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A wave breaking criterion for a modified periodic two-component Camassa-Holm system

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Abstract

In this paper, a wave-breaking criterion of strong solutions is acquired in the Soblev space $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s > \frac{3}{2}$ by employing the localization analysis in the transport equation theory, which is different from that of the two-component Camassa-Holm system.

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Keywords: a modified periodic two-component Camassa-Holm system; wave-breaking criterion; localization analysis

1 Introduction

The classical two-component Camassa-Holm system takes the form

$$\begin{cases} (1-\partial_x^2)u_t + u(1-\partial_x^2)u_x + 2u_x(1-\partial_x^2)u + \rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \end{cases} \tag{1}$$

where the variable $u(t,x)$ represents the horizontal velocity of the fluid, and $\rho(t,x)$ is related to the free surface elevation from equilibrium with the boundary assumptions $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. System (1) was found originally in [1], but it was firstly derived rigorously by Constantin and Ivanov [2]. The system has bi-Hamiltonian structure and is completely integrable. Since the birth of the system, a large number of literature was devoted to the study of the two-component Camassa-Holm system. Some mathematical and physical properties of the system have been obtained. Chen *et al.* [3] established a reciprocal transformation between the two-component Camassa-Holm system and the first negative flow of the AKNS hierarchy. Escher *et al.* [4] used Kato's theory to establish local well-posedness for the two-component system and presented some precise blow-up scenarios for strong solutions of the system. In [2], Constantin and Ivanov described sufficient conditions for wave-breaking and global solution to the system. Dynamics in the periodic case for system (1) were considered in [5]. It is worth mentioning that the wave-breaking criteria of strong solutions is determined in the lowest Soblev space H^s with $s > \frac{3}{2}$ by applying the localization analysis in the transport equation theory [6]. The other results related to the system can be found in [7–15].

Inspired by the works mentioned, in this article, we consider a modified periodic two-component Camassa-Holm system on the circle $\mathbb{S}$ with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ (the circle of unit lengh):

$$\begin{cases} m_t + um_x + 2u_x m + \rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ m(0,x) = m_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ m(t,x+1) = m(t,x), & t > 0, x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t > 0, x \in \mathbb{R}, \end{cases} \tag{2}$$

where $m = (1 - \partial_x^2)^2 u$, and $\mathbb{R}$ is the real number set. In fact, system (2) is a two-component generalization of the equation (if $\rho = 0$ in system (2))

$$m_t + um_x + 2u_x m = 0, \qquad m = \left(1 - \partial_x^2\right)^2 u. \tag{3}$$

Equation (3) was first derived as the Euler-Poincaré differential equation on the Bott-Virasoro group with respect to the H^2 metric [16], and it is known as a modified Camassa-Holm equation and also viewed as a geodesic equation on some diffeomorphism group [16]. It is shown in [16] that the well-posedness and dynamics of Eq. (3) on the unit circle $\mathbb{S}$ are significantly different from that of the Camassa-Holm equation. For example, Eq. (3) does not conform with blow-up solution in finite time.

As we know, differently from the Camassa-Holm equation, Eq. (3) has not blow-up solution. The motivation of the present paper is to find out whether or not system (2) has some similar dynamics as the classical two-component Camassa-Holm equation and Eq. (3) mathematically, for example, wave-breaking and global solution. One of the difficulties is the acquisition of the a priori estimates of $\|u_{xx}\|_{L^\infty}$ and $\|u_{xxx}\|_{L^\infty}$. This difficulty has been overcome by Lemmas 3.4 and 3.5. We mainly use the ideas of [6] to derive a wave-breaking criterion (see Theorem 1) of strong solutions for system (2) in the low Sobolev spaces $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s > \frac{3}{2}$, where a new conservation law is necessary. We need to point out that in the Sobolev spaces $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$, the wave-breaking of the solution for system (1) only depends on the slope of the component u of the solution [6]. However, since the slope of the component u of the solution is bounded by the Sobolev imbedding theorem $H^1 \hookrightarrow L^\infty$, the wave-breaking of the solution for system (2) is determined only by the slope of the component ρ of solution definitely in the low Sobolev spaces $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s > \frac{3}{2}$ (see Theorem 1). This implies that there exists some difference between system (2) and the two-component Camassa-Holm equation. Moreover, this is quite different from Eq. (3) because Eq. (3) does not admit a blow-up solution in infinite time.

2 The main results

We denote by $*$ the convolution. Note that if $g(x) := 1 + 2\sum_{n=1}^{\infty} \frac{1}{1+2n^2+n^4} \cos(nx)$, then $(1 - \partial_x^2)^{-2} f = g * f$ for all $f \in L^2(\mathbb{R})$, and $g * m = u$. We let C denote all of different positive constants that depend on initial data. To investigate dynamics of system (2), we can rewrite system (2) in the form

$$\begin{cases} u_t + uu_x + \partial_x g * [u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2], & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t > 0, x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t > 0, x \in \mathbb{R}. \end{cases} \tag{4}$$

The main result of the present paper is as follows.

