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A new integral formula for the angle between intersected submanifolds

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Abstract

Let G be a Lie group and H its subgroup, and M^q, N^r two submanifolds of dimensions q, r, respectively, in the Riemannian homogeneous space G/H. A kinematic integral formula for the angle between the two intersected submanifolds is obtained.

MSC: 52A20; 53C65

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1 Introduction

Kinematic formulas in integral geometry are important and useful. At the beginning of [1], Chern said: 'one of the basic problems in integral geometry is to find explicit formulas for the integrals of geometric quantities over the kinematic density in terms of known integral invariants'. He proved the fundamental kinematic formula in n-dimensional Euclidean space $\mathbb{R}^n$ in [2]. In [1], he provided integral formulas for the quantities introduced by Weyl for the volume of tubes. These formulas complement the fundamental kinematic formula, which only deals with hypersurfaces. They can be found in [3] and [4].

Now we state the aim of kinematic formulas in general homogeneous spaces. Let G be a unimodular Lie group with kinematic density dg and H a closed subgroup of G. Assume that there exists an invariant Riemannian metric in the homogeneous space G/H. Let M^q, N^r be two compact submanifolds of dimensions q, r in G/H, respectively, M^q fixed and gN^r the image of N under a motion $g \in G$. Let $I(M^q \cap gN^r)$ denote a certain invariant of $M^q \cap gN^r$, which may be a volume, a curvature integral, *etc.* Then the purpose of the kinematic formula related to the invariant $I(M^q \cap gN^r)$ is to evaluate the following integral:

$$\int_{\{g\in G:M^q\cap gN^r\neq\emptyset\}} I\bigl(M^q \cap gN^r\bigr)\,dg \tag{1}$$

by the well-known integral invariants of M^q and N^r.

The integral formulas have been studied by many geometers from various viewpoints. For example, in the case that G is the group of motions in $\mathbb{R}^n$, M^q, and N^r are submanifolds of $\mathbb{R}^n$ and

$$I\bigl(M^q \cap gN^r\bigr) = \operatorname{vol}\bigl(M^q \cap gN^r\bigr), \tag{2}$$

the evaluation of (1) leads to the formulas due to Poincaré, Blaschke, Santaló, and others (see [3–5] and the references therein). Let $I(M \cap gN) = \chi(M \cap gN)$ be the Euler characteristic of $M \cap N$ of domains M and N in $\mathbb{R}^n$, then $\int_G \chi(M \cap gN)\, dg$ can be expressed explicitly by the integrals of elementary symmetric functions of principal curvatures over the boundaries and the Euler characteristics of M, N. This well-known kinematic formula in integral geometry is due to Chern [1, 2]. Next, assume that $I(M^q \cap gN^r) = \mu(M^q \cap gN^r)$ is one of the integral invariants from the Weyl tube formula, then (1) leads to the Chern-Federer kinematic formula for submanifolds of $\mathbb{R}^n$ [6]. Furthermore, Howard defined integral invariants induced from an invariant homogeneous polynomial of the second fundamental form of $M^q \cap gN^r$, and he achieved a general kinematic formula, where G is unimodular and acts transitively on the sets of tangent spaces to each of M^q and N^r. Finally, he put the kinematic formulas listed above into a uniform shape. Most existed kinematic formulas are intrinsic, and only a few of them are extrinsic, for example, C-S Chen's formula. In [7], Zhou presented an extrinsic type kinematic formula for mean curvature powers which is a generalization of the formulas in [3, 8]. This is a typical work in which the moving frame method is used effectively. By means of some kinematic formulas, sufficient conditions for one domain to contain or to be contained in another domain can be obtained. See [5, 9–13] for more kinematic formulas and their applications.

In addition, Zhou stated in [7] that an important unsolved problem is whether an invariant $I(M^q \cap gN^r)$ (either intrinsic or extrinsic) can be expressed by invariants of submanifolds M^q and N^q. At least we are not aware of letting $I(M \cap gN) = \operatorname{diam}(M \cap gN)$, the diameter of intersection $I(M \cap gN)$ of two domains M and N in $\mathbb{R}^n$. Let $I(M^q \cap gN^r)$ be the angle between two intersected submanifolds (the angle between M^q and gN^r is an integral invariant [1]) and the explicit formula is still obscure to us. In this paper, we will discuss the second problem and obtain an extrinsic kinematic formula.

