3) can be improved.

By the definition of the Gamma function we see that $\Gamma(-n) = \infty$ holds for all nonnegative integer numbers n. According to the proof of Theorem 1 we know that for all $n \ge 2$, $\lim_{K\to 1^+} \Gamma(1 + \frac{1}{K} - n) = \infty$. Therefore

$$\lim_{K\to 1^+} B_n(K) = 0$$

holds for all $n \ge 2$.

Let $t = 0$ in equation (21). Then we have $\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = 1$. The sharp coefficient estimate of a_1 and b_1 shows that if $K \to 1^+$ then $|a_1|^2 + |b_1|^2 \ge B_1(K) \to 1$. This shows that under the assumptions of Theorem 1 if additionally $w(z)$ is a conformal self-mapping of $\mathbb{D}$ satisfying $w(0) = 0$, then all the coefficients b_n for $n \ge 1$ and a_n for $n \ge 2$ are zeros and $|a_1| = 1$, that is, $w(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

Remark 2 In [1] the authors showed that an asymptotically sharp inequality holds for all z in $\mathbb{D}$. Our Theorem 1, however, gives an estimate at $z = 0$ only. In this sense, Theorem 1 partly improves the former results.

Theorem 2 shows that $n^2(|a_n|^2 + |b_n|^2)$ is less than or equal to a positive number determined by K.

Theorem 2 *Under the assumption of Theorem* 1, *the coefficients of* $w(z)$ *satisfy the following inequality*:

$$|a_n|^2 + |b_n|^2 \le \frac{16}{n^2\pi^2} K^{6K} 2^{5(K-1/K)}, \quad n = 1, 2, \ldots.$$

Proof For every $z = re^{i\theta} \in \mathbb{D}$,

$$w(re^{i\theta}) = \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \bar{b}_n r^n e^{-in\theta},$$

hence

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta}) e^{-in\theta}\,d\theta, \quad n = 1, 2, \ldots,$$
$$\bar{b}_n r^n = \frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta}) e^{in\theta}\,d\theta, \quad n = 1, 2, \ldots.$$

For every n we set $a_n = |a_n| e^{i\alpha_n}$, $b_n = |b_n| e^{i\beta_n}$, and $\theta_n = \frac{\alpha_n + \beta_n}{2n}$. Then

$$\begin{aligned}
(|a_n| + |b_n|) r^n &= \left| \frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta}) \left[e^{-i\alpha_n} e^{-in\theta} + e^{i\beta_n} e^{in\theta}\right] d\theta \right| \\
&= \left| \frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta}) \left[e^{-in(\theta+\theta_n)} + e^{in(\theta+\theta_n)}\right] d\theta \right| \\
&= \left| \frac{1}{\pi}\int_0^{2\pi} w(re^{i\theta}) \cos n(\theta + \theta_n)\, d\theta \right|.
\end{aligned}$$

Integrating by parts we have

$$\big(|a_n| + |b_n|\big)r^n = \left| \frac{1}{n\pi} \int_0^{2\pi} w_\theta\big(re^{i\theta}\big) \sin n(\theta + \theta_n)\, d\theta \right|. \tag{23}$$

In Theorem 2.8 of [10], Kalaj proved that the radial limits of w_θ and w_r exist almost everywhere and

$$\lim_{r\to 1^-} \partial_\theta w\big(re^{i\theta}\big) = \frac{df(e^{i\theta})}{d\theta}$$

for almost every $z = e^{i\theta} \in \mathbf{T}$. Here f is the boundary function of w. Hence, letting $r \to 1^-$ and using (13), (23) we see that

$$|a_n| + |b_n| \le \frac{1}{n\pi} \int_0^{2\pi} \big|f'\big(e^{i\theta}\big)\big| \big|\sin n(\theta + \theta_n)\big|\, d\theta = \frac{4K^{3K} 2^{5(K-1/K)/2}}{n\pi}.$$

It shows that $|a_n|^2 + |b_n|^2 \le (|a_n| + |b_n|)^2 \le \frac{16K^{6K} 2^{5(K-1/K)}}{n^2\pi^2} := A_n(K)$.

The proof is completed. □

Competing interests
The author declares that she has no competing interests.

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