Theorem 1 *Let $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$, and T be the maximal existence time of the solution $z = (u, \rho)$ to system* (4). *Assume that $m_0 \in L^2(\mathbb{S})$ and $T < \infty$. Then*

$$\int_0^T \|\partial_x \rho(\tau)\|_{L^\infty(\mathbb{S})}\, d\tau = \infty.$$

3 Preliminaries

In order to prove Theorem 1, we first give some lemmas.

Lemma 3.1 ([6, 17]) (1-D Moser-type estimates) *The following estimates hold*:

(i) *For $s \geq 0$*,

$$\|fg\|_{H^s} \leq C\big(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}\big). \tag{5}$$

(ii) *For $s > 0$*,

$$\|f\partial_x g\|_{H^s} \leq C\big(\|f\|_{H^{s+1}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\partial_x g\|_{H^s}\big). \tag{6}$$

(iii) *For $s_1 \leq \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $s_1 + s_2 > 0$*,

$$\|f\partial_x g\|_{H^{s_1}} \leq C\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}, \tag{7}$$

where C is a constant independent of f and g.

Lemma 3.2 ([17, 18]) *Suppose that $s > -\frac{d}{2}$. Let v be a vector field such that ∇v belongs to $L^1([0,T]; H^{s-1})$ if $s > 1 + \frac{d}{2}$ or to $L^1([0,T]; H^{\frac{d}{2}} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in H^s$, $F \in L^1([0,T]; H^s)$, and that $f \in L^\infty([0,T]; H^s) \cap C([0,T]; S')$ solves the d-dimensional linear transport equation*

$$\begin{cases} f_t + v \cdot \nabla f = F, \\ f|_{t=0} = f_0. \end{cases} \tag{8}$$

Then $f \in C([0,T]; H^s)$. More precisely, there exists a constant C depending only s, p, and d, and such that the following statements hold:

(1) *If $s \neq 1 + \frac{d}{2}$, then*

$$\|f\|_{H^s} \leq \|f_0\|_{H^s} + C\int_0^t \|F(\tau)\|_{H^s}\, d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{H^s}\, d\tau, \tag{9}$$

or

$$\|f\|_{H^s} \leq e^{CV(t)}\left(\|f_0\|_{H^s} + \int_0^t e^{-CV(t)}\|F(\tau)\|_{H^s}\,d\tau\right) \tag{10}$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{\frac{d}{2}}\cap L^\infty}\,d\tau$ *if* $s < 1 + \frac{d}{2}$ *and* $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{s-1}}\,d\tau$ *else.*

(2) *If* $f = v$, *then for all* $s > 0$, *estimates* (9) *and* (10) *hold with* $V(t) = \int_0^t \|\partial_x u(\tau)\|_{L^\infty}\,d\tau$.

Lemma 3.3 ([6]) *Let* $0 < \sigma < 1$. *Suppose that* $f_0 \in H^\sigma$, $g \in L^1([0,T];H^\sigma)$, $v, \partial_x v \in L^1([0,T];L^\infty)$, *and* $f \in L^\infty([0,T];H^\sigma) \cap C([0,T];S')$ *solves the* 1*-dimensional linear transport equation*

$$\begin{cases} f_t + v\partial_x f = g, \\ f|_{t=0} = f_0. \end{cases} \tag{11}$$

Then $f \in C([0,T];H^\sigma)$. *More precisely, there exists a constant* C *depending only* σ *and such that the following statement holds*:

$$\|f\|_{H^\sigma} \leq \|f_0\|_{H^\sigma} + C\int_0^t \|g(\tau)\|_{H^\sigma}\,d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{H^\sigma}\,d\tau, \tag{12}$$

or

$$\|f\|_{H^\sigma} \leq e^{CV(t)}\left(\|f_0\|_{H^\sigma} + \int_0^t C\|g(\tau)\|_{H^\sigma}\,d\tau\right) \tag{13}$$

with $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{L^\infty})\,d\tau$.

Lemma 3.4 *For all* $x \in \mathbb{R}$, *the following statements hold*:

$$\text{(i)}\quad \|\partial_x^2 g\|_{L^\infty(\mathbb{R})} \leq 1 + \frac{\pi}{4} \tag{14}$$

and

$$\text{(ii)}\quad \|\partial_x^3 g\|_{L^\infty(\mathbb{R})} \leq 2 + \ln 2 + \pi. \tag{15}$$

Proof Let $g(x)$ be the Green function for the operator $(1-\partial_x^2)^2$. Then from

$$\left(1 - 2\partial_x^2 + \partial_x^4\right)g(x) = \delta(x) = \sum_{n=-\infty}^{\infty} e^{inx}$$

we get

$$g(x) = \sum_{n=-\infty}^{\infty} \frac{1}{1+2n^2+n^4}e^{inx} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{1+2n^2+n^4}\cos(nx).$$

Hence,

$$g_{xx}(x) = -2\sum_{n=1}^{\infty}\frac{n^2}{1+2n^2+n^4}\cos(nx),$$

which results in

$$\left|g_{xx}(x)\right| \le 2\sum_{n=1}^{\infty}\frac{n^2}{1+2n^2+n^4}\left|\cos(nx)\right| \le 2\sum_{n=1}^{\infty}\frac{n^2}{1+2n^2+n^4}.$$