Let $G(n)$ be the group of rigid motions in $\mathbb{R}^n$ and $O(n)$ the group of rotations in $\mathbb{R}^n$. Denote by dg the invariant measure of the group $G(n)$ which is the product measure of the Lebesgue measure of $\mathbb{R}^n$ and the invariant measure of $SO(n)$, where the invariant measure of $SO(n)$ is normalized so that the total measure is $O_{n-1} \cdots O_1$. Let M^q and N^r be two submanifolds in $\mathbb{R}^n$, M^q fixed, and gN^r moving under the rigid motion g of $\mathbb{R}^n$ with the kinematic density dg. Denote by $d\sigma$ the volume element.

Theorem 1 *Let M^q and N^r be two intersected submanifolds in $\mathbb{R}^n$ ($n \geq 3$). Denote by $G(n)$ the group of rigid motions in $\mathbb{R}^n$ and Δ the angle between M^q and gN^r. Then, for any positive integer k,*

$$\int_{G(n)} \int_{M^q \cap gN^r} \Delta^k \, d\sigma_{r+q-n} \, dg = C(n) \sigma_q\big(M^q\big) \sigma_r\big(N^r\big), \tag{3}$$

where $C(n) = \frac{O_{k+n} \cdots O_{k+q+1} O_{q-1} \cdots O_{r+q-n} O_{r-1} \cdots O_1}{O_{k+r} \cdots O_{k+r+q-n+1} O_0}$.

2 Preliminaries

In this section, we review some basic facts about the angle between intersected submanifolds in $\mathbb{R}^n$ and present an important density formula.

2.1 The angle between intersected submanifolds

We first introduce the angle between two vector subspaces below, which will be useful for later purposes. Let V and W be vector subspaces of dimensions p and q, respectively.

Let $v_{p+1}, \ldots, v_n$ be an orthonormal basis of $N(V)$ and $w_{q+1}, \ldots, w_n$ an orthonormal basis of $N(W)$, that is,

$$N(V) = \mathbf{span}\{v_{p+1}, \ldots, v_n\};$$
$$N(W) = \mathbf{span}\{w_{q+1}, \ldots, w_n\},$$

the normal spaces to V, W, respectively. The angle between subspaces V and W is defined by

$$\Delta(V, W) = \|v_{p+1} \wedge \cdots \wedge v_n \wedge w_{q+1} \wedge \cdots \wedge w_n\|,$$

where

$$\|x_1 \wedge \cdots \wedge x_k\|^2 = \left|\mathbf{det}\big((x_i, x_s)\big)\right|, \quad 1 \leq i, s \leq k.$$

If V, W are both $(n-1)$-dimensional then $\Delta(V, W) = |\cos\theta|$, where θ is the angle between the normals of V and W. It is obvious that

$$0 \leq \Delta(V, W) \leq 1,$$

with

$$\Delta(V, W) = 0 \quad \text{if and only if} \quad V \cap W \neq \{0\},$$
$$\Delta(V, W) = 1 \quad \text{if and only if} \quad V \perp W.$$

Similarly if g is an isometry of $\mathbb{R}^n$, then $\Delta(gV, gW) = \Delta(V, W)$.

Let M^q and N^r be two submanifolds of dimensions q, r in $\mathbb{R}^n$, respectively. We assume M^q fixed and N^r moving under the rigid motion g of $\mathbb{R}^n$ with the kinematic density dg. dg is the invariant measure of $G(n)$ and has the decomposition $dg = dx\,d\gamma$, where dx is the Lebesgue measure of $\mathbb{R}^n$ and $d\gamma$ is the invariant measure of $SO(n)$. Consider generic positions gN^r so that the intersection $M^q \cap gN^r$ is a $(q+r-n)$-dimensional manifold. We make use of the following convention on the ranges of indices:

$$1 \leq A \leq n; \qquad 1 \leq \alpha \leq q+r-n; \qquad q+r-n+1 \leq a \leq q;$$
$$q+1 \leq \lambda \leq n; \qquad q+r-n+1 \leq h \leq r; \qquad r+1 \leq \rho \leq n.$$