From Cauchy integral test we have

$$\sum_{n=2}^{\infty}\frac{n^2}{1+2n^2+n^4} \le \lim_{n\to\infty}\int_1^n \frac{x^2}{(1+x^2)^2}\,dx = \frac{1}{4}+\frac{\pi}{8}.$$

It follows that

$$|g_{xx}| \le 2\sum_{n=1}^{\infty}\frac{n^2}{1+2n^2+n^4} \le 1+\frac{\pi}{4}.$$

Now, we prove (ii). From the Fourier series we have

$$h(x) = \sum_{n=1}^{\infty}\frac{\sin(nx)}{n} = \frac{\pi}{2}\left(1-\frac{x}{\pi}\right) \quad \text{for } 0 < x < 2\pi,$$

from which we get

$$\begin{aligned}\left|2h(x)-g_{xxx}\right| &= \left|2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{n^3}{1+2n^2+n^4}\right)\sin(nx)\right| \\ &\le 2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{n^3}{1+2n^2+n^4}\right) \\ &= 2\sum_{n=1}^{\infty}\left(\frac{1}{n(1+n^2)}+\frac{n}{(1+n^2)^2}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{n=2}^{\infty}\left(\frac{1}{n(1+n^2)}+\frac{n}{(1+n^2)^2}\right) &\le \lim_{n\to\infty}\int_1^n\left(\frac{1}{x}-\frac{x}{1+x^2}+\frac{x}{(1+x^2)^2}\right)dx \\ &= \frac{1}{2}\ln 2+\frac{1}{4}.\end{aligned}$$

Hence, we have

$$\left\|\partial_x^3 g\right\|_{L^\infty} \le 2+\ln 2+\pi.$$ □

Lemma 3.5 *Let* $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ *with* $s > \frac{3}{2}$. *Suppose that* T *is the maximal existence time of solution* $z = (u, \rho)$ *of system* (4) *with the initial data* z_0. *Then, for all* $t \in [0, T)$, *the following conservation law holds*:

$$H = \int_{\mathbb{S}}\left(u^2+2u_x^2+u_{xx}^2+\rho^2\right)dx = \int_{\mathbb{S}}\left(u_0^2+2u_{0x}^2+u_{0xx}^2+\rho_0^2\right)dx. \tag{16}$$

Moreover, assume that $m_0 \in L^2$. *Then*

$$\begin{aligned}\|u_{xx}\|_{L^\infty(\mathbb{S})} &\le \left(1+\frac{\pi}{4}\right)\left(\|m_0\|_{L^2}^2 + \|z_0\|_{H^2\times L^2}\int_0^t \|\rho_x\|_{L^\infty}\,d\tau\right)^{\frac{1}{2}} \\ &\quad\times \exp\left[\frac{1}{2}\|z_0\|_{H^2\times L^2}\int_0^t \left(3+\|\rho_x\|_{L^\infty}\right)d\tau\right] \\ &\triangleq L(t) \qquad (17)\end{aligned}$$

and

$$\begin{aligned}\|u_{xxx}\|_{L^\infty(\mathbb{S})} &\le (2+\ln 2+\pi)\left(\|m_0\|_{L^2}^2 + \|z_0\|_{H^2\times L^2}\int_0^t \|\rho_x\|_{L^\infty}\,d\tau\right)^{\frac{1}{2}} \\ &\quad\times \exp\left[\frac{1}{2}\|z_0\|_{H^2\times L^2}\int_0^t \left(3+\|\rho_x\|_{L^\infty}\right)d\tau\right] \\ &\triangleq M(t). \qquad (18)\end{aligned}$$

Proof Multiplying the first equation of system (2) by u and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}\left(u^2+2u_x^2+u_{xx}^2\right)dx + \int_{\mathbb{S}} u\rho\rho_x\,dx = 0. \qquad (19)$$

Multiplying the second equation of system (2) by ρ and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}\rho^2\,dx - \int_{\mathbb{S}} u\rho\rho_x\,dx = 0, \qquad (20)$$

which, together with (19), yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}\left(u^2+2u_x^2+u_{xx}^2+\rho^2\right)dx = 0, \qquad (21)$$

which implies (16).

Next, we prove (17). Multiplying the first equation of system (2) by m and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} m^2\,dx = -\int_{\mathbb{S}} umm_x\,dx - 2\int_{\mathbb{S}} u_x m^2\,dx - \int_S m\rho\rho_x\,dx, \qquad (22)$$

which results in

$$\frac{d}{dt}\int_{\mathbb{S}} m^2\,dx = -3\int_{\mathbb{S}} u_x m^2\,dx - 2\int_{\mathbb{S}} m\rho\rho_x\,dx. \qquad (23)$$