Let $\{x; e_A\}$ be a local orthonormal frame at $x \in M^q$, and $e_1, \ldots, e_q$ are tangent to M^q at x. Similarly, let $\{x'; e'_A\}$ be a local orthonormal frame at $x' \in gN^r$, and $e'_1, \ldots, e'_r$ are tangent to gN^r at x'. Suppose that g is generic, so that $M^q \cap gN^r$ is of dimension $q+r-n$. We restrict the above families of frames by the condition

$$x = x', \qquad e_\alpha = e'_\alpha.$$

Geometrically the latter means that $x \in M^q \cap gN^r$ and e_α's are tangent to $M^q \cap gN^r$ at x. The two submanifolds M^q and N^r at x have a scalar invariant, the angle between M^q and

N^r, *i.e.*,

$$\Delta^2 = \left|\mathbf{det}\left(e_\lambda, e'_\rho\right)\right| = \left|\mathbf{det}\left(e_a, e'_h\right)\right|.$$

In the case that M^q and N^r are both hypersurfaces ($q = r = n-1$) it is the absolute value of the cosine of the angle between their normal vectors.

2.2 Some relations between densities of linear subspaces

Let O be a fixed point (origin) and let $L_{q[O]}$ be a fixed q-plane through O. Let $L_{r[O]}$ be a moving r-plane through O and assume that $q + r > n$, so that $L_{q[O]} \cap L_{r[O]}$ is, in general, a $(r + q - n)$-plane through O, which we represent by $L_{r+q-n[O]}$. We can express $dL_{r[O]}$ as a product of $dL_{r[r+q-n]}$ (density of L_r about $L_{r+q-n[O]}$) and $dL^{(q)}_{r+q-n[O]}$ (density of $L_{r+q-n[O]}$ as subspace of the fixed about $L_{q[O]}$). We consider the following two orthonormal moving frames:

Moving frame 1:

- $\operatorname{span}\{e_1, e_2, \dots, e_{r+q-n}\} = L_{q[O]} \cap L_{r[O]}$;
- $e_{r+q-n+1}, \dots, e_r$ lie on $L_{r[O]}$;
- $\operatorname{span}\{e_{r+1}, \dots, e_n\}$ are arbitrary unit vector that complete orthonormal frame 1.

Moving frame 2:

- $\operatorname{span}\{e_1, e_2, \dots, e_{r+q-n}\} = L_{q[O]} \cap L_{r[O]}$;
- $e'_{r+q-n+1}, \dots, e'_r$ are constant unit vectors in the $(n-q)$-plane $L_{n-q}[O]$ perpendicular to $L_{q[O]}$;
- $e'_{q+1}, \dots, e'_n$ are contained in $L_{q[O]}$ such that they form an orthonormal frame in $L_{q[O]}$ together with $e_1, e_2, \dots, e_{r+q-n}$.

By these notations, we have

$$dL_{r[O]} = \bigwedge_{\alpha,i}(e_{r+\alpha}, de_i)\bigwedge_{\alpha,h}(e_{r+\alpha}, de_h) \tag{4}$$

with the following ranges of indices, which will be used throughout the rest of this section:

$$\alpha = 1, 2, \dots, n-q; \qquad i = 1, 2, \dots, r+q-n;$$
$$h = r+q-n+1, \dots, q; \qquad k = r+1, r+2, \dots, n.$$

The total measure of the unoriented r-planes of $\mathbb{R}^n$ through a fixed point (*i.e.*, the volume of the Grassmann manifold $G_{r,n-r}$) is

$$m(G_{r,n-r}) = m(G_{n-r,r}) = \int_{G_{r,n-r}} dL_{r[O]} = \frac{O_{n-1}O_{n-2}\cdots O_{n-r}}{O_{r-1}O_{r-2}\cdots O_1 O_0}, \tag{5}$$

where O_i is the surface area of the i-dimensional unit sphere.

We also have the following density formulas:

$$dL_{r[r+q-n]} = \bigwedge_{\alpha,h}(e_{r+\alpha}, de_h),$$
$$dL^{(q)}_{r+q-n[O]} = \bigwedge_{\alpha,i}\left(e'_{r+\alpha}, de_i\right).$$

Put

$$e'_{r+\alpha} = \sum_h u_{r+\alpha} e'_h + \sum_k u_{r+\alpha,k} e'_k.$$

Since e'_h's are constant vectors, we have $(e'_h, de_i) = -(e_i, de'_h) = 0$, and thus

$$(e_{r+\alpha}, de_i) = \sum_k u_{r+\alpha,k} \bigl(e'_k, de_i\bigr). \tag{6}$$

From (4) and (6) we have the following formula (see [3]):

$$dL_{r[O]} = \Delta^{r+q-n}\, dL_{r[r+q-n]} \wedge dL^{(q)}_{r+q-n[O]}, \tag{7}$$

where $\Delta = \mathbf{det}(e'_{r+\alpha}, e'_k)$.

2.3 An important differential formula

Let M^q be a fixed q-dimensional manifold and N^r a moving one of dimension r, both assumed smooth of class C^1, having finite volumes $\sigma_q(M^q)$ and $\sigma_r(N^r)$, respectively. Let $q + r \geq n$ and consider positions of N^r such that $M^q \cap N^r \neq \phi$. Let $x \in M^q \cap N^r$ and choose the orthonormal vectors $e_1, e_2, \ldots, e_n$ such that $e_1, e_2, \ldots, e_{r+q-n}$ are tangent to $M^q \cap N^r$ and $e_{r+q-n+1}, \ldots, e_r$ are tangent to N^r. Let $e'_1, \ldots, e'_{n-r}$ be orthonormal vectors such that $e_1, e_2, \ldots, e_{r+q-n}, e'_1, \ldots, e'_{n-r}$ span the tangent q-plane to M^q at x.

Since $x \in M^q$, we have

$$dx = \sum_{h=1}^{r+q-n} \alpha_h e_h + \sum_{j=1}^{n-r} \beta_j e'_j, \tag{8}$$

where α_h and β_j are 1-forms. Thus

$$\omega_{r+h} = dx \cdot e_{r+h} = \sum_{j=1}^{n-r} \beta_j \bigl(e'_j, e_{r+h}\bigr), \quad h = 1, 2, \ldots, n-r \tag{9}$$

and

$$\bigwedge_{h=1}^{n-r} \omega_{r+h} = \Delta \bigwedge_{j=1}^{n-r} \beta_j, \tag{10}$$

where Δ is the $(n-r) \times (n-r)$ determinant

$$\Delta = \bigl|\bigl(e'_j, e_{r+h}\bigr)\bigr| \quad (j, h = 1, 2, \ldots, n-r). \tag{11}$$

The exterior product $\bigwedge \beta_j$ $(j = 1, 2, \ldots, n-r)$ is the $(n-r)$-dimensional volume element on M^q in the direction of the tangent $(n-r)$-plane orthogonal to $M^q \cap N^r$. Therefore, denote by $d\sigma_{q+r-n}(x)$ the volume element of $M^q \cap N^r$ at x, then

$$d\sigma_{r+q-n}(x) \bigwedge_{j=1}^{n-r} \beta_j = d\sigma_q(x), \tag{12}$$

where $d\sigma_q(x)$ is the q-dimensional volume element of M^q at x.

On the other hand, the exterior product $\omega_1 \wedge \omega_2 \cdots \wedge \omega_r$ is equal to $d\sigma_r(x)$ of N^r at x. Thus, multiplying (9) by $d\sigma_{r+q-n}(x) \wedge \omega_1 \wedge \omega_2 \cdots \wedge \omega_r$ and taking (10) into account, we obtain

$$d\sigma_{r+q-n}(x) \bigwedge_{i=1}^{n} \omega_i = \Delta\, d\sigma_q(x) \wedge d\sigma_r(x). \tag{13}$$

Multiplying by $dK_{[x]} = \bigwedge \omega_{jh}$ ($j < h; j, h = 1, 2, \dots, n$), we have (see [3])

$$d\sigma_{r+q-n}(x) \wedge dK = \Delta\, d\sigma_q(x) \wedge d\sigma_r(x) \wedge dK_{[x]}, \tag{14}$$

where $d\sigma_{r+q-n}(x)$ is the volume element of $M^q \cap N^r$ at x, $d\sigma_r(x)$, $d\sigma_q(x)$ express the volume element of M^q, N^r at x, respectively. This density formula expresses the relation of the volume element between M^q, N^r and $M^q \cap gN^r$.