By the Hölder inequality we get from (23) that

$$\begin{aligned}\frac{d}{dt}\|m\|_{L^2(\mathbb{S})}^2 &\le 3\|u_x\|_{L^\infty}\|m\|_{L^2}^2 + 2\|m\|_{L^2}\|\rho\|_{L^2}\|\rho_x\|_{L^\infty} \\ &\le 3\|u_x\|_{L^\infty}\|m\|_{L^2}^2 + \left(1+\|m\|_{L^2}^2\right)\|\rho\|_{L^2}\|\rho_x\|_{L^\infty} \\ &\le \|m\|_{L^2}^2\left(3\|u_x\|_{L^\infty} + \|\rho\|_{L^2}\|\rho_x\|_{L^\infty}\right) + \|\rho\|_{L^2}\|\rho_x\|_{L^\infty}.\end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\|m\|_{L^2}^2 \le \left(\|m_0\|_{L^2}^2 + \int_0^t \|\rho\|_{L^2}\|\rho_x\|_{L^\infty}\,d\tau\right)\exp\left[\int_0^t \left(3\|u_x\|_{L^\infty} + \|\rho\|_{L^2}\|\rho_x\|_{L^\infty}\right)d\tau\right],$$

which, together with (16), yields

$$\|m\|_{L^2}^2 \le \left(\|m_0\|_{L^2}^2 + \|z_0\|_{H^2\times L^2}\int_0^t \|\rho_x\|_{L^\infty}\,d\tau\right) \times \exp\left[\|z_0\|_{H^2\times L^2}\int_0^t \left(3 + \|\rho_x\|_{L^\infty}\right)d\tau\right]. \tag{24}$$

On the other hand, from Lemma 3.4 we deduce

$$\|u_{xx}\|_{L^\infty} = \|g_{xx} * m\|_{L^\infty} \le \|g_{xx}\|_{L^\infty}\|m\|_{L^1} \le \left(1 + \frac{\pi}{4}\right)\|m\|_{L^2}. \tag{25}$$

It follows from (24) that

$$\|u_{xx}\|_{L^\infty} \le \left(1 + \frac{\pi}{4}\right)\left(\|m_0\|_{L^2}^2 + \|z_0\|_{H^2\times L^2}\int_0^t \|\rho_x\|_{L^\infty}\,d\tau\right)^{\frac{1}{2}} \times \exp\left[\frac{1}{2}\|z_0\|_{H^2\times L^2}\int_0^t \left(3 + \|\rho_x\|_{L^\infty}\right)d\tau\right]. \tag{26}$$

Similarly, we can obtain (18).

This completes the proof of Lemma 3.5. □

4 Proof of main theorem

Proof of Theorem 1 Using the maximal principle to the transport equation about ρ,

$$\rho_t + u\rho_x = -u_x\rho,$$

we have

$$\|\rho(t)\|_{L^\infty(\mathbb{S})} \le \|\rho_0\|_{L^\infty(\mathbb{S})} + C\int_0^t \|\partial_x u(\tau)\|_{L^\infty}\|\rho(\tau)\|_{L^\infty}\,d\tau.$$

Applying Gronwall's inequality yields

$$\|\rho(t)\|_{L^\infty(\mathbb{S})} \le \|\rho_0\|_{L^\infty}\exp\left[C\int_0^t \|\partial_x u(\tau)\|_{L^\infty}\,d\tau\right].$$

Using the Sobolev embedding theorem $H^s \hookrightarrow L^\infty$ ($s > \frac{1}{2}$), we get from Lemma 3.5 that

$$\|u_x\|_{L^\infty(\mathbb{S})} \le C\left(\|u_0\|_{H^2} + \|\rho_0\|_{L^2}\right).$$

Therefore, we have

$$\|\rho(t)\|_{L^\infty(\mathbb{S})} \le \|\rho_0\|_{L^\infty}e^{Ct(\|u_0\|_{H^2}+\|\rho_0\|_{L^2})} = \|\rho_0\|_{L^\infty}e^{CT\|z_0\|_{H^2\times L^2}}. \tag{27}$$

Next, we split the remaining proof of Theorem 1 into five steps.

Step 1. For $s \in (\frac{3}{2}, 2)$, applying Lemma 3.3 to the second equation, we have

$$\begin{aligned}\|\rho\|_{H^{s-1}(\mathbb{S})} &\leq \|\rho_0\|_{H^{s-1}} + C\int_0^t \|u_x\rho\|_{H^{s-1}}\,d\tau \\ &\quad + C\int_0^t \|\rho\|_{H^{s-1}}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau.\end{aligned}$$

From Lemma 3.1 (5) we get

$$\|\rho u_x\|_{H^{s-1}(\mathbb{S})} \leq C\big(\|u_x\|_{H^{s-1}}\|\rho\|_{L^\infty} + \|\rho\|_{H^{s-1}}\|u_x\|_{L^\infty}\big). \tag{28}$$

From (28) we obtain

$$\begin{aligned}\|\rho\|_{H^{s-1}(\mathbb{S})} &\leq \|\rho_0\|_{H^{s-1}} + C\int_0^t \|u\|_{H^s}\|\rho\|_{L^\infty}\,d\tau \\ &\quad + C\int_0^t \|\rho\|_{H^{s-1}}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau.\end{aligned} \tag{29}$$

On the other hand, using Lemma 3.2, we get from the first equation of system (4) that

$$\begin{aligned}\big\|u(t)\big\|_{H^s(\mathbb{S})} &\leq C\int_0^t \left\|\partial_x g * \left[u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2\right]\right\|_{H^s} d\tau \\ &\quad + \|u_0\|_{H^s} + C\int_0^t \big\|u(t)\big\|_{H^s}\big\|\partial_x u(\tau)\big\|_{L^\infty}\,d\tau.\end{aligned}$$

From Lemma 3.4(b) of [19], we have

$$\begin{aligned}&\left\|\partial_x g * \left[u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2\right]\right\|_{H^s} \\ &\quad \leq C\left\|u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2\right\|_{H^{s-3}} \\ &\quad \leq C\big(\|u\|_{H^{s-3}}\|u\|_{L^\infty} + \|u_x\|_{H^{s-3}}\|u_x\|_{L^\infty} + \|u_{xx}\|_{H^{s-3}}\|u_{xx}\|_{L^\infty} \\ &\qquad + \|u_{xxx}\|_{H^{s-3}}\|u_x\|_{L^\infty} + \|\rho\|_{H^{s-3}}\|\rho\|_{L^\infty}\big).\end{aligned}$$

Hence, we get

$$\begin{aligned}\big\|u(t)\big\|_{H^s(\mathbb{S})} &\leq \|u_0\|_{H^s(\mathbb{S})} + C\int_0^t \big\|\rho(\tau)\big\|_{H^{s-1}}\big\|\rho(\tau)\big\|_{L^\infty}\,d\tau \\ &\quad + C\int_0^t \|u\|_{H^s}\big(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty}\big)\,d\tau,\end{aligned} \tag{30}$$

which, together with (29), ensures that

$$\begin{aligned}&\big\|u(t)\big\|_{H^s(\mathbb{S})} + \big\|\rho(t)\big\|_{H^{s-1}(\mathbb{S})} \\ &\quad \leq \|u_0\|_{H^s(\mathbb{S})} + \|\rho_0\|_{H^{s-1}(\mathbb{S})} + C\int_0^t \big(\|u\|_{H^s} + \big\|\rho(t)\big\|_{H^{s-1}}\big) \\ &\qquad \times \big(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \|\rho\|_{L^\infty}\big)\,d\tau.\end{aligned} \tag{31}$$

Using Gronwall's inequality, we have

$$\begin{aligned}&\|u(t)\|_{H^s(\mathbb{S})} + \|\rho(t)\|_{H^{s-1}(\mathbb{S})}\\&\quad\leq \left(\|u_0\|_{H^s(\mathbb{S})} + \|\rho_0\|_{H^{s-1}(\mathbb{S})}\right)\\&\qquad\times \exp\left[C\int_0^t \left(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \|\rho\|_{L^\infty}\right) d\tau\right].\end{aligned} \tag{32}$$

From (27) and Lemma 3.5 we get

$$\begin{aligned}&\|u(t)\|_{H^s(\mathbb{S})} + \|\rho(t)\|_{H^{s-1}(\mathbb{S})}\\&\quad\leq \left(\|u_0\|_{H^s(\mathbb{S})} + \|\rho_0\|_{H^{s-1}(\mathbb{S})}\right)\\&\qquad\times \exp\left(C\int_0^t \left(L(t) + \|z_0\|_{H^2\times L^2} + \|\rho_0\|_{L^\infty} e^{CT\|z_0\|_{H^2\times L^2}}\right) d\tau\right).\end{aligned} \tag{33}$$

Therefore, if the maximal existence time $T < \infty$ satisfies $\int_0^t \|\rho_x\|_{L^\infty}\, d\tau < \infty$, then we get from (33) that

$$\limsup_{t\to T}\left(\|u(t)\|_{H^s(\mathbb{S})} + \|\rho(t)\|_{H^{s-1}(\mathbb{S})}\right) < \infty, \tag{34}$$

which completes the proof of Theorem 1 for $s \in (\frac{3}{2}, 2)$.

Step 2. For $s \in [2, \frac{5}{2})$, applying Lemma 3.2 to the second equation of system (4), we get

$$\begin{aligned}\|\rho\|_{H^{s-1}(\mathbb{S})} \leq{}& \|\rho_0\|_{H^{s-1}} + C\int_0^t \|u_x\rho\|_{H^{s-1}}\, d\tau\\&+ C\int_0^t \|\rho\|_{H^{s-1}}\|\partial_x u\|_{L^\infty\cap H^{\frac{1}{2}}}\, d\tau.\end{aligned}$$

Using (28) results in

$$\|\rho\|_{H^{s-1}(\mathbb{S})} \leq \|\rho_0\|_{H^{s-1}} + C\int_0^t \|u_x\|_{H^{s-1}}\|\rho\|_{L^\infty}\, d\tau + C\int_0^t \|\rho\|_{H^{s-1}}\|\partial_x u\|_{L^\infty\cap H^{\frac{1}{2}}}\, d\tau,$$

which, together with (30), yields

$$\begin{aligned}&\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\\&\quad\leq \|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + C\int_0^t \left(\|u\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\right)\\&\qquad\times \left(\|u\|_{L^\infty} + \|u\|_{H^{\frac{3}{2}+\varepsilon}} + \|u_{xx}\|_{L^\infty} + \|\rho\|_{L^\infty}\right) d\tau,\end{aligned} \tag{35}$$

where $\varepsilon \in (0, \frac{1}{2})$, and we used the fact that $H^{\frac{1}{2}+\varepsilon} \hookrightarrow L^\infty \cap H^{\frac{1}{2}}$.