3 Main theorem and proof

Lemma 1 *Let $L_{q[x]}$ be a fixed q-plane through a fixed point x and $L_{r[x]}$ a moving r-plane through x in $\mathbb{R}^n$. Let Δ be the angle between the two linear subspaces. Let $dL_{n-q[x]}^{(2n-r-q)}$ denote the density of $dL_{n-q[x]}$ as a subspace of the fixed $dL_{2n-q-r[x]}$. Assume that $r + q > n$. Then, for any positive integer k,*

$$\int_{G_{n-q,n-r}} \Delta^k\, dL_{n-q[x]}^{(2n-r-q)} = \frac{O_{k+2n-q-r-1} \cdots O_{k+n-r}}{O_{k+n-q-1} \cdots O_k}, \tag{15}$$

where O_i is the surface area of the i-dimensional unit sphere.

Proof By using (7) we have

$$dL_{r[x]} = \Delta^{r+q-n}\, dL_{r[r+q-n]} \wedge dL_{r+q-n[x]}^{(q)}, \tag{16}$$

where $dL_{r[r+q-n]}$ is the density of L_r about $L_{r+q-n[x]}$ and $dL_{r+q-n[x]}^{(q)}$ is the density of $L_{r+q-n[x]}$ as subspace of the fixed $L_{q[x]}$. Integrating (16) over all $L_{r[x]}$, we obtain on the left-hand side the volume of the Grassmann manifold $G_{r,n-r}$ and on the right-hand side we can integrate $dL_{r+q-n}^{(q)}$, applying the same formula (5) for $n \to q$, $r \to r + q - n$, since Δ depends only on $L_{r[r+q-n]}$, so we obtain

$$\int_{G_{n-q},n-r} \Delta^{r+q-n}\, dL_{r[r+q-n]} = \frac{O_{n-1} O_{n-2} \cdots O_q}{O_{r-1} O_{r-2} \cdots O_{r+q-n}}, \tag{17}$$

where the integral is extended over all $L_{r[r+q-n]}$.

According to $dL_{r[r+q-n]} = dL_{n-q[x]}^{(2n-r-q)}$, (17) can be reformulated as

$$\int_{G_{n-q},n-r} \Delta^{r+q-n}\, dL_{n-q[x]}^{2n-r-q} = \frac{O_{n-1} \cdots O_q}{O_{r-1} \cdots O_{r+q-n}}. \tag{18}$$

Suppose $a \in [n - r - q + 1, +\infty]$ is an integer and make the change of notation $n \to n + a$, $r \to r + a$, $q \to q + a$. Since the increased dimension is the dimension of the intersected part, the angle remains unchanged.

Therefore (18) is rewritten as

$$\int_{G_{n-q,n-r}} \Delta^{r+q-n+a}\, dL^{(2n-r-q)}_{n-q[x]} = \frac{O_{n+a-1}\cdots O_{q+a}}{O_{r+a-1}\cdots O_{r+q-n+a}}. \tag{19}$$

Let $k = r + q - n + a$ in (19) be a positive integer, and by (18), we arrive at

$$\int_{G_{n-q,n-r}} \Delta^{k}\, dL^{(2n-r-q)}_{n-q[x]} = \frac{O_{k+2n-r-q-1}\cdots O_{k+n-r}}{O_{k+n-q-1}\cdots O_{k}}. \tag{20}$$

□

Lemma 2 *Let $L_{r[x]}$ be the r-plane through x spanned by $e_1, e_2, \ldots, e_r$ in $\mathbb{R}^n$, $dg^r_{[x]}$ denote the kinematic density of the group of special rotations about x in L_r, and $dg^{n-r}_{[x]}$ denote the kinematic density about x in the $(n-r)$-plane orthogonal to $L_{r[x]}$, then*

$$dg_{[x]} = dg^r_{[x]} \wedge dL_{r[x]} \wedge dg^{n-r}_{[x]},$$

where $dg_{[x]}$ is the kinematic density of the group of special rotations about x.