Using Gronwall's inequality, we have

$$\begin{aligned}&\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\\&\quad\leq \left(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\right) \exp\left[C\int_0^t \left(\|u\|_{L^\infty} + \|u\|_{H^{\frac{3}{2}+\varepsilon}} + \|u_{xx}\|_{L^\infty} + \|\rho\|_{L^\infty}\right) d\tau\right].\end{aligned} \tag{36}$$

From (27) and Lemma 3.5 we get

$$\begin{aligned}&\|u(t)\|_{H^s}+\|\rho(t)\|_{H^{s-1}}\\&\quad\le\left(\|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}\right)\\&\qquad\times\exp\left(C\int_0^t\left(L(t)+\|z_0\|_{H^2\times L^2}+\|\rho_0\|_{L^\infty}e^{CT\|z_0\|_{H^2\times L^2}}\right)d\tau\right).\end{aligned}\tag{37}$$

Applying the argument as in step 1, we complete the proof of Theorem 1 for $s\in[2,\frac{5}{2})$.

Step 3. For $s\in(2,3)$, differentiating once the second equation of system (4) with respect to x, we have

$$\partial_t\rho_x+u\partial_x\rho_x+2u_x\rho_x+u_{xx}\rho=0.\tag{38}$$

Using Lemma 3.3, we get

$$\begin{aligned}\|\rho_x\|_{H^{s-2}(\mathbf{S})}&\le\|\rho_{0x}\|_{H^{s-2}}+C\int_0^t\|u\|_{H^s}\|\rho\|_{L^\infty}\,d\tau\\&\quad+C\int_0^t\|\rho\|_{H^{s-1}}\left(\|u\|_{L^\infty}+\|\partial_x u\|_{L^\infty}\right)d\tau,\end{aligned}\tag{39}$$

where we used the estimates

$$\|u_x\rho_x\|_{H^{s-2}}\le C\left(\|u_x\|_{H^{s-1}}\|\rho\|_{L^\infty}+\|\rho_x\|_{H^{s-2}}\|u_x\|_{L^\infty}\right)$$

and

$$\|\rho u_{xx}\|_{H^{s-2}}\le C\left(\|\rho\|_{H^{s-1}}\|u_x\|_{L^\infty}+\|u_{xx}\|_{H^{s-2}}\|\rho\|_{L^\infty}\right),$$

where Lemma 3.1 (6) was used.

Using (39), (30), and (29) (where $s-1$ is replaced by $s-2$) yields

$$\begin{aligned}\|u(t)\|_{H^s}+\|\rho(t)\|_{H^{s-1}}&\le\|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}+C\int_0^t\left(\|u\|_{H^s}+\|\rho(t)\|_{H^{s-1}}\right)\\&\quad\times\left(\|u\|_{L^\infty}+\|u_x\|_{L^\infty}+\|u_{xx}\|_{L^\infty}+\|\rho\|_{L^\infty}\right)d\tau.\end{aligned}\tag{40}$$

Applying Gronwall's inequality, we have

$$\begin{aligned}&\|u(t)\|_{H^s}+\|\rho(t)\|_{H^{s-1}}\\&\quad\le\left(\|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}\right)\exp\left[C\int_0^t\left(\|u\|_{L^\infty}+\|u_x\|_{L^\infty}+\|u_{xx}\|_{L^\infty}+\|\rho\|_{L^\infty}\right)d\tau\right].\end{aligned}\tag{41}$$

From (27) and Lemma 3.5 we get

$$\begin{aligned}&\|u(t)\|_{H^s}+\|\rho(t)\|_{H^{s-1}}\\&\quad\le\left(\|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}\right)\\&\qquad\times\exp\left(C\int_0^t\left(L(t)+\|z_0\|_{H^2\times L^2}+\|\rho_0\|_{L^\infty}e^{CT\|z_0\|_{H^2\times L^2}}\right)d\tau\right).\end{aligned}\tag{42}$$

Using the argument as in step 1, we complete the proof of Theorem 1 for $s \in (2, 3)$.