Proof By the density of linear space, we have

$$dL_{r[x]} = \bigwedge_{\alpha,\beta} \omega_{\alpha\beta}, \quad \alpha = 1, 2, \ldots, r; \beta = r+1, \ldots, n. \tag{21}$$

Let $dg^{n-r}_{[x]}$ denote the kinematic density about x in $(n-r)$-plane orthogonal to $L_{r[x]}$, then

$$dg^{n-r}_{[x]} = \bigwedge_{\lambda<\mu} \omega_{\lambda\mu}, \quad \lambda, \mu = r+1, \ldots, n \tag{22}$$

and

$$dg^{r}_{[x]} = \bigwedge_{c<f} \omega_{cf}, \quad c, f = 1, \ldots, r. \tag{23}$$

The kinematic density of the group of special rotations about x, $dg_{[x]}$ can be expressed as

$$dg_{[x]} = \bigwedge_{B<C} \omega_{BC}, \quad B, C = 1, \ldots, n. \tag{24}$$

From (21), (22), (23), and (24) it follows that

$$dg_{[x]} = dg^r_{[x]} \wedge dL_{r[x]} \wedge dg^{n-r}_{[x]}.$$

□

Proof of Theorem 1 By (14) and applying Lemma 2, we have

$$\begin{aligned}\int_{G(n)} \int_{M^q \cap gN^r} \Delta^{k}\, d\sigma_{r+q-n}(x) \wedge dg &= \int_{G(n)} \int_{M^q \cap gN^r} \Delta^{k+1}\, d\sigma_q(x) \wedge d\sigma_r(x) \wedge dg_{[x]} \\ &= C(n)\sigma_q\big(M^q\big)\sigma_r\big(N^r\big),\end{aligned} \tag{25}$$

where $C(n)$ is a constant (independent of x and the manifolds M^q, N^r) that depends on the dimensions q, r and is given by the integral

$$C(n) = \int \Delta^{k+1}\, dg_{[x]}$$

taken over all positions of N^r about x.

Next, we turn our attention to the computation of the coefficient $C(n)$. Notice that

$$C(n) = \int \Delta^{k+1}\, dg_{[x]} = \int \Delta^{k+1}\, dg^r_{[x]} \wedge dL_{r[x]} \wedge dg^{n-r}_{[x]}$$

and

$$\int dg^r_{[x]} = O_{r-1} \cdots O_1, \qquad \int dg^{n-r}_{[x]} = O_{n-r-1} \cdots O_1,$$

where the integrals are taken over all positions of N^r about x, so

$$\int \Delta^{k+1}\, dg_{[x]} = O_{r-1} \cdots O_1 O_{n-r-1} \cdots O_1 \int \Delta^{k+1}\, dL_{r[x]}. \tag{26}$$

By the density formula (16), we have

$$\begin{aligned} \int \Delta^{k+1}\, dL_{r[x]} &= \int \Delta^{k+1} \Delta^{r+q-n}\, dL_{r[r+q-n]} \wedge dL^{(q)}_{r+q-n[x]} \\ &= \frac{O_{q-1} \cdots O_{n-r}}{O_{r+q-n-1} \cdots O_1 O_0} \int \Delta^{k+r+q-n+1}\, dL^{(2n-r-q)}_{n-q[x]}. \end{aligned} \tag{27}$$

From Lemma 1, we have

$$\int \Delta^{k+r+q-n+1}\, dL^{(2n-r-q)}_{n-q[x]} = \frac{O_{k+n} \cdots O_{k+q+1}}{O_{k+r} \cdots O_{k+r+q-n+1}}. \tag{28}$$

Combining (26), (27), and (28), we obtain the constant in (25),

$$C(n) = \frac{O_{k+n} \cdots O_{k+q+1} O_{q-1} \cdots O_{r+q-n} O_{r-1} \cdots O_1}{O_{k+r} \cdots O_{k+r+q-n+1} O_0}.$$

Then (25) is rewritten as

$$\begin{aligned} &\int_{G(n)} \int_{M^q \cap gN^r} \Delta^k\, d\sigma_{r+q-n} \wedge dg \\ &\quad = \sigma_q\bigl(M^q\bigr) \sigma_r\bigl(N^r\bigr) \frac{O_{k+n} \cdots O_{k+q+1} O_{q-1} \cdots O_{r+q-n} O_{r-1} \cdots O_1}{O_{k+r} \cdots O_{k+r+q-n+1} O_0}. \end{aligned}$$

Thus we complete the proof of Theorem 1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by CZ and she prepared the manuscript initially and performed all the proofs. SB and YT helped to revise the paper. All authors read and approved the final manuscript.

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