Step 4. For $s = k \in \mathbf{N}$, $k \geq 3$, differentiating $k - 2$ times the second equation of system (4) with respect to x, we obtain

$$(\partial_t + u\partial_x)\partial_x^{k-2}\rho + \sum_{l_1+l_2=k-3, l_1, l_2 \geq 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho + \rho\partial_x\big(\partial_x^{k-2}u\big) = 0. \tag{43}$$

Using Lemma 3.2, we get from (43) that

$$\begin{aligned}\big\|\partial_x^{k-2}\rho\big\|_{H^1} \leq{}& \big\|\partial_x^{k-2}\rho_0\big\|_{H^1} + C\int_0^t \big\|\partial_x^{k-2}\rho\big\|_{H^1}\|\partial_x u\|_{H^{\frac{1}{2}}\cap L^\infty}\,d\tau \\ &+ C\int_0^t \bigg\|\sum_{l_1+l_2=k-3, l_1, l_2 \geq 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho + \rho\partial_x^{k-1}u\bigg\|_{H^1}\,d\tau.\end{aligned} \tag{44}$$

Since H^1 is an algebra, we have

$$\big\|\rho\partial_x^{k-1}u\big\|_{H^1} \leq C\|\rho\|_{H^1}\big\|\partial_x^{k-1}u\big\|_{H^1} \leq C\|\rho\|_{H^1}\|u\|_{H^s}$$

and

$$\bigg\|\sum_{l_1+l_2=k-3, l_1, l_2 \geq 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho\bigg\|_{H^1} \leq C\|\rho\|_{H^{s-1}}\|u\|_{H^{s-1}}.$$

It follows that

$$\begin{aligned}\big\|\partial_x^{k-2}\rho\big\|_{H^1} \leq{}& \big\|\partial_x^{k-2}\rho_0\big\|_{H^1} + C\int_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}}\big) \\ &\times \big(\|u\|_{H^{s-1}} + \|\rho\|_{H^1}\big)\,d\tau.\end{aligned} \tag{45}$$

From the Gagliardo-Nirenberg inequality we have that, for $\sigma \in (0, 1)$,

$$\|\rho\|_{H^{s-1}} \leq C\big(\|\rho\|_{H^\sigma} + \big\|\partial_x^{k-2}\rho\big\|_{H^1}\big). \tag{46}$$

On the other hand, for $\sigma \in (0, 1)$, rewrite (29) as

$$\begin{aligned}\|\rho\|_{H^\sigma(\mathbf{S})} \leq{}& \|\rho_0\|_{H^\sigma} + C\int_0^t \|u\|_{H^{\sigma+1}}\|\rho\|_{L^\infty}\,d\tau \\ &+ C\int_0^t \|\rho\|_{H^\sigma}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau,\end{aligned} \tag{47}$$

which, together with (45), yields

$$\begin{aligned}\|\rho\|_{H^{s-1}} \leq{}& C\|\rho_0\|_{H^{s-1}} + C\int_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}}\big) \\ &\times \big(\|u\|_{H^{s-1}} + \|\rho\|_{H^1}\big)\,d\tau,\end{aligned} \tag{48}$$

where (46) was used.

Using Lemma 3.1 (5), we get

$$\|u(t)\|_{H^s(\mathbf{S})} \le \|u_0\|_{H^s} + C\int_0^t \|u\|_{H^s}\big(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \|u_{xxx}\|_{L^\infty}\big)\,d\tau + C\int_0^t \|\rho(\tau)\|_{H^{s-1}}\|\rho(\tau)\|_{L^\infty}\,d\tau, \tag{49}$$

which, together with (48), results in

$$\begin{aligned}&\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\\ &\quad\le C\big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big) + C\int_0^t \big(\|u\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\big)\\ &\qquad\times\big(\|u\|_{H^{s-1}} + \|\rho\|_{H^1} + \|u_{xx}\|_{L^\infty} + \|u_{xxx}\|_{L^\infty}\big)\,d\tau.\end{aligned} \tag{50}$$

Using Gronwall's inequality, we get

$$\begin{aligned}&\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\\ &\quad\le C\big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big)\\ &\qquad\times\exp\bigg[C\int_0^t \big(\|u\|_{H^{s-1}} + \|\rho\|_{H^1} + \|u_{xx}\|_{L^\infty} + \|u_{xxx}\|_{L^\infty}\big)\,d\tau\bigg].\end{aligned} \tag{51}$$

If $T < \infty$ satisfies $\int_0^T \|\rho_x\|_{L^\infty}\,d\tau < \infty$, applying step 2 and the induction assumption, we obtain from Lemma 3.5 that $\|u\|_{H^{s-1}} + \|\rho\|_{H^1} + \|u_{xx}\|_{L^\infty} + \|u_{xxx}\|_{L^\infty}$ is uniformly bounded. From (51) we get

$$\limsup_{t\to T}\big(\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\big) < \infty,$$

which contradicts the assumption that $T < \infty$ is the maximal existence time. This completes the proof of Theorem 1 for $s = k \in N$ and $k \ge 3$.

Step 5. For $s \in (k, k+1)$, $k \in N$, and $k \ge 3$, differentiating $k-1$ times the second equation of system (4) with respect to x, we obtain

$$(\partial_t + u\partial_x)\partial_x^{k-1}\rho + \sum_{l_1+l_2=k-2,\,l_1,l_2\ge 0} C_{l_1,l_2}\partial_x^{l_1+1}u\,\partial_x^{l_2+1}\rho + \rho\partial_x\big(\partial_x^{k-1}u\big) = 0. \tag{52}$$

Using Lemma 3.3 with $s-k \in (0,1)$, we get from (52) that

$$\begin{aligned}\big\|\partial_x^{k-1}\rho\big\|_{H^{s-k}} &\le \big\|\partial_x^{k-1}\rho_0\big\|_{H^{s-k}} + C\int_0^t \big\|\partial_x^{k-1}\rho\big\|_{H^{s-k}}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau\\ &\quad + C\int_0^t \bigg\|\sum_{l_1+l_2=k-2,\,l_1,l_2\ge 0} C_{l_1,l_2}\partial_x^{l_1+1}u\,\partial_x^{l_2+1}\rho + \rho\partial_x^k u\bigg\|_{H^{s-k}}\,d\tau.\end{aligned} \tag{53}$$

For each $\varepsilon \in (0, \frac{1}{2})$, using Lemma 3.1 (6) and the fact that $H^{\frac{1}{2}+\varepsilon} \hookrightarrow L^\infty$, we have

$$\begin{aligned}\big\|\rho\partial_x^k u\big\|_{H^{s-k}} &\le C\big(\|\rho\|_{H^{s-k+1}}\big\|\partial_x^{k-1}u\big\|_{L^\infty} + \big\|\partial_x^k u\big\|_{H^{s-k}}\|\rho\|_{L^\infty}\big)\\ &\le C\big(\|\rho\|_{H^{s-k+1}}\|u\|_{H^{k-\frac{1}{2}+\varepsilon}} + \|u\|_{H^{s-1}}\|\rho\|_{L^\infty}\big)\end{aligned} \tag{54}$$

and

$$\begin{aligned}
&\Big\| \sum_{l_1+l_2=k-2, l_1,l_2\geq 0} C_{l_1,l_2} \partial_x^{l_1+1} u \partial_x^{l_2+1} \rho \Big\|_{H^{s-k}} \\
&\quad \leq C \sum_{l_1+l_2=k-2, l_1,l_2\geq 0} C_{l_1,l_2} \big(\big\| \partial_x^{l_1+1} u \big\|_{H^{s-k+1}} \big\| \partial_x^{l_2} \rho \big\|_{L^\infty} \\
&\qquad + \big\| \partial_x^{l_1+1} u \big\|_{L^\infty} \big\| \partial_x^{l_2+1} \rho \big\|_{H^{s-k}} \big) \\
&\quad \leq C \big(\|u\|_{H^s} \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon}} + \|u\|_{H^{k-\frac{1}{2}+\varepsilon}} \|\rho\|_{H^{s-1}} \big).
\end{aligned} \tag{55}$$

Therefore, from (53), (54), and (55) we get

$$\begin{aligned}
\big\| \partial_x^{k-1} \rho \big\|_{H^{s-k}} \leq \big\| \partial_x^{k-1} \rho_0 \big\|_{H^{s-k}} + C \int_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}} \big) \\
\times \big(\|u\|_{H^{k-\frac{3}{2}+\varepsilon}} + \|\rho\|_{H^{k-\frac{1}{2}+\varepsilon}} \big)\, d\tau.
\end{aligned} \tag{56}$$

Applying Lemma 3.2 to the first equation of system (4) for $s \in (k, k+1)$ with $k \geq 3$, we obtain

$$\begin{aligned}
\big\| u(t) \big\|_{H^s(\mathbf{S})} \leq \|u_0\|_{H^s} + C \int_0^t \big\| \rho(\tau) \big\|_{H^{s-1}} \big\| \rho(\tau) \big\|_{L^\infty}\, d\tau \\
+ C \int_0^t \|u\|_{H^s} \big(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} \big)\, d\tau,
\end{aligned} \tag{57}$$

which, together with (56) and (29) (where $s-1$ is replaced by $s-k$), gives

$$\begin{aligned}
&\big\| u(t) \big\|_{H^s} + \big\| \rho(t) \big\|_{H^{s-1}} \\
&\quad \leq C \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} \big) + C \int_0^t \big(\|u\|_{H^s} + \big\| \rho(t) \big\|_{H^{s-1}} \big) \\
&\quad \times \big(\|u\|_{H^{k-\frac{1}{2}+\varepsilon}} + \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon}} \big)\, d\tau.
\end{aligned} \tag{58}$$

Using Gronwall's inequality, we get

$$\begin{aligned}
&\big\| u(t) \big\|_{H^s} + \big\| \rho(t) \big\|_{H^{s-1}} \\
&\quad \leq C \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} \big) \exp \bigg[C \int_0^t \big(\|u\|_{H^{k-\frac{1}{2}+\varepsilon}} + \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon}} \big)\, d\tau \bigg].
\end{aligned} \tag{59}$$

Noting that $k - \frac{1}{2} + \varepsilon < k$, $k - \frac{3}{2} + \varepsilon < k-1$, and $k \geq 3$ and applying step 4, we obtain that $\|u\|_{H^{k-\frac{1}{2}+\varepsilon}} + \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon}}$ is uniformly bounded. Therefore, we complete the proof of Theorem 1 for $s \in (k, k+1)$, $k \in N$, and $k \geq 3$.

So, the proof of Theorem 1 is completed. □

Competing interests

The author declares that they have no competing interests.